Can. J. Math. Vol. 44 (3), 1992 pp. 561–590

HERMITE AND HERMITE-FEJER INTERPOLATION AND ASSOCIATED PRODUCT INTEGRATION RULES ON THE REAL LINE: THE L_1 THEORY

D. S. LUBINSKY AND P. RABINOWITZ

ABSTRACT. We investigate convergence in a weighted L_1-norm of Hermite-Fejér and Hermite interpolation at the zeros of orthogonal polynomials associated with weights on the real line. The results are then applied to convergences of product integration rules. From the point of view of orthogonal polynomials, the new feature is that Freud and Erdös weights are treated simultaneously and that relatively few assumptions are placed on the weight. From the point of view of product integration, the rules exhibit convergence for highly oscillatory kernels (for example) and for functions of rapid growth at infinity.

1. **Introduction.** In this paper, we shall study the convergence of *Hermite-Fejér* and *related interpolation operators* in a weighted L_1 norm,

$$\|g\| := \int_{-\infty}^{\infty} |g(x)|\, W^2(x)\, dx. \tag{1.1}$$

Here $W := e^{-Q}$, and $Q\colon \mathbb{R} \longrightarrow \mathbb{R}$ is even, continuous and of at least polynomial growth at infinity. The interpolation takes place at the zeros $\{x_{kn}\}_{k=1}^{n} = \{x_{kn}(W^2)\}_{k=1}^{n}$ of the nth orthonormal polynomial

$$p_n(x) := p_n(W^2, x) := \gamma_n x^n + \cdots, \quad \gamma_n > 0, \tag{1.2}$$

for W^2, defined by the condition

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) W^2(x)\, dx = \delta_{mn}. \tag{1.3}$$

The zeros are ordered so that

$$-\infty < x_{nn} < x_{n-1,n} < x_{n-2,n} < \cdots < x_{1n} < \infty.$$

It is a classical result of Erdös-Turan that the natural setting for studying convergence of *Lagrange interpolation* at the zeros of $\{p_n\}_1^{\infty}$ is the L_2-setting. Recent results of Nevai and Vértesi [21] show that the natural analogue for Hermite-Fejér interpolation is the L_1-setting. They showed that if w is a non-negative weight with support in the finite interval $[a, b]$, so that in particular

$$0 < \int_a^b w(x)\, dx < \infty,$$

Received by the editors November 22, 1990.
AMS subject classification: Primary: 41A05, 65D05, 65D30; secondary: 42C05.
Key words and phrases: Hermite-Fejér, Hermite, interpolation, product integration, convergence, infinite intervals, Freud weights, Erdös weights.

and if $H_n(w,f,x)$ is the nth associated *Hermite-Fejér interpolation polynomial* at the zeros $\{x_{jn}\}_{j=1}^n$ of $p_n(w,x)$, defined by

$$(1.4)\qquad \begin{aligned} H_n(w,f,x_{jn}) &= f(x_{jn}), \quad 1 \le j \le n, \\ H_n'(w,f,x_{jn}) &= 0, \quad 1 \le j \le n, \end{aligned}$$

then

$$\lim_{n\to\infty} \int_a^b |f(x) - H_n(w,f,x)|\, w(x)\, dx = 0,$$

for each polynomial f. Furthermore, if $[a,b] = [-1,1]$, and w is a generalized Jacobi weight of the form

$$w(x) = g(x)(1-x)^\alpha(1+x)^\beta, \quad \alpha,\beta > -1,$$

where g is positive on $[-1,1]$ and g' satisfies a Lipschitz condition of order 1 on $[-1,1]$, then this limit remains valid for each f continuous on $[-1,1]$.

Their careful analysis of the related convergence questions in $L_p, p \ne 1$, shows that these cases are inherently more complicated, and that much more needs to be assumed about the weight.

It is partly for this reason that we shall concentrate on the L_1-case in this paper. An interesting feature of the results is that simultaneously they treat Freud and Erdös weights. These are respectively the cases where Q above is of polynomial, and of faster than polynomial growth, at infinity. Previously these have been treated separately [8–13, 15, 18, 20].

