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ELEMENTS OF FINITE ORDER IN STONE-ČECH COMPACTIFICATIONS

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Let S be a free semigroup (on any set of generators). When S is given the discrete topology, its Stone–Čech compactification has a natural semigroup structure. We give two results about elements p of finite order in βS. The first is that any continuous homomorphism of βS into any compact group must send p to the identity. The second shows that natural extensions, to elements of finite order, of relationships between idempotents and sequences with distinct finite sums, do not hold.

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1. Introduction

One of the unsolved problems about the semigroup $\beta\mathbb{N}$ is whether it contains an element of finite order, that is to say a p such that $p+p+\cdots+p=p$ where there cannot be fewer than three terms on the left-hand side. Here we present two main results about elements of finite order which the sceptical reader may interpret as evidence that they do not exist. The first is easy to state: any continuous homomorphism from $\beta\mathbb{N}$ to a compact topological group must send each element of finite order to the zero of the group. The second concerns the relationship between elements of finite order and finite sums of terms in sequences. The parallel with the theory of idempotents necessarily fails—see Theorem 3.2. In order to make the discussion of finite sums more natural, we broaden our context from $\beta\mathbb{N}$ to compactifications of general free semigroups.

We introduce our notation and some basic known results. The operation in all semigroups, whether commutative or not, will be denoted by $+$ and we shall use additive terminology in all respects save one: an *idempotant* p satisfies $p+p=p$. A sum $p+p+\cdots+p$, with k terms, is denoted by kp. To say p *has order* k is to say $(k+1)p=p$ and $rp\neq p$ if $r\leqq k$; it is easy to see that this implies that kp is idempotent and $\{p,2p,\ldots,kp\}$ is a group.

If S is a discrete semigroup than the Stone–Čech compactification βS (which we regard as containing S) has a unique semigroup operation $+$ with the properties (i) $x\to x+y$ is continuous on βS for each $y\in\beta S$ and (ii) $y\to s+y$ is continuous on βS for each $s\in S$. Since S is dense in βS the sum $x+y$ of $x,y\in\beta S$ is determined by S and the following consequence of (i) and (ii): (iii) if (s_α), (t_β) are nets in S with $s_\alpha\to x$, $t_\beta\to y$ in βS

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then $x+y=\lim_\alpha \lim_\beta(s_\alpha+t_\beta)$ (and the order of the limits here is crucial). From (iii) it is easy to deduce (iv) if $f:S\to T$ is a homomorphism of discrete semigroups then the unique continuous extension $f^\beta:\beta S\to\beta T$ is a homomorphism, and (v) if $g:S\to K$ is a homomorphism into a compact separately continuous semigroup K than $g^\beta:\beta S\to K$ is a homomorphism.

We shall need two facts about a compact jointly continuous semigroup K. The first is immediate from joint continuity: (vi) if $x_\alpha\to x$ in K then $kx_\alpha\to kx$ for any positive integer k. The second is well known in compact semigroup theory. (vii) If K is a group and S is a commutative subsemigroup, then $\mathrm{cl}_K S$ is a group. (**Proof.** The closure $\mathrm{cl}_K S$ is a commutative semigroup since S is, and it is compact, therefore its minimal ideal is a group G. The idempotent in G can only be the zero of K, and hence (as G is an ideal) $G=\mathrm{cl}_K S$.)

(Standard results about compact semigroups in general and βS—or, more particularly, $\beta\mathbb{N}$—can be found in many places, e.g. [**1**, **4**, **6**, **11**].)

2. Images of elements of finite order

Let X be a set. Let $F(X)$ be the free semigroup on the set X of generators with the discrete topology. In this section we fix $k\in\mathbb{N}$ and fix $p\in\beta F(X)$ with the property that kp is idempotent (so p could be of order k).

