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OPERATORS ON TENSOR PRODUCTS OF $\mathscr{L}_1$, $\mathscr{L}_2$ AND $\mathscr{L}_\infty$ SPACES

Dedicated to Professor Yoneichiro Sakaki on his 60th birthday, March 5, 1973

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Introduction

Let A and B be densely defined closed linear operators in complex Banach spaces X, Y, respectively, with nonempty resolvent sets. Then for a class of polynomials $P(\xi,\eta)$ the spectral mapping theorem has been established by the author [9] (cf. [7], [8]):

$$P(\sigma(A),\sigma(B)) = \sigma(P(A\otimes I, I\otimes B)) = \sigma(\tilde{P}(A\otimes I, I\otimes B)) ,$$

where $\tilde{P}(A\otimes I, I\otimes B)$ is a maximal extension of $P(A\otimes I, I\otimes B)$ in $X\hat{\otimes}_\alpha Y$, the completion of $X\otimes Y$ with respect to a uniform reasonable norm α. Another investigation has recently been made by M. Reed and B. Simon [17].

The aim of this work is to extend the spectral mapping theorem to a much larger class of polynomials $P(\xi,\eta)$, when both X and Y are $\mathscr{L}_1$, $\mathscr{L}_2$ or $\mathscr{L}_\infty$ spaces of J. Lindenstrauss and A. Pełczyński [13] and one of A and B is a scalar type spectral operator (see [2]). In contrast to the results in [9], the set $P(\sigma(A),\sigma(B))$ may not always be closed (cf. [17]).

The theory applies to the operators of the form $A\otimes I + I\otimes B$, which include not only the elliptic and parabolic differential operators but also the hyperbolic differential operators. A new meaning is given thereby to the method of separation of variables for partial differential equations (cf. [3]).

Section 1 is concerned with some definitions and results on $\mathscr{L}_p$ spaces and tensor products. Our main results are formulated in Section 2. Section 3 is devoted, in particular, to the operators of the forms $A\otimes I + I\otimes B$ and $A\hat{\otimes}_\alpha I + I\hat{\otimes}_\alpha B$. In Section 4, we refer to some applications of the results.

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For the basic facts on topological linear spaces and tensor products, see [11], [4], [5] and [18].

1. Preliminary Results

We shall start by summarizing some useful results on $\mathscr{L}_p$ spaces and tensor products.

1.1. $\mathscr{L}_p$ spaces

J. Lindenstrauss and A. Pełczyński [13] have introduced the $\mathscr{L}_p$ spaces. We shall only concern with the cases $p = 1$, 2 and ∞ in this work.

Two Banach spaces E and F are isomorphic if there exists an invertible bounded linear operator of E onto F. The distance $d(E, F)$ of two isomorphic Banach spaces is defined by $\inf(\|T\|\,\|T^{-1}\|)$, where the infimum is taken over all invertible bounded linear operators T of E onto F.

By $L_p(\mu) = L_p(\Omega, \mathscr{B}, \mu)$, $1 \leq p \leq \infty$, we denote the Banach space of all equivalence classes of measurable functions on some measure space $(\Omega, \mathscr{B}, \mu)$ whose p-th power is integrable (resp. essentially bounded if $p = \infty$). If $(\Gamma, \mathscr{B}, \mu)$ is the discrete measure space on a set Γ with $\mu(\{\gamma\}) = 1$ for every $\gamma \in \Gamma$, we denote $L_p(\mu)$ by $\ell_p(\Gamma)$. If $\Gamma = \{1, 2, \cdots, n\}$, we denote $\ell_p(\Gamma)$ by ℓ_p^n. The subspace of $\ell_\infty(\Gamma)$ of those $f \in \ell_\infty(\Gamma)$ for which the set $\{\gamma \in \Gamma; |f(\gamma)| \geq \varepsilon\}$ is finite for every $\varepsilon > 0$ is denoted by $c_0(\Gamma)$. For a compact Hausdorff space K, we denote by $C(K)$ the Banach space of all continuous functions on K.

A Banach space E is said to be an *$\mathscr{L}_{p,\lambda}$ space*, $1 \leq p \leq \infty$, $1 \leq \lambda < \infty$, if for each finite-dimensional subspace $F \subset E$ there exists a finite-dimensional subspace G with $F \subset G \subset E$ such that $d(G, \ell_p^n) \leq \lambda$, where $n = \dim G$, the dimension of G,

A Banach space E is said to be an *$\mathscr{L}_p$ space*, $1 \leq p \leq \infty$, if it is an $\mathscr{L}_{p,\lambda}$ space for some $\lambda \geq 1$.

It is known [13] that the Banach spaces $L_p(\mu)$ (resp. $C(K)$) are $\mathscr{L}_{p,\lambda}$ (resp. $\mathscr{L}_{\infty,\lambda}$) spaces for every $\lambda > 1$, but for $1 \leq p \leq \infty$, $p \neq 2$, there exist $\mathscr{L}_p$ spaces which are not isomorphic to the spaces $L_p(\mu)$. For $p = 2$, the class of $\mathscr{L}_2$ spaces coincides with the class of spaces isomorphic to Hilbert spaces, so that an $\mathscr{L}_2$ space can be considered to be endowed with an inner product.

A closed subspace F of a Banach space E is said to be a *complemented* subspace if there is a continuous projection of E onto F.