Once we have the convergence of Hermite-Fejér interpolation, we can consider convergence of the associated *product quadrature rules*. These rules involve approximation of

$$(1.5)\qquad I[k;f] := \int_{-\infty}^{\infty} k(x)f(x)\,dx$$

by quadrature rules

$$(1.6)\qquad I_n[k;f] := \sum_{j=1}^n w_{jn}(k)f(x_{jn})$$

where the weights $\{w_{jn}\}$ are usually determined by integration of some (typically polynomial) approximation to f. The philosophy is to split the integrand kf into a difficult component k with known types of singularities or oscillatory behaviour, and such that the weights $w_{jn}(k)$ can be evaluated explicitly, and a smooth component f that is treated numerically.

To develop this theme further, we need some more notation. Let $\{\ell_{jn}(x)\}_{j=1}^n = \{\ell_{jn}(W^2,x)\}_{j=1}^n$ denote the *fundamental polynomials of Lagrange interpolation* for $\{x_{jn}\}_{j=1}^n$, so that $\ell_{jn}(x)$ has degree $n-1$, and

$$(1.7)\qquad \ell_{jn}(x_{kn}) = \delta_{jk}, \quad 1 \le j,k \le n.$$

The *fundamental polynomials of Hermite interpolation* are then

$$h_{jn} := \left\{1 - \frac{p_n''(x_{jn})}{p_n'(x_{jn})}(x - x_{jn})\right\} \ell_{jn}^2(x), \tag{1.8}$$

and

$$\hat{h}_{jn}(x) := (x - x_{jn})\ell_{jn}^2(x), \tag{1.9}$$

$1 \leq j \leq n$. The *Hermite-Fejér polynomial* $H_n(W^2, f, x)$ satisfying (1.4) with $w = W^2$ is given by

$$H_n(W^2, f, x) := \sum_{j=1}^{n} f(x_{jn})h_{jn}(x). \tag{1.10}$$

If $f'(x_{jn})$ is defined, $1 \leq j \leq n$, one may also define the *Hermite (or osculatory) interpolation polynomial*

$$\hat{H}_n(W^2, f, x) := \sum_{j=1}^{n} f(x_{jn})h_{jn}(x) + \sum_{j=1}^{n} f'(x_{jn})\hat{h}_{jn}(x), \tag{1.11}$$

satisfying

$$\hat{H}_n^{(\ell)}(W^2, f, x_{jn}) = f^{(\ell)}(x_{jn}), \quad 1 \leq j \leq n, \ell = 0, 1. \tag{1.12}$$

Both H_n and $\hat{H}_n$ are special cases of the operator

$$H_n^*(W^2, f, \{d_{\ell n}\}, x) := \sum_{j=1}^{n} f(x_{jn})h_{jn}(x) + \sum_{j=1}^{n} d_{jn}\hat{h}_{jn}(x), \tag{1.13}$$

for which

$$\left.\begin{aligned} H_n^*(W^2, f, \{d_{\ell n}\}, x_{jn}) &= f(x_{jn}), \\ H_n^{*\prime}(W^2, f, \{d_{\ell n}\}, x_{jn}) &= d_{jn}, \end{aligned}\right\} \quad 1 \leq j \leq n \tag{1.14}$$

In several classical cases, and in those treated in this paper, the contribution from $p_n''(x_{jn})/p_n'(x_{jn})(x - x_{jn})$ in $h_{jn}(x)$ turns out to be negligible. One is then tempted to introduce the very simple operator

$$Y_n(W^2, f, x) := \sum_{j=1}^{n} f(x_{jn})\ell_{jn}^2(x). \tag{1.15}$$

It has the advantage of being *positive*, that is

$$f \geq 0 \text{ in } \mathbb{R} \text{ implies } Y_n(W^2, f, \cdot) \geq 0 \text{ in } \mathbb{R}.$$

Grünwald [7] investigated Y_n for "ρ-normal sets" and Rabinowitz and Vértesi considered Y_n in the context of product integration [27]. It is also a close cousin of Nevai's G_n operator [19, p. 74].