Lemma 2.1. *Let K be a compact commutative jointly continuous semigroup. Let $f:F(X)\to K$ be a homomorphism and suppose that every element of $f(X)$ is divisible by k (that is, for each $x\in X$ the equation $ky=f(x)$ has at least one solution y in K). Then $f^\beta(p)$ is idempotent.*

Proof. For each $x\in X$ pick $\bar{x}\in K$ with $k\bar{x}=f(x)$. Define $g:F(X)\to K$ to be the unique homomorphism satisfying $g(x)=\bar{x}$ for $x\in X$. For any $x_1,\ldots,x_r\in X$ we then have (using the commutativity of K for the second equality)

$$kg(x_1+\cdots+x_r)=k(g(x_1)+\cdots+g(x_r))=kg(x_1)+\cdots+kg(x_r)$$

$$=f(x_1)+\cdots+f(x_r)=f(x_1+\cdots+x_r),$$

that is, $kg(x)=f(x)$ for $x\in F(X)$. Taking limits using 1(vi) gives $kg^\beta(x)=f^\beta(x)$ for $x\in\beta F(X)$. By 1(v), g^β is a homomorphism, so $kg^\beta(x)=g^\beta(kx)$ for $x\in\beta F(X)$. Since kp is idempotent, so is $g^\beta(kp)=f^\beta(p)$.

Theorem 2.2. *Let G be a compact commutative topological group and let $f:F(X)\to G$ be a homomorphism. Then $f^\beta(p)=0$, the zero of G.*

Proof. If $G=T$, the circle group, Lemma 2.1 gives the result immediately. Otherwise, by the Pontjagin Duality Theorem [**3**, Theorem 24.8], G is embeddable in some product

T^I of circle groups. Following f by the projection into the ith component of this product shows that the ith coordinate of $f^\beta(p)$ is 0. So $f^\beta(p)$ is 0.

Corollary 2.3. *If G is any compact topological group and $f\colon \mathbb{N} \to G$ is a homomorphism then $f^\beta(p)=0$ if p has order k in $\beta\mathbb{N}$.*

Proof. Since $f(\mathbb{N})$ is a commutative subsemigroup of G, i(vii) shows that $\mathrm{cl}_G f(\mathbb{N}) = f^\beta\ (\beta\mathbb{N})$ is a commutative group. Theorem 2.2 applies to $f\colon \mathbb{N} \to f^\beta(\beta\mathbb{N})$ since $\mathbb{N}$ is the free semigroup on one generator.

Corollary 2.4. *For $w \in F(X)$ denote by $l(w)$ the length of the word w. Let $m \in \mathbb{N}$ be given. Let (w_α)in $F(X)$ satisfy $w_\alpha \to p$. Then $l(w_\alpha)$ is eventually a multiple of m.*

Proof. Write $l_m(w)=l(w) \pmod m$, so that $l_m\colon F(X) \to \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Since l is a homomorphism into $\mathbb{N}$, l_m is also a homomorphism. Theorem 2.2 tells us that $l_m^\beta(p)=0$. Since $l_m(w_\alpha) \to l_m^\beta(p)=0$ and $\mathbb{Z}_m$ is discrete, eventually $l_m(w_\alpha)=0$, as required.

3. Finite sums

If $\langle x_n \rangle_{n=1}^\infty$ is a sequence in a semigroup S we write $FS\langle x_n \rangle_{n=1}^\infty$ (or simply $FS\langle x_n \rangle$) for $\{x_{n_1}+\cdots+x_{n_r}\colon r \in \mathbb{N}, n_1<n_2<\cdots<n_r\}$, the *set of finite sums* from $\langle x_n \rangle$. We say $\langle x_n \rangle$ has *distinct finite sums* if $x_{m_1}+\cdots+x_{m_r}=x_{n_1}+\cdots+x_{n_s}$, with the suffices on each side increasing, implies $r=s$ and $m_1=n_1,\dots,m_r=n_r$.

Under certain circumstances, it is easy to find sequences with distinct finite sums. For example, if S is cancellative every sequence $\langle x_n \rangle$ of distinct elements has a subsequence with distinct finite sums. (**Proof.** For any y, x in a cancellative S, the set $-y+x = \{z\colon x=y+z\}$ contains either no elements or just one. Given $N \in \mathbb{N}$, the sets

$$\Sigma_N=\{x_{n_1}+\cdots+x_{n_r}\colon n_1<\cdots<n_r \leqq N\}, \quad \Delta_N = \cup\{-y+x\colon y, x \in \Sigma_N\}$$

are finite. We can therefore find $x_m \in \langle x_n \rangle \backslash (\Sigma_N \cup \Delta_N)$ with $m>N$. Then for any $x, y \in \Sigma_N$ both $x \neq x_m$ and $x \neq y+x_m$, that is, no sum involving x_m and x_n's with $n \leqq N$ can be equal to a sum in Σ_N. It is now easy to see how to produce inductively a subsequence $\langle x_{n_r} \rangle$ with distinct finite sums, starting with $x_{n_1}=x_1$.)