A complemented subspace of an $\mathscr{L}_p$ space $(1 \leq p \leq \infty)$ which is not isomorphic to a Hilbert space is an $\mathscr{L}_p$ space [14]. Every $\mathscr{L}_p$ space $(1 \leq p \leq \infty)$ is isomorphic to a subspace of a space $L_p(\mu)$ for some measure μ[13]. Every Banach space E is an $\mathscr{L}_p$ space $(1 \leq p \leq \infty)$ if and only if its dual X' is an $\mathscr{L}_{p'}$ space with $1/p + 1/p' = 1$ [14].

By a bounded Boolean algebra $\mathscr{E}$ of projections in a Banach space Z, we mean a Boolean algebra of commuting continuous projection E in Z such that $\sup \|E\| < K$ for all $E \in \mathscr{E}$ (see [2]).

We shall make use of a result of J. Lindenstrauss and A. Pełczyński on unconditional Schauder decompositions of $\mathscr{L}_1$ and $\mathscr{L}_\infty$ spaces [13, Corollary 8 to Theorem 6.1], stated in the following form we need.

THEOREM 1.1. *Let Z be an $\mathscr{L}_1$ (resp. $\mathscr{L}_\infty$) space and let $\mathscr{E}$ be a bounded Boolean algebra of projections. Then there exists a positive constant M_1 (resp. M_∞) such that for every finite family $\{E_k\}_{k=1}^s$ of disjoint projections in $\mathscr{E}$*

$$\sum_{k=1}^s \|E_k z\| \leq M_1 \| \sum_{k=1}^s E_k z\| , \qquad z \in Z$$
$$(\textit{resp. } \|\sum_{k=1}^s E_k z\| \leq M_\infty \max_{1\leq k \leq s} \|E_k z\|, z \in Z).$$

1.2. Tensor Products

Let X and Y be complex Banach spaces and X', Y' their dual spaces. Let $X \otimes Y$ be the algebraic tensor product of X and Y and $X \hat{\otimes}_\alpha Y$ its completion with respect to a uniform reasonable norm α on $X \otimes Y$.

Suppose that $A: \mathrm{D}[A] \subset X \to X$ and $B: D[B] \subset Y \to Y$ are densely defined closed linear operators with nonempty resolvent sets $\rho(A)$, $\rho(B)$ and with spectra $\sigma(A)$, $\sigma(B)$. Assume further that B is a scalar type spectral operator with the countably additive resolution E of the identity [2]. We may assume $\sigma(B)$ nonempty. The identity operators in both X and Y will be denoted by the same I.

To each polynomial of degrees m in ξ and n in η

$$P(\xi, \eta) = \sum c_{jk}\xi^j \cdot \eta^k , \tag{1.1}$$

we assign two kinds of polynomial operators defined densely in $X \hat{\otimes}_\alpha Y$

$$P(A \otimes I, I \otimes B) = \sum c_{jk} A^j \otimes B^k \tag{1.2}$$

with domain $D[A^m] \otimes D[B^n]$ and

$$\sum c_{jk} A^j \hat{\otimes}_\alpha B^k \tag{1.3}$$

with domain $\bigcap_{j,k; c_{jk}\neq 0} D[A^j \hat{\otimes}_\alpha B^k]$, where $A^j \hat{\otimes}_\alpha B^k$ denotes a maximal

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extension of $A^j \otimes B^k$ in $X \hat{\otimes}_\alpha Y$. Maximal extensions of the operators (1.2) and (1.3) are denoted by $\tilde{P}(A \otimes I, I \otimes B)$, $(\sum c_{jk} A^j \hat{\otimes}_\alpha B^k)^\sim$, respectively (see [12], [7], [8] and [9]).

In order to establish our main results, we shall restrict the Banach spaces $X \hat{\otimes}_\alpha Y$ concerned to the following three cases:

(α_1) *both X and Y are $\mathscr{L}_1$ spaces and α is the greatest reasonable norm π;*

(α_2) *both X and Y are $\mathscr{L}_2$ spaces and α is the uniform crossnorm α_0 for which $X \hat{\otimes}_\alpha Y$ is an $\mathscr{L}_2$ space;*

(α_∞) *both X and Y are $\mathscr{L}_\infty$ spaces and α is the smallest reasonable norm ε.*

In cases (α_2) and (α_∞), the norms α_0 and ε are faithful. For case (α_1), however, the author is unaware whether or not the norm π there is faithful. This is certainly true if X or Y satisfies the condition of approximation [4]. It is known [10] that a separable $\mathscr{L}_p$ space ($1 \le p \le \infty$) has a Schauder basis. Therefore we can assert in case (α_1) that if X and Y are besides separable, then the norm π there is faithful (e.g. [7]).

Thus, as is the case in all applications, we assume for simplicity further that *the norm π is faithful in case (α_1) whenever both X and Y are $\mathscr{L}_1$ spaces.*

For faithful α, $A^j \hat{\otimes}_\alpha B^k$ is nothing but the closure of $A^j \otimes B^k$ in $X \hat{\otimes}_\alpha Y$, and the same is true for $\tilde{P}(A \otimes I, I \otimes B)$ and $(\sum c_{jk} A^j \hat{\otimes}_\alpha B^k)^\sim$.

We remark (cf. [14], [6]) that under the condition (α_1) (resp. (α_2), (α_∞)), $X \hat{\otimes}_\alpha Y$ is also an $\mathscr{L}_1$ (resp. $\mathscr{L}_2$, $\mathscr{L}_\infty$) space.