The operator H_n generates the product quadrature rule

$$I_n[k;f] := \int_{-\infty}^{\infty} H_n(W^2, f, x)k(x)\,dx = \sum_{j=1}^{n} w_{jn}(k)f(x_{jn}), \tag{1.16}$$

where

$$w_{jn}(k) := \int_{-\infty}^{\infty} h_{jn}(x)k(x)\,dx, \quad 1 \le j \le n. \tag{1.17}$$

Obvious analogues are generated by $\hat{H}_n$ and H_n^*. For example, in the case of $\hat{H}_n$, we obtain the product quadrature rule

$$\hat{I}_n[k;f] := I_n[k;f] + \sum_{j=1}^{n} \hat{w}_{jn}(k)f'(x_{jn}),$$

where

$$\hat{w}_{jn}(k) := \int_{-\infty}^{\infty} \hat{h}_{jn}(x)k(x)\,dx, \quad 1 \le j \le n.$$

In this paper, under suitable conditions on W^2, the condition

$$A := \sup_{x\in\mathbb{R}} |k(x)/W^2(x)| < \infty, \tag{1.18}$$

will guarantee convergence, that is,

$$\lim_{n\to\infty} I_n[k;f] = I[k;f], \tag{1.19}$$

for functions that may grow almost as fast as W^{-2}. We typically require for some $\alpha = \alpha(W)$,

$$\sup_{x\in\mathbb{R}} |f(x)|\, W^2(x)\big[1 + |Q'(x)|\big]^{\alpha+1} < \infty. \tag{1.20}$$

By contrast, when product quadrature rules based on Lagrange interpolation [8] were considered for weights on $\mathbb{R}$, f was required to grow somewhat slower than W^{-1}. Of course, the condition on k was correspondingly weaker.

Algorithmic aspects, such as evaluation of the weights $w_{jn}(k)$ via modified moments $I[k;p_n]$, have been considered in [25]. Another relevant reference for product quadrature rules based on Hermite-Fejér interpolation is [1]. For product quadrature on infinite intervals, see [8, 14, 28].

We remark that the approximation

$$J_n[k;f] := \int_{-\infty}^{\infty} Y_n(W^2, f, x)k(x)\,dx = \sum_{j=1}^{n} v_{jn}(k)f(x_{jn}), \tag{1.21}$$

where

$$v_{jn}(k) := \int_{-\infty}^{\infty} \ell_{jn}^2(x)k(x)\,dx, \quad 1 \le j \le n, \tag{1.22}$$

has several advantages. First, if $k \geq 0$, then all the weights $v_{jn}(k)$ are non-negative. Furthermore, the algorithm to evaluate $v_{jn}(k)$ is simpler than that for $w_{jn}(k)$. In the special case $k \equiv W^2$, we recover the Gauss weights

$$\lambda_{jn} := \lambda_{jn}(W^2) := \int_{-\infty}^{\infty} \ell_{jn}^2(x)W^2(x)\,dx = \int_{-\infty}^{\infty} \ell_{jn}(x)W^2(x)\,dx, \quad 1 \leq j \leq n. \tag{1.23}$$

Finally, under the condition (1.18), we obtain

$$|v_{jn}(k)| \leq A\lambda_{jn}, \quad 1 \leq j \leq n, \tag{1.24}$$

so that many of the convergence properties of the Gauss case carry over to this product quadrature rule. For example, when f has some singularity, the process of "avoiding the singularity", well known to converge for the Gauss rule [12] also guarantees convergence for the product quadrature rule.

This paper is organised as follows: In Section 2, we introduce our class of weights, and state our main results. In Section 3, we present some technical estimates, perhaps proved for the first time simultaneously for Freud and Erdös weights. In Section 4, we prove some infinite-finite range inequalities. In Section 5, we estimate quantities relating to a differential equation, developed by Shohat, Nevai, Bonan, Bauldry, and Mhaskar, enabling us to bound $|p_n''(x_{kn})/p_n'(x_{kn})|$. In Section 6, we obtain estimates for Christoffel functions. Finally, in Section 7, we prove the results of Section 2.