When $X=\{x_1, x_2, \dots\}$, $FS\langle x_n \rangle$ in $F(X)$ is called an *abstract distinct finite sum system* (or ADFSS); as $F(X)$ is free on X, these finite sums are obviously distinct. When S is a semigroup, a map $f\colon X \to S$ extends in a unique way to a homomorphism, which we again denote by f, of $F(X)$ into S, and obviously $FS\langle x_n \rangle$ maps onto $FS\langle f(x_n) \rangle$. The sequence $\langle f(x_n) \rangle$ has distinct finite sums if and only if f is injective on $FS\langle x_n \rangle$. One point of considering ADFSS's (which were called non-commutative oids in [9]) is that in many situations f is injective on $FS\langle x_n \rangle$ but not on $F(X)$ (see the examples in [10]). In such cases concepts applicable to free semigroups can sometimes be used in the ADFSS. For example length, l, makes sense for elements of $FS\langle f(x_n) \rangle$ if f is injective on $FS\langle x_n \rangle$.

When $f: F(X) \to S$ is a homomorphism and $F(X)$ and S have their discrete topologies, then by 1(iv) f^β is a homomorphism from $\beta F(X)$ to βS. If in addition the restriction of f to $FS\langle x_n \rangle$ is injective, then f^β is bijective from $\mathrm{cl}_{\beta F(X)} FS\langle x_n \rangle$ into $\mathrm{cl}_{\beta S} FS\langle f(x_n) \rangle$, and therefore f^β is an isomorphism of whatever algebraic structure $\beta F(X)$ induces on $\mathrm{cl}_{\beta F(X)} FS\langle x_n \rangle$. Now in fact there is a considerable amount. Write $H = \bigcap_r \mathrm{cl}_{\beta F(X)} FS\langle x_n \rangle_{n=r}^{\infty}$. If $u \in FS\langle x_n \rangle_{n=r}^{\infty}$ then so is $u + w$ whenever $w \in FS\langle x_n \rangle_{n=s}^{\infty}$ and s is greater than the largest suffix occurring in the sum for u. Using 1(ii) we conclude that $u + H \subseteq \mathrm{cl}_{\beta F(X)} FS\langle x_n \rangle_{n=r}^{\infty}$. Taking limits again using 1(i) gives $H + H \subseteq H$, that is, H is a subsemigroup of $\beta F(X)$. Thus f^β is an isomorphism of H onto a subsemigroup of βS.

Proposition 3.1. *Let S be a semigroup, and let $\langle y_n \rangle$ be a sequence in S with distinct finite sums. Define a homomorphism $f: F(X) \to S$ by writing $f(x_n) = y_n$ for all n. If $p \in f^\beta(\mathrm{H})$ is of finite order, (w_α) is a net in $FS\langle y_n \rangle$ and $w_\alpha \to p$, then for any given positive integer m the length $l(w_\alpha)$ is eventually a multiple of m.*

Proof. This is immediate from Corollary 2.4 and the above remarks.

A long time ago some basic relationships between idempotents and finite sums were established. To describe them, let S be a discrete semigroup. An element $p \in \beta S$ will be considered to be an ultrafilter on S, so that $A \in p$ implies that $\mathrm{cl}_{\beta S} A$ is a neighbourhood of p in βS and, on the other hand, if V is a neighbourhood of p in βS then $V \cap S \in p$. Then

(1) if p is idempotent and $A \in p$ there is $\langle x_n \rangle$ with $FS\langle x_n \rangle \subseteq A$, (see[**7**]) and

(2) for every sequence $\langle x_n \rangle$, $\bigcap_m \mathrm{cl}_{\beta S} FS\langle x_n \rangle_{n=m}^{\infty}$ contains idempotents (and so if p is such an idempotent $FS\langle x_n \rangle \in p$), see [**5**].

Recently a new link between (1) and (2) was found.

(3) It is consistent with ZFC that there are idempotents p such that for each $A \in p$ it is possible to find $\langle x_n \rangle$ satisfying (1) and also $p \in \mathrm{cl}_{\beta S} FS\langle x_n \rangle$.

(3) has been proved using Martin's Axiom [**7**]. It has also been shown [**2**, **8**] that (3) is independent of ZFC.