To prove Proposition 2.1, we shall need

LEMMA 1.2. *Suppose the space $X \hat{\otimes}_\alpha Y$ satisfies the condition (α_1), (α_2) or (α_∞). If Y_1 is a complemented subspace of Y, then $X \hat{\otimes}_\alpha Y_1$ is a complemented subspace of $X \hat{\otimes}_\alpha Y$. Therefore, if $Y = \sum_{k=1}^s \oplus Y_k$ is a topological direct sum, then $X \hat{\otimes}_\alpha Y = \sum_{k=1}^s \oplus (X \hat{\otimes}_\alpha Y_k)$.*

Proof. It is trivial for case (α_2). For cases (α_1) and (α_∞), note that the norms π and ε are $\otimes$-norms in the sense of A. Grothendieck [5], so that they are defined for every pair of Banach spaces X and Y. Let P be the continuous projection of Y onto Y_1, and J the injection of Y_1 into Y. Then, since α is a $\otimes$-norm, $I \hat{\otimes}_\alpha P$ (resp. $I \hat{\otimes}_\alpha J$) is a continuous linear operator of $X \hat{\otimes}_\alpha Y$ (resp. $X \hat{\otimes}_\alpha Y_1$) into $X \hat{\otimes}_\alpha Y_1$ (resp. $X \hat{\otimes}_\alpha Y$). We have $I \hat{\otimes}_\alpha Pu = (I \otimes P)u = u$ on $X \otimes Y_1$. Since $(I \otimes P)(I \otimes J)u = u$ on

$X \otimes Y_1$, it follows by continuity that $(I \hat{\otimes}_\alpha P)(I \hat{\otimes}_\alpha J)u = u$ on $X \hat{\otimes}_\alpha Y_1$, which implies that the range of $I \hat{\otimes}_\alpha P$ is $X \hat{\otimes}_\alpha Y_1$. Clearly $(I \hat{\otimes}_\alpha P)^2 = I \hat{\otimes}_\alpha P$. Thus $I \hat{\otimes}_\alpha P$ is a continuous projection of $X \hat{\otimes}_\alpha Y$ onto $X \hat{\otimes}_\alpha Y_1$, so that $X \hat{\otimes}_\alpha Y_1$ is a complemented subspace of $X \hat{\otimes}_\alpha Y$. Q.E.D.

2. Spectral Mapping Theorem

For the spaces $X \hat{\otimes}_\alpha Y$ and the operators A and B described in Section 1, we shall formulate the spectral mapping theorem for the polynomial operators (1.2) and (1.3).

Throughout, the following convention will be used. Given two subsets F_1 and $F_2 \neq \emptyset$ of the complex plane C and a polynomial $P(\xi, \eta)$ of degrees $m \geq 1$ in ξ and n in η, we can define $P(F_1, F_2)$ and its closure $\overline{P(F_1, F_2)}$ in an obvious way if F_1 is not empty, and otherwise we set $P(F_1, F_2) = \overline{P(F_1, F_2)} = \emptyset$.

PROPOSITION 2.1. *Let α be a faithful uniform reasonable norm on $X \otimes Y$. Suppose that the space $X \hat{\otimes}_\alpha Y$ satisfies the condition (α_1), (α_2) or (α_∞), and that A and B are those operators described in Section* 1. *Let $P(\xi, \eta)$ be a polynomial of degrees $m \geq 1$ in ξ and n in η such that if $\sigma(A)$ is nonempty, $\overline{P(\sigma(A), \sigma(B))}$ does not cover the whole complex plane C. Then for $\lambda \notin \overline{P(\sigma(A), \sigma(B))}$, we have*

$$\lambda \notin \sigma(P(A \otimes I, I \otimes B)) = \sigma(\tilde{P}(A \otimes I, I \otimes B)) ,$$

provided that $P(A, \eta) - \lambda I$, with $\eta \in \sigma(B)$, has an everywhere defined continuous inverse $(P(A, \eta) - \lambda I)^{-1}$ which is uniformly bounded on $\sigma(B)$.

Proof. First note that the complement of $\sigma(B)$ has E-measure zero so that $E(\sigma(B)) = I$. We shall show for $\lambda \notin \overline{P(\sigma(A), \sigma(B))}$ that $\tilde{P}(A \otimes I, I \otimes B) - \lambda I \hat{\otimes}_\alpha I$ has an everywhere defined continuous inverse. To do so, we must establish that the improper Riemann integral

$$\int_{\sigma(B)} [(P(A, \eta) - \lambda I)^{-1} \otimes E(d\eta)]v , \qquad v \in X \otimes Y , \tag{2.1}$$

exists a an element of $X \hat{\otimes}_\alpha Y$ and defines a continuous linear operator of $X \otimes Y \subset X \hat{\otimes}_\alpha Y$ into $X \hat{\otimes}_\alpha Y$, taking the condition for the spaces $X \hat{\otimes}_\alpha Y$ into consideration.

Let σ be an arbitrary Borel set and $e = \sigma \cap \sigma(\mathrm{B})$. Let $\{e_1, \cdots, e_s\}$ be an arbitrary finite decomposition of σ into disjoint Borel sets. We may

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assume $e_k \cap e \neq \emptyset$ for all k, $1 \leq k \leq s$. Set $e'_k = e_k \cap e$ and $\eta_k \in e'_k$ for $k = 1, 2, \cdots, s$. By assumption, $(P(A, \eta_k) - \lambda I)^{-1}$ is then uniformly bounded for all k and for all the decompositions $\{e_k\}_{k=1}^s$ of σ into disjoint Borel sets. Clearly $E(e) = \sum_{k=1}^s E(e'_k)$. It follows in virtue of Lemma 1.2 that the $I \hat{\otimes}_\alpha E(e'_k)$ are mutually disjoint continuous projections in $X \hat{\otimes}_\alpha Y$ and of σ into $I \hat{\otimes}_\alpha E(e) = \sum_{k=1}^s I \hat{\otimes}_\alpha E(e'_k)$.