2. **Main results.** The theory of orthogonal polynomials for weights on the real line, and its associated approximation theory, has in recent years been developed primarily for weights $W^2 := e^{-2Q}$, where Q is even, and of polynomial growth at infinity [20]—the so-called Freud case. Quite recently, the case where Q is of faster than polynomial growth—the so-called Erdös case, has also received attention. The following definition allows both cases:

DEFINITION 2.1. We write $W \in \mathcal{W}$ if the following conditions are satisfied:

(a) $W = e^{-Q}$, where $Q: \mathbb{R} \longrightarrow \mathbb{R}$ is even, continuously differentiable, Q'' exists in $(0, \infty)$, and

$$Q(0) = 0. \tag{2.1}$$

(b) For $x \in (0, \infty)$,

$$Q'(x) > 0 \text{ and } Q''(x) \geq 0. \tag{2.2}$$

(c) There exist $C_1, C_2 > 0$ and $\eta > 0$, such that for $x \in (C_1, \infty)$,

$$\eta \leq xQ''(x)/Q'(x) \leq C_2\big(\log Q'(x)\big)^2. \tag{2.3}$$

The normalization $Q(0) = 0$ is merely for convenience, and can be achieved by multiplying W by a suitable constant. The condition (b) forces Q to be convex and Q' to be increasing in $(0, \infty)$. The left inequality in (2.3), combined with (2.2), ensures that $Q(x)$ grows at least as fast as $|x|^{1+\eta}$, as $x \longrightarrow \infty$. In this connection, we remark that this "lower growth condition" is not so severe; the set of all polynomials is not dense in the space of continuous functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ with norm

$$\|f\| := \int_{-\infty}^{\infty} |f(x)|\, W^2(x)\, dx < \infty, \tag{2.4}$$

if $W(x) = \exp\big(-Q(x)\big)$, when $Q(x) := |x|^{\beta}, \beta < 1$.

The right inequality in (2.3) is a weak regularity condition. In fact, for regularly growing Q, one has for each $\epsilon > 0$, as $x \longrightarrow \infty$,

$$xQ''(x)/\,Q'(x) = O[\log Q'(x)]^{1+\epsilon}.$$

In the general case, one can show that this relation holds outside a "small" set of x.

As examples of Q satisfying (a)—(c), we mention

$$Q(x) := |x|^{\beta}, \quad \beta > 1, \tag{2.5}$$

or

$$Q(x) := \exp_k(|x|^{\beta}) - \exp_k(0), \quad \beta \geq 1, \tag{2.6}$$

where $\exp_k(u) := \exp\big(\exp\big(\exp\big(\cdots \exp(u) \cdots\big)\big)\big)$ denotes the kth iterated exponential.

We shall also need the *Mhaskar-Rahmanov-Saff* number $a_u = a_u(Q)$, defined as the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t)(1 - t^2)^{-1/2}\, dt, \quad u > 0. \tag{2.7}$$

Since $sQ'(s)$ is positive and strictly increasing for $s \in (0, \infty)$, with limits 0 and ∞ at 0 and ∞ respectively, a_u is uniquely defined. Moreover,

$$\lim_{u \to 0+} a_u = 0; \quad \lim_{u \to \infty} a_u = \infty. \tag{2.8}$$

The significance of a_u in studying weights on $\mathbb{R}$ has become clear in recent years [16, 17].

The class of polynomials of degree at most n with real coefficients is denoted by $\mathcal{P}_n$. Constants independent of n, x and $P \in \mathcal{P}_n$ are denoted by $C, C_1, C_2, \ldots$. The same symbol does not necessarily denote the same constant in different occurrences. To emphasise that C does not depend on a particular parameter α, we write $C \neq C(\alpha)$, etc.