We might expect there to be analogues of these results for elements of order k. Write $FS_{i(\mathrm{mod}\,k)}\langle x_n \rangle$ for the subset of $FS\langle x_n \rangle$ consisting of sums of the form $x_{n_1} + \cdots + x_{n_r}$ with $r \equiv i (\mathrm{mod}\, k)$. Then these two conjectures seem natural.

(1_k) If $p \in \beta S$ has order k and $A_i \in ip$ for $1 \leqq i \leqq k$, there is a sequence $\langle x_n \rangle$ with $FS_{i(\mathrm{mod}\,k)}\langle x_n \rangle \subseteq A_i$ for each i.

(2_k) For every sequence $\langle x_n \rangle$, $\bigcap_m \mathrm{cl}_{\beta S} FS_{1(\mathrm{mod}\,k)} \langle x_n \rangle_{n=m}^{\infty}$ contains elements of order k.

More generally, we might expect $\bigcap_m \mathrm{cl}_{\beta S} FS_{i(\mathrm{mod}\,k)} \langle x_n \rangle_{n=m}^{\infty}$ to contain an element whose order is the order of i in the group of integers mod k. In fact, we have the following result.

Theorem 3.2. *(1_k) is true but (2_k) is not. Indeed, for any sequence $\langle x_n \rangle$ with distinct finite sums, $\bigcap_m \mathrm{cl}_{\beta S} FS_{i\,(\mathrm{mod}\,k)} \langle x_n \rangle_{n=m}^{\infty}$ contains no element of finite order* if $i \not\equiv 0 \pmod k$.

Proof. As we wish to end on a positive note, we first consider (2_k). Let $\langle x_n \rangle$ have distinct finite sums. Let $w_\alpha \in FS_{i(\mathrm{mod}\,k)} \langle x_n \rangle$, $w_\alpha \to p \in \bigcap_r \mathrm{cl}_{\beta S} FS_{i(\mathrm{mod}\,k)} \langle x_n \rangle_{n=r}^{\infty}$, where p has finite order. Then Proposition 3.1 tells us, in particular, that $l(w_\alpha)$ is eventually a multiple of k, but in fact $l(w_\alpha) = i(\mathrm{mod}\,k)$ for all α.

We establish (1_k) by the following inductive construction. We claim that we can define decreasing sequences $\langle V_n(i) \rangle_{n=0}^{\infty}$ of neighbourhoods of ip, for $1 \leqq i \leqq k$, and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ of elements of S with the following properties:

$$V_n(i) \subseteq \mathrm{cl}_{\beta S} A_i, \qquad x_n \in V_n(1),$$

$$x_n + ip \in V_{n-1}(i+1), \qquad \text{and} \qquad x_m + V_n(1) \subseteq V_{m-1}(i+1),$$

for all relevant values of i, m, n with $m \leqq n$. To see this, first choose $V_0(i) = \mathrm{cl}_{\beta S} A_i$ for each i. Next, choose $x_1 \in S \cap V_0(1)$ satisfying the conditions that $x_1 + ip \in V_0(i+1)$ for $1 \leqq i \leqq k$, where addition in $\{1, 2, \ldots, k\}$ is taken as addition modulo k. Now suppose that we have chosen $V_n(i)$ and x_{n+1} with the required properties, for $0 \leqq n \leqq r$ and $1 \leqq i \leqq k$. Since, for every $n = 1, 2, \ldots, r+1$ and $1 \leqq i \leqq k$, $x_n + ip \in V_{n-1}(i+1)$, it is possible, for each i, to choose a neighbourhood $V_{r+1}(i)$ of ip satisfying $V_{r+1}(i) \subseteq V_r(i)$, and $x_n + V_{r+1}(i) \subseteq V_{n-1}(i+1)$ for all $n = 1, 2, \ldots, r+1$. Having done this, we then choose $x_{r+2} \in S \cap V_{r+1}(1)$ satisfying $x_{r+2} + ip \in V_{r+1}(i+1)$ for $1 \leqq i \leqq k$.

Having constructed these sequences, observe that if $n_1 < n_2 < \cdots < n_r$ then $x_{n_1} + x_{n_2} + \cdots + x_{n_r} \in V_{n_r} \in V_{n_1 - 1}(r(\mathrm{mod}\,k))$. From this it is clear that $\langle x_n \rangle$ has the required properties.

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