In case (α_1) where $X \hat{\otimes}_\alpha Y$ in an $\mathscr{L}_1$ space, we obtain by Theorem 1.1 for $v \in X \otimes Y$

$$\begin{aligned}
&\|\textstyle\sum_{k=1}^s [(P(A, \eta_k) - \lambda I)^{-1} \otimes E(e'_k)]v\|_\pi \\
&\quad = \|\textstyle\sum_{k=1}^s [(P(A, \eta_k) - \lambda I)^{-1} \otimes I][I \otimes E(e'_k)]v\|_\pi \\
&\quad \leq \textstyle\sum_{k=1}^s \|(P(A, \eta_k) - \lambda I)^{-1}\| \, \|[I \otimes E(e'_k)]v\|_\pi \\
&\quad \leq C_1 \textstyle\sum_{k=1}^s \|[I \otimes E(e'_k)]v\|_\pi \\
&\quad \leq C_1 M_1 \|\textstyle\sum_{k=1}^s [I \otimes E(e'_k)]v\|_\pi \\
&\quad = C \|[I \otimes E(e)]v\|_\pi \, .
\end{aligned}$$

In case (α_2), since $X \hat{\otimes}_\alpha Y$ is considered as a Hilbert space, we obtain for v, $w \in X \otimes Y$

$$\begin{aligned}
&|(\textstyle\sum_{k=1}^s [(P(A, \eta_k) - \lambda I)^{-1} \otimes E(e'_k)]v, w)]| \\
&\quad = |\textstyle\sum_{k=1}^s ([(P(A, \eta_k) - \lambda I)^{-1} \otimes I][I \otimes E(e'_k)]v, [I \otimes E(e'_k)]w)| \\
&\quad \leq \textstyle\sum_{k=1}^s \|(P(A, \eta_k) - \lambda I)^{-1}\| \, \|[I \otimes E(e'_k)]v\|_{\alpha_0} \cdot \|[I \otimes E(e'_k)]w\|_{\alpha_0} \\
&\quad \leq C_1 \{\textstyle\sum_{k=1}^s ([I \otimes E(e'_k)]v, v)\}^{1/2} \{\textstyle\sum_{k=1}^s ([I \otimes E(e'_k)]w, w)\}^{1/2} \\
&\quad = C \|[I \otimes E(e)]v\|_{\alpha_0} \cdot \|w\|_{\alpha_0} \, .
\end{aligned}$$

In case (α_∞) where $X \hat{\otimes}_\varepsilon Y$ is an $\mathscr{L}_\infty$ space, by Theorem 1.1 we obtain for $v \in X \otimes Y$

$$\begin{aligned}
&\|\textstyle\sum_{k=1}^s [(P(A, \eta_k) - \lambda I)^{-1} \otimes E(e'_k)]v\|_\varepsilon \\
&\quad = \|\textstyle\sum_{k=1}^s [I \otimes E(e'_k)](\textstyle\sum_{j=1}^s [(P(A, \eta_j) - \lambda I)^{-1} \otimes E(e'_j)]v)\|_\varepsilon \\
&\quad \leq M_\infty \max_{1 \leq k \leq s} \|[(P(A, \eta_k) - \lambda I)^{-1} \otimes E(e'_k)]v\|_\varepsilon \\
&\quad \leq M_\infty \max_{1 \leq k \leq s} \|(P(A, \eta_k) - \lambda I)^{-1}\| \, \|[I \otimes E(e'_k)]v\|_\varepsilon \\
&\quad \leq C_1 \max_{1 \leq k \leq s} \|[I \otimes E(e'_k)]v\|_\varepsilon \\
&\quad \leq C \|[I \otimes E(e)]v\|_\varepsilon \, .
\end{aligned}$$

Here we have used the same symbol C to express different constants independent of the decompositions.

This assures that the integral (2.1) defines a continuous linear operator

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of $X \otimes Y \subset X \hat{\otimes}_\alpha Y$ into $X \hat{\otimes}_\alpha Y$. We denote its continuous extension to $X \hat{\otimes}_\alpha Y$ by $\widetilde{P_\lambda^{-1}}(A \otimes I, I \otimes B)$ (with $P_\lambda = P - \lambda$).

Recalling the definition of the integral with respect to the measure E, we can show easily that for $u \in D[A^m] \otimes D[B^n]$

$$\begin{aligned}
&\widetilde{P_\lambda^{-1}}(A \otimes I, I \otimes B)[\tilde{P}(A \otimes I, I \otimes B) - \lambda I \hat{\otimes}_\alpha I]u \\
&\quad = \int_{\sigma(B)} [(P(A,\eta) - \lambda I)^{-1} \otimes I][P(A \otimes I, I \otimes B) - \lambda I \otimes I][I \otimes E(d\eta)]u \\
&\quad = \int_{\sigma(B)} [(P(A,\eta) - \lambda I)^{-1} \otimes I][(P(A,\eta) - \lambda I) \otimes I][I \otimes E(d\eta)]u \\
&\quad = \int_{\sigma(B)} [I \otimes E(d\eta)]u \\
&\quad = u\,.
\end{aligned}$$

It follows by the continuity of $\widetilde{P_\lambda^{-1}}(A \otimes I, I \otimes B)$ that

$$\widetilde{P_\lambda^{-1}}(A \otimes I, I \otimes B)[\tilde{P}(A \otimes I, I \otimes B) - \lambda I \hat{\otimes}_\alpha I]u = u$$

for all u in the domain of $\tilde{P}(A \otimes I, I \otimes B)$.