Following is our result for polynomials:

THEOREM 2.2. *Let $W := e^{-Q} \in \mathcal{W}$ and $a_n = a_n(Q)$, $n \geq 1$. Then there exist n_1 and C_1 such that for $n \geq n_1$ and $R_n \in \mathcal{P}_n$,*

$$\int_{-\infty}^{\infty} |R_n(x) - H_n(W^2, R_n, x)|\, W^2(x)\, dx \leq C_1 \frac{a_n^{3/2}(\log n)^{1/2}}{n} \left\{ \int_{-\infty}^{\infty} (R_n' W)^2(x)\, dx \right\}^{1/2}. \tag{2.9}$$

Here $C_1 \neq C_1(n, R_n)$. In particular, if for some $\epsilon > 0$,

$$Q(x) \geq x^{3/2+\epsilon}, \quad x \text{ large enough}, \tag{2.10}$$

then for every polynomial P,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |P(x) - H_n(W^2, P, x)|\, W^2(x)\, dx = 0. \tag{2.11}$$

The $3/2$ in (2.10) can possibly be replaced by 1, but not using our present method of proof, which involves the Cauchy-Schwarz inequality. With additional assumptions, it should be possible to prove this.

Next, we turn to general functions:

THEOREM 2.3. *Let $W := e^{-Q} \in \mathcal{W}$ and let $a_n = a_n(Q)$ satisfy*

$$a_n = o(n/\sqrt{\log n})^{2/3}, \quad n \to \infty. \tag{2.12}$$

Choose $\alpha > 0$ such that for some $\sigma > 0$,

$$(a_n/n)^{\alpha} = O(n^{-1-\sigma}), \quad n \to \infty. \tag{2.13}$$

Then, if $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and Riemann-integrable in each finite interval, and

$$\sup_{x\in\mathbb{R}} |f(x)|\, W^2(x)\left[1 + |Q'(x)|\right]^{\alpha+1} < \infty, \tag{2.14}$$

then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - H_n(W^2, f, x)|\, W^2(x)\, dx = 0, \tag{2.15}$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - Y_n(W^2, f, x)|\, W^2(x)\, dx = 0. \tag{2.16}$$

In particular, (2.12) holds if (2.10) is satisfied, and in this case, we may choose any $\alpha > 3$.

https://doi.org/10.4153/CJM-1992-036-2 Published online by Cambridge University Press

In the special case $Q(x) := |x|^{\beta}$, $\beta > 3/2$, it is easily seen that for some $A > 0$,

$$a_n = An^{1/\beta}, \quad n \geq 1, \tag{2.17}$$

and (2.13) becomes

$$\alpha > \beta/(\beta - 1). \tag{2.18}$$

We may reformulate condition (2.14) in this case as

$$\sup_{x\in\mathbb{R}} |f(x)|\ W^2(x)[1 + |x|^{2\beta-1+\epsilon}] < \infty, \tag{2.19}$$

some $\epsilon > 0$.

We note that (2.14) can be somewhat weakened if one places additional assumptions on the weight. If for example,

$$a_n = o(\sqrt{n/\log n}), \quad n \longrightarrow \infty,$$

one can replace (2.14) by

$$\sup_{x\in\mathbb{R}} |fQ'(x)|\ W^2(x)(1 + |x|)\{\log(2 + |x|)\}^2 < \infty,$$

as is evident from the proofs.

We next turn to the operators $\hat{H}_n$ and H_n^*, defined by (1.11) and (1.13) respectively:

THEOREM 2.4. *Assume the hypotheses of Theorem 2.3 on f, W^2, and α. For $n \geq 1$, let there be given $\{d_{jn}\}_{j=1}^{n}$, satisfying*

$$\sup_{1\leq j\leq n} |d_{jn}|\ W^2(x_{jn})\big(1 + |Q'(x_{jn})|\big)^{-\alpha} \leq C < \infty. \tag{2.20}$$

Then the generalized Hermite operators $\{H_n^*\}$ satisfy

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - H_n^*(W^2, f, \{d_{jn}\}, x)|\ W^2(x)\, dx = 0. \tag{2.21}$$

In particular, if f' exists in $\mathbb{R}$ and

$$\sup_{x\in\mathbb{R}} |f'(x)|\ W^2(x)\big(1 + |Q'(x)|\big)^{-\alpha} < \infty, \tag{2.22}$$

then the Hermite interpolation polynomials $\{\hat{H}_n\}$ satisfy

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - \hat{H}_n(W^2, f, x)|\ W^2(x)\, dx = 0. \tag{2.23}$$

We now turn to the product quadrature rules. For simplicity, we discuss only the rules $I_n[k;f]$ and $J_n[k;f]$ defined by (1.16–17) and (1.21–22) respectively. Recall too the notation (1.5).