Just in the same way, using the closedness of $\tilde{P}(A \otimes I, I \otimes B)$, we can show that

$$[\tilde{P}(A \otimes I, I \otimes B) - \lambda I \hat{\otimes}_\alpha I]\widetilde{P_\lambda^{-1}}(A \otimes I, I \otimes B)v = v$$

for all $v \in X \hat{\otimes}_\alpha Y$.

Thus, $\widetilde{P_\lambda^{-1}}(A \otimes I, I \otimes B)$ is the everywhere defined continuous inverse of $\tilde{P}(A \otimes I, I \otimes B) - \lambda I \hat{\otimes}_\alpha I$. Q.E.D.

In order to state the spectral mapping theorem for the operator (1.2) and its closure, we introduce a class of polynomials, larger than the one in [9], which will turn out to satisfy the assumptions of Proposition 2.1.

Let $\mathscr{P}'(A, B)$ be the class of polynomials $P(\xi, \eta)$ of degrees $m \geq 1$ in ξ and n in η satisfying the following condition: for any open neighbourhood W in $\boldsymbol{C}$ of the closure of $P(\sigma(A), \sigma(B))$ (when $\sigma(A)$ is empty, take $W = \complement K(0; R)$ for any $R > 0$, where $K(0; R)$ is the closed disc $\{\zeta; |\zeta| \leq R\}$), there exists a nonempty open set U whose complement $\complement U$ is contained in $\rho(A)$ (resp. $\rho(B)$) such that

(i)′ $P(U, \sigma(B)) \subset W$, and

(ii)′ the resolvent $R(\zeta; A)$ is uniformly bounded in $\complement U$.

We note that the set $P(\sigma(A), \sigma(B))$ is not necessarily closed in $\boldsymbol{C}$ (cf. [9] and [17]).

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Then we have

THEOREM 2.2. *Let* α *be a faithful uniform reasonable norm on* $X \otimes Y$. *Suppose the space* $X \hat{\otimes}_\alpha Y$ *satisfies the condition* (α_1), (α_2) *or* (α_∞). *Let* $A: D[A] \subset X \to X$ *be a densely defined closed linear operator with* $\rho(A) \neq \emptyset$ *and let* $B: D[B] \subset Y \to Y$ *be a densely defined, closed, scalar type spectral operator with* $\rho(B) \neq \emptyset$ *and* $\sigma(B) \neq \emptyset$. *Then for* $P \in \mathscr{P}'(A, B)$ *it holds*

$$\overline{P(\sigma(A), \sigma(B))} = \sigma(P(A \otimes I, I \otimes B)) = \sigma(\tilde{P}(A \otimes I, I \otimes B)) . \tag{2.2}$$

This means that (2.2) *holds valid if* $\sigma(A)$ *is not empty, and that the spectra of* $P(A \otimes I, I \otimes B)$ *and its closure* $\tilde{P}(A \otimes I, I \otimes B)$ *are empty if and only if* $\sigma(A)$ *is empty.*

Proof. Let $P \in \mathscr{P}'(A, B)$ be of the form

$$P(\xi, \eta) = c_m(\eta)\xi^m + c_{m-1}(\eta)\xi^{m-1} + \cdots + c_0(\eta) , \tag{2.3}$$

where $c_m(\eta) \not\equiv 0$. When $\sigma(A)$ and $\sigma(B)$ are nonempty, the inclusion

$$\overline{P(\sigma(A), \sigma(B))} \subset \sigma(P(A \otimes I, I \otimes B)) = \sigma(\tilde{P}(A \otimes I, I \otimes B))$$

is already shown ([8], [9]).

The proof of the rest of Theorem 2.2 will be reduced to Proposition 2.1. Since the resolvent set $\rho(A)$ is not empty, for η fixed

$$P(A, \eta) = \sum_{j=0}^m c_j(\eta) A^j$$

is a densely defined closed linear operator in X with domain $D[A^{m(\eta)}]$, where $m(\eta)$ is the greatest integer, $0 \leq m(\eta) \leq m$, for which $c_{m(\eta)}(\eta) \neq 0$. When $\sigma(A)$ is not empty, we may assume $\overline{P(\sigma(A), \sigma(B))} \neq \boldsymbol{C}$. Then for $\lambda \notin \overline{P(\sigma(A), \sigma(B))}$ (when $\sigma(A)$ is empty, λ shall be an arbitrary complex number), we have only to show that $(P(A, \eta) - \lambda I)^{-1}$ is a continuous linear operator defined on the whole X for each $\eta \in \sigma(B)$ and is uniformly bounded on $\sigma(B)$.

Since $P \in \mathscr{P}'(A, B)$, there exists by assumption a nonempty open set U such that $|P(\xi, \eta) - \lambda|$ is bounded away from zero on $U \times \sigma(B)$, and such that $R(\xi; A)$ is uniformly bounded in $\complement U$.

Choose a sufficiently large $R > 0$ such that the polynomial $c_m(\eta)$ in (2.3) has no zero on $\sigma(B) \cap \complement K(0; R-1)$. Then we have $c_m = \inf |c_m(\eta)| > 0$ for $\eta \in \sigma(B) \cap \complement K(0; R)$. Since $P(A, \eta) - \lambda I$ is a closed operator, we have by

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the usual spectral mapping theorem $\sigma(P(A,\eta)) = P(\sigma(A),\eta)$ for each $\eta \in \sigma(B)$. It follows that $P(A,\eta) - \lambda I$ has an everywhere defined continuous inverse in X for each $\eta \in \sigma(B)$. It is clear that $(P(A,\eta) - \lambda I)^{-1}$ is uniformly bounded on the compact set $\sigma(B) \cap K(0;R)$.

Further for $\eta \in \sigma(B) \cap \complement K(0;R)$ we have

$$P(\xi,\eta) - \lambda = c_m(\eta) \prod_{j=1}^{m} (\xi - \xi_j(\eta)) ,$$

where none of the $\xi_j(\eta)$ lie in U. Since $R(\xi;A)$ is uniformly bounded in $\complement U$, we obtain for $\eta \in \sigma(B) \cap \complement K(0;R)$ and for $x \in D[A^m]$

$$\begin{aligned} \|(P(A,\eta) - \lambda I)x\| &= \|c_m(\eta) \prod_{j=1}^{m} (A - \xi_j(\eta)I)x\| \\ &\geq c_m \| \prod_{j=1}^{m} (A - \xi_j(\eta)I)x\| \\ &\geq C\|x\| , \end{aligned}$$

with a positive constant C independent of η. Since $D[A^m]$ is the domain of $P(A,\eta)$ for these η, $(P(A,\eta) - \lambda I)^{-1}$ is also uniformly bounded on $\sigma(B) \cap \complement K(0;R)$. This proves uniform boundedness of $(P(A,\eta) - \lambda I)^{-1}$ on $\sigma(B)$. Q.E.D.

To establish the spectral mapping theorem for the operator (1.3) and its closure, we shall show

THEOREM 2.3. *Let $X \hat{\otimes}_\alpha Y$ and A, B be as in Theorem 2.2. For a polynomial $P(\xi,\eta)$ of degrees $m \geq 1$ in ξ and n in η, if there is a complex number λ such that the closed operator $P(A,\eta) - \lambda I$ has an everywhere defined continuous inverse for each $\eta \in \sigma(B)$ which is uniformly bounded on $\sigma(B)$, then the closures of the polynomial operators* (1.2) *and* (1.3) *coincide.*

In particular, for $P \in \mathscr{P}'(A,B)$ the above assertion is valid, provided that $\overline{P(\sigma(A),\sigma(B))} \neq \mathbf{C}$.

Proof. We must show the closed operator $(\sum c_{jk} A^j \hat{\otimes}_\alpha B^k)^{\sim} - \lambda I \hat{\otimes}_\alpha I$ is one-to-one. Since the norm α considered is always faithful, it suffices to prove that $\tilde{P}(A' \otimes I', I' \otimes B') - \lambda I' \hat{\otimes}_{\alpha'} I'$ has a dense range in $X' \hat{\otimes}_{\alpha'} Y'$ (cf. [9]).

Let us note the following facts. The (Banach-space-) adjoint B' of B is also a scalar type spectral operator in the dual space Y' with the resolution E' (=the adjoint of E) of the identity. The spectrum of a densely defined operator coincides with that of its adjoint, and

$$P(A',\eta) - \lambda I' = (P(A,\eta) - \lambda I)' \qquad \text{for } \eta \in \sigma(B) = \sigma(B') .$$

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In case (α_1) (resp. (α_2)), $X' \hat{\otimes}_{\alpha'} Y'$ is an $\mathscr{L}_\infty$ (resp. $\mathscr{L}_2$) space, because $\pi' = \varepsilon$ (resp. $\alpha'_0 = \alpha_0$). Then we can show just in the same way as in the proof of Proposition 2.1 that $\tilde{P}(A' \otimes I', I' \otimes B') - \lambda I' \hat{\otimes}_{\alpha'} I'$ has the range $X' \hat{\otimes}_{\alpha'} Y'$. In case (α_∞), $X' \hat{\otimes}_\pi Y'$ is an $\mathscr{L}_1$ space. Similarly, $\tilde{P}(A' \otimes I', I' \otimes B') - \lambda I' \hat{\otimes}_\pi I'$ is seen to have the range $X' \hat{\otimes}_\pi Y'$, so that $P(A' \otimes I', I' \otimes B') - \lambda I' \otimes I'$ maps $D[(A')^m] \otimes D[(B')^n]$ onto a dense subspace of $X' \otimes Y'$ in the norm π. Since we have $\varepsilon' \leq \pi$, it follows that $\tilde{P}(A' \otimes I', I' \otimes B') - \lambda I' \hat{\otimes}_{\varepsilon'} I'$ has a dense range in $X' \hat{\otimes}_{\varepsilon'} Y'$. Thus in all the three cases, $\tilde{P}(A' \otimes I', I' \otimes B') - \lambda I' \hat{\otimes}_{\alpha'} I'$ is shown to have a dense range in $X' \hat{\otimes}_{\alpha'} Y'$. Q.E.D.

Since the spectrum is unchanged under the closure operation, the following theorem is a direct consequence of Theorems 2.2 and 2.3.