COROLLARY 2.5. *Assume the hypotheses of Theorem 2.3 on f, W^2 and α. Let $k\colon \mathbb{R} \longrightarrow \mathbb{R}$ be measurable, and satisfy*

$$\sup_{x\in\mathbb{R}} |k(x)/W^2(x)| < \infty. \tag{2.24}$$

Then

$$\lim_{n\to\infty} I_n[k;f] = I[k;f], \tag{2.25}$$

and

$$\lim_{n\to\infty} J_n[k;f] = I[k;f]. \tag{2.26}$$

We note that Corollary 2.5 may be extended to handle functions f with finitely many singularities, if we modify the rules so as to "avoid the singularity". To be more precise, we introduce classes of functions $M_d(\xi\,;k)$ and $\bar{M}_d(\xi\,;k)$ that have a dominated growth at a point $\xi \in \mathbb{R}$:

DEFINITION 2.6. (a) We say that $f\colon \mathbb{R} \longrightarrow \mathbb{R} \in M_d(\xi\,;k)$ if f is continuous on $(\xi\,,\infty)$ and $\exists F\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

(i) $\int_{-\infty}^{\infty} |kF|(x)\,dx < \infty$,

(ii) $F = 0$ on $(-\infty, \xi\,]$, and is continuous and monotone decreasing on $(\xi\,,\infty)$,

(iii) $|f(x)| \leq F(x)$, $x \in \mathbb{R}$.

(b) We say that $f\colon \mathbb{R} \longrightarrow \mathbb{R} \in \bar{M}_d(\xi\,;k)$ if f is continuous on $(-\infty, \xi\,)$ and $\exists F\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

(i) and (iii) above hold;

(ii) $F = 0$ on $[\xi\,,\infty)$ and is continuous and monotone increasing on $(-\infty, \xi\,)$.

The set of singular functions $\mathcal{L}$ that we shall consider consists of those functions that can be expressed as a finite linear combination of functions in $M_d(\xi_j;k)$, $j = 1,2,3,\ldots m$, and $\bar{M}_d(\eta_j;k)$, $j = 1,2,3,\ldots p$, and functions bounded and Riemann integrable in every finite interval satisfying (2.14). Here the sets $X := \{\,\xi_j : 1 \leq j \leq m\}$ and $Y := \{\,\eta_j : 1 \leq j \leq p\}$ need not be distinct. Note that elements of $M_d(\xi\,;k)$ and $\bar{M}_d(\eta\,;k)$ are bounded outside a finite interval, and so functions in $\mathcal{L}$ satisfy (2.14) if we omit a suitable finite interval.

Let us now define the modified quadrature rule $J_n^*[k;f]$ from $J_n[k;f]$ by dropping the at most $m+p$ points x_{jn} in the rule $J_n[k;f]$ that are closest to the points of $X \cup Y$. For such rules, we have the following corollary:

COROLLARY 2.7. *Assume the hypotheses of Theorem 2.3 on W^2 and α, and let k*

satisfy (2.24). Then for any $f \in \mathcal{L}$,

$$\lim_{n\to\infty} J_n^*[k;f] = I[k;f]. \tag{2.27}$$

3. **Technical estimates.** *Throughout this section, we assume that $W = e^{-Q} \in \mathcal{W}$ and we define*

$$T(x) := 1 + xQ''(x)/Q'(x) = \frac{d}{dx}\big(xQ'(x)\big)/Q'(x), \quad x \in (0,\infty). \tag{3.1}$$

Note that from (2.2) and (2.3),

$$1 + \eta \le T(x) \le C_3\{\log Q'(x)\}^2, \quad x \ge C_4. \tag{3.2}$$

We shall derive several upper and lower bounds for $Q^{(\ell)}(a_u)$, $\ell = 0, 1, 2$, a_{2u}/a_u, and other quantities. The estimates are essentially proved in the same way as those in [10], but since only Erdös weights were treated there, we include the proofs.