THEOREM 2.4. *Under the same assumption as in Theorem 2.2, we have for* $P \in \mathscr{P}'(A, B)$

$$\begin{aligned}\overline{P(\sigma(A), \sigma(B))} &= \overline{P(\sigma(A \hat{\otimes}_\alpha I), \sigma(I \hat{\otimes}_\alpha B))} \\ &= \sigma(P(A \otimes I, I \otimes B)) = \sigma(\tilde{P}(A \otimes I, I \otimes B)) \\ &= \sigma(\textstyle\sum c_{jk} A^j \hat{\otimes}_\alpha B^k) = \sigma((\textstyle\sum c_{jk} A^j \hat{\otimes}_\alpha B^k)^\sim)\,.\end{aligned}$$

3. Operators $A \otimes I + I \otimes B$ and $A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B$

In this section, we consider in particular the operators of the forms $A \otimes I + I \otimes B$ and $A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B$, which are of especial importance in applications.

As a direct consequence of Theorem 2.4 for $P(\xi, \eta) = \xi + \eta$, the results of Ju. M. Berezanskiĭ [1] and L. and K. Maurin [15] for selfadjoint operators are generalized as follows (cf. [9]).

THEOREM 3.1. *Let α, $X \hat{\otimes}_\alpha Y$ and A, B be those described in Section 1. Suppose further that if $\sigma(A)$ is not empty, we have $\|R(\zeta; A)\| \leq C_\delta$ outside $U_\delta = \{\zeta; \mathrm{dist}(\zeta, \sigma(A)) < \delta\}$ for any $\delta > 0$ and that if $\sigma(A)$ is empty, for any $R > 0$ there exists a nonempty open set U for which $U + \sigma(B) \subset \complement K(0; R)$ and $\|R(\zeta; A)\| \leq C_R$ in $\complement U$. Here, C_δ and C_R are constants depending only on δ, R, respectively. Then the spectra of $A \otimes I + I \otimes B$, $A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B$ and their closures are empty if and only if $\sigma(A)$ is empty. If $\sigma(A)$ is not empty, it holds*

$$\begin{aligned}\overline{\sigma(A) + \sigma(B)} &= \overline{\sigma(A \hat{\otimes}_\alpha I) + \sigma(I \hat{\otimes}_\alpha B)} \\ &= \sigma(A \otimes I + I \otimes B) = \sigma((A \otimes I + I \otimes B)^\sim) \\ &= \sigma(A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B) = \sigma((A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B)^\sim)\,.\end{aligned}$$

Proof. For $\sigma(A)$ empty, the assertion of Theorem 3.1 is clear from Theorems 2.2 and 2.4 for $P(\xi, \eta) = \xi + \eta$. If $\sigma(A)$ is not empty, we may assume $\overline{\sigma(A) + \sigma(B)} \neq C$. Let $\lambda \notin \overline{\sigma(A) + \sigma(B)}$, so that $\delta = \text{dist}(\lambda, \sigma(A) + \sigma(B)) > 0$. Choose $U = U_{\delta/2} \equiv \{\xi; \text{dist}(\xi, \sigma(A)) < \delta/2\}$. Then $|\xi + \eta - \lambda|$ is bounded away from zero on $U \times \sigma(B)$, and by assumption $R(\xi; A)$ is uniformly bounded in $\complement U$. Thus the same argument as in the proof of Theorem 2.2 yields the desired assertion. Q.E.D.

We consider now when the closure of $A \otimes I + I \otimes B$ coincides with $A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B$. They coincide if and only if $A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B$ is closed in $X \hat{\otimes}_\alpha Y$.

The following theorem is an extension of the part (1) of Theorem 4.6 in [9]. The sector $\{\zeta; |\arg \zeta| \leq \theta\}$ is denoted by $S(\theta)$.

THEOREM 3.2. *Let α, $X \hat{\otimes}_\alpha Y$ and A, B be those described in Section 1. Suppose, for some θ_A and θ_B with $0 \leq \theta_A, \theta_B < \pi$ and $0 \leq \theta_A + \theta_B < \pi$, that $\rho(A)$ contains the complement of the sector $S(\theta_A)$ and $\|\zeta R(\zeta; A)\| \leq M_\theta$ outside $S(\theta)$ for each θ with $\theta_A < \theta < \pi$, where M_θ is a constant depending only on θ, and that $\rho(B)$ contains the complement of the sector $S(\theta_B)$. Then the closure of $A \otimes I + I \otimes B$ coincides with $A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B$. The spectra of $A \otimes I + I \otimes B$ and $A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B$ are empty if and only if $\sigma(A)$ is empty.*

If $\sigma(A)$ is not empty, it holds

$$\begin{aligned}\sigma(A) + \sigma(B) &= \sigma(A \hat{\otimes}_\alpha I) + \sigma(I \hat{\otimes}_\alpha B) \\ &= \sigma(A \otimes I + I \otimes B) = \sigma(A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B) .\end{aligned}$$

Therefore, for any $\lambda \notin \sigma(A) + \sigma(B)$ (when $\sigma(A)$ is empty, λ shall be an arbitrary complex number) and for any $f \in X \hat{\otimes}_\alpha Y$ there exists a unique $u \in D[A \hat{\otimes}_\alpha I] \cap D[I \hat{\otimes}_\alpha B]$ which satisfies

$$(A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B - \lambda I \hat{\otimes}_\alpha I)u = f .$$

Moreover the following inequality holds:

$$\|u\|_\alpha + \|A \hat{\otimes}_\alpha Iu\|_\alpha + \|I \hat{\otimes}_\alpha Bu\|_\alpha \leq C \|f\|_\alpha ,$$

with a constant C independent of u and f.

Proof. First note the condition of Theorem 3.1 is satisfied. Since $\sigma(A) + \sigma(B)$ is closed and does not cover the whole complex plane, it follows by Theorem 2.3 that $(A \otimes I + I \otimes B)^{\sim} = (A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B)^{\sim}$. So, to prove Theorem 3.2, it suffices to establish the inequality

$$\|A \hat{\otimes}_\alpha Iu\|_\alpha \leq C[\|(A \hat{\otimes}_\alpha I + I \hat{\otimes}_\alpha B)u\| + \|u\|_\alpha] \tag{3.1}$$

for $u \in D[A \otimes I + I \otimes B] \equiv D[A] \otimes D[B]$.

Clearly, $-1 \notin \sigma(A) + \sigma(B)$. Then in virtue of Theorem 3.1, $(A \otimes I + I \otimes B)^{\sim} + I \hat{\otimes}_\alpha I$ has an everywhere defined continuous inverse.

Since $A(A+(1+\eta)I)^{-1} = I - (1+\eta)(A+(1+\eta)I)^{-1}$ is uniformly bounded on $\sigma(B)$, the same argument as in the proof of Proposition 2.1 shows that the integral

$$\begin{aligned} &\int_{\sigma(B)} [A(A + (1+\eta)I)^{-1} \otimes E(d\eta)]f \\ &\qquad = \int_{\sigma(B)} (A \otimes I)[(A + (1+\eta)I)^{-1} \otimes E(d\eta)]f \end{aligned} \tag{3.2}$$

defines a continuous linear operator of $X \otimes Y \subset X \hat{\otimes}_\alpha Y$ into $X \hat{\otimes}_\alpha Y$. It follows by the definition of the integral and by the closedness of $A \hat{\otimes}_\alpha I$ that $f \in X \otimes Y$ implies

$$[(A \otimes I + I \otimes B)^{\sim} + I \hat{\otimes}_\alpha I]^{-1} f \in D[A \hat{\otimes}_\alpha I] \tag{3.3}$$

and the integral (3.2) equals

$$(A \hat{\otimes}_\alpha I)[(A \otimes I + I \otimes B)^{\sim} + I \hat{\otimes}_\alpha I]^{-1} f .$$

Then we see that, for all $f \in X \hat{\otimes}_\alpha Y$, (3.3) holds valid and

$$\|(A \hat{\otimes}_\alpha I)[(A \otimes I + I \otimes B)^{\sim} + I \hat{\otimes}_\alpha I]^{-1} f\|_\alpha \leq C \|f\|_\alpha ,$$

whence follows immediately the inequality (3.1). Q.E.D.

Remark. In case (α_2), since an $\mathscr{L}_2$ space is considered as a Hilbert space and a normal operator is a scalar type spectral operator, it is possible to state the same assertion as in Theorem 3.2 for B being a normal operator in Y. It should be emphasized that A need not be m-accretive, in contrast to Theorem 4.6 (1) in [9]; this fact suggests that Theorem 3.2 is of wider application.

4. Applications

Theorem 3.2 is well applied to the first boundary value problem of a class of quasi-elliptic differential equations which includes especially the Laplace and heat equations (cf. [16], [9]).

However, our theory is true in a far wider application; it includes the hyperbolic differential equations.

For an application of Theorem 3.1, we consider the initial value problem of the wave equation

$$[\partial^2/\partial t^2 - \partial^2/\partial x^2 - \lambda]u(t,x) = f(t,x) \tag{4.1}$$

in the strip $S = \{(t,x) \in \boldsymbol{R}^2 ; 0 < t < T\} = \boldsymbol{I} \times \boldsymbol{R}, I = (0,T)$, with the initial condition

$$u(0,x) = u_t(0,x) = 0 \ . \tag{4.2}$$

Then for any $\lambda \in \boldsymbol{C}$ and for any $f \in L_2(S) = L_2(I) \hat{\otimes}_{\alpha_0} L_2(\boldsymbol{R})$ there exists a unique solution u in $L_2(S)$ of the initial value problem (4.1) and (4.2); moreover we have $\|u\| \leq C \|f\|$, with a constant C.

To show this, let $A = d^2/dt^2$ in $L_2(I)$ with domain

$$D[A] = \{\varphi \in L_2(I) ; d^2/dt^2\varphi \in L_2(I), \varphi(0) = \varphi'(0) = 0\}$$

and let $B = -d^2/dx^2$ in $L_2(\boldsymbol{R})$ with domain

$$D[B] = \{\psi \in L_2(\boldsymbol{R}) ; -d^2/dx^2\psi \in L_2(\boldsymbol{R})\} \ .$$

Then B is a selfadjoint operator in $L_2(\boldsymbol{R})$ with the spectrum $\sigma(B)$ being the nonnegative real line. A is a densely defined closed linear operator in $L_2(I)$ with empty spectrum $\sigma(A)$. For $R > 0$, let $U = \{\zeta ; |\mathrm{Im}\,\zeta| > R$ or $\mathrm{Rom}\,\zeta > R\}$.

Then we can show easily that $U + \sigma(B) = U \subset \complement K(0;R)$ and that $\|R(\zeta;A)\| \leq C_R$ on $\complement U$, with a constant C_R depending only on R. We denote by L the operator $A \otimes I + I \otimes B$ with domain $D[A] \otimes D[B]$. Applying Theorem 3.1 yields the emptiness of the spectrum of the closure of $L = \partial^2/\partial t^2 - \partial^2/\partial x^2$, whence follows the desired assertion. Here, we see that the solution u lies in the domain of the closure of L.

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