LEMMA 3.1. *(a) For $x > 0$, $L \ge 1$,*

$$Q(x) \le L^{-1}Q(Lx). \tag{3.3}$$

(b) $\exists C > 0$ such that

$$Q(a_u) \le Cu, \quad u \in (0,\infty). \tag{3.4}$$

(c) $\exists C_1$, C_2, C_3 such that

$$Q'(x) \ge C_1 x^{\eta}, \quad x \ge C_3. \tag{3.5}$$

$$Q(x) \ge C_2 x^{1+\eta}, \quad x \ge C_3. \tag{3.6}$$

(d)

$$a_u \le C_4 u^{1/(1+\eta)}, \quad u \ge C_5. \tag{3.7}$$

PROOF. (a)

$$Q(t) = \int_0^t Q'(s)\,ds \le tQ'(t), \quad t \in (0,\infty)$$

as Q' is non-decreasing. Then

$$Q(Lx)/Q(x) = \exp\left(\int_x^{Lx} \frac{Q'(t)}{Q(t)}\,dt\right)$$
$$\ge \exp\left(\int_x^{Lx} \frac{dt}{t}\right) = L.$$

(b) Now

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)(1-t^2)^{-1/2}\,dt$$
$$\geq \frac{2}{\pi}\int_{1/\sqrt{2}}^1 a_u Q'(a_u t)\,dt = \frac{2}{\pi}[Q(a_u) - Q(a_u/\sqrt{2})],$$

since $t/\sqrt{1-t^2} \geq 1$, $t \in [1/\sqrt{2}, 1)$. Applying (a) yields

$$u \geq \frac{2}{\pi} Q(a_u)[1 - 1/\sqrt{2}].$$

(c) From (2.3),

$$Q''(x)/Q'(x) \geq \eta/x, \quad x \geq C_1.$$

Integrating from C_1 to s yields

$$\log[Q'(s)/Q'(C_1)] \geq \eta \log[s/C_1],$$

and (3.5) follows for $x = s \geq C_3$. Integrating (3.5) similarly yields (3.6).

(d) This follows directly from (3.4) and (3.6). ∎

LEMMA 3.2. *There exists C_1, C_2, C_3 such that for $v \geq u \geq C_3$,*

$$(1 + C_1/u)(v/u)^{1/(1+\eta)} \geq a_v/a_u \geq (v/u)^{C_2/(\log Q'(a_v))^2}. \tag{3.8}$$

PROOF. Differentiating (2.7) with respect to u yields

$$1 = \frac{2}{\pi}\int_0^1 \frac{a_u' t Q'(a_u t) + a_u t Q''(a_u t) a_u' t}{\sqrt{1-t^2}}\,dt$$
$$= \frac{a_u'}{a_u}\frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t) T(a_u t)}{\sqrt{1-t^2}}\,dt.$$

Now give $B > 1$, we have for large u,

$$\int_0^{B/a_u} \frac{a_u t Q'(a_u t) T(a_u t)}{\sqrt{1-t^2}}\,dt = \frac{1}{a_u}\int_0^B \frac{sQ'(s)T(s)}{\sqrt{1-(s/a_u)^2}}\,ds$$
$$\leq C_4/a_u \int_0^B \big(sQ'(s) + s^2 Q''(s)\big)\,ds$$
$$\leq C_5/a_u.$$

Using our upper bound in (3.2) with a suitable value of B, and using the monotonicity of Q', we then obtain

$$1 \leq \frac{a_u'}{a_u}\left[\frac{2}{\pi}C_5/a_u + C_6\big(\log Q'(a_u)\big)^2 \frac{2}{\pi}\int_{B/a_u}^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}\,dt\right]$$
$$\leq \frac{a_u'}{a_u} C_7 \big(\log Q'(a_u)\big)^2 u. \tag{3.9}$$

Similarly using our lower bound in (3.2),

$$1 \geq \frac{a'_u}{a_u}(1+\eta)\frac{2}{\pi}\int_{B/a_u}^{1}\frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}\,dt$$
$$= \frac{a'_u}{a_u}(1+\eta)\Big[\frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}\,dt + O(1/a_u)\Big],$$

whence

$$\frac{a'_u}{a_u} \leq \frac{1}{1+\eta}\frac{1}{u-C_8}.$$

Then for large enough C_3, $v \geq u \geq C_3$ implies

$$a_v/a_u = \exp\Big(\int_u^v \frac{a'_t}{a_t}\,dt\Big)$$
$$\leq \exp\Big(\frac{1}{1+\eta}\log\Big(\frac{v-C_8}{u-C_8}\Big)\Big)$$
$$= \exp\Big(\frac{1}{1+\eta}\Big[\log\Big(\frac{v}{u}\Big)+\log\Big(\frac{1-C_8/v}{1-C_8/u}\Big)\Big]\Big)$$
$$\leq (v/u)^{1/(1+\eta)}(1+C_9/u).$$

Integrating (3.9) similarly yields

$$a_v/a_u = \exp\Big(\int_u^v \frac{a'_t}{a_t}\,dt\Big)$$
$$\geq \exp\Big(\frac{1}{C_7\big(\log Q'(a_v)\big)^2}\int_u^v \frac{dt}{t}\Big). \qquad \blacksquare$$

LEMMA 3.3. *(a)*

$$\frac{sQ'(s)}{tQ'(t)} \geq \Big(\frac{s}{t}\Big)^{\max\{T(u):u\in[s,t]\}}, \quad t > s > 0. \tag{3.10}$$

(b) For $u \geq C_2$,

$$u \leq a_u Q'(a_u) \leq C_1 u \log u. \tag{3.11}$$

(c) Given fixed $r > 1$,

$$a_{ru}/a_u \geq 1 + C_3/(\log u)^2, \quad u \geq C_4. \tag{3.12}$$

PROOF. (a)

$$\frac{sQ'(s)}{tQ'(t)} = \exp\Big(-\int_s^t \frac{\frac{d}{du}\big(uQ'(u)\big)}{uQ'(u)}\,du\Big)$$
$$= \exp\Big(-\int_s^t \frac{T(u)}{u}\,du\Big) \geq \exp\Big(-\max\{T(u) : u\in[s,t]\}\int_s^t \frac{du}{u}\Big).$$

https://doi.org/10.4153/CJM-1992-036-2 Published online by Cambridge University Press

(b) From the monotonicity of $sQ'(s)$,

$$u = \frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}\,dt \le a_u Q'(a_u)\frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^2}} = a_u Q'(a_u).$$

So we have the lower bound in (3.11). Next using (3.10),

$$\begin{aligned}\frac{u}{a_u Q'(a_u)} &= \frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{a_u Q'(a_u)}\frac{dt}{\sqrt{1-t^2}}\\ &\ge \frac{2}{\pi}\int_{1-\{\log Q'(a_u)\}^{-2}}^1 t^{\max\{T(v):v\in[a_u t, a_u]\}}\frac{dt}{\sqrt{1-t^2}}\\ &\ge \frac{2}{\pi}\left(1-\{\log Q'(a_u)\}^{-2}\right)^{C_3\{\log Q'(a_u)\}^2}\int_{1-\{\log Q'(a_u)\}^{-2}}^1 \frac{dt}{\sqrt{1-t^2}}\\ &\qquad \text{(by (3.2))}\\ &\ge C_4\left(\log Q'(a_u)\right)^{-1},\end{aligned}$$

by the inequality

$$(1-1/s)^s \ge e^{-1}/2,\ s \text{ large enough.}$$

Then the upper bound in (3.11) follows.

(c) From (3.8),

$$a_{ru}/a_u \ge r^{C_2/\{\log Q'(a_{ru})\}^2} \ge \exp\left(C_3/(\log u)^2\right),$$

by (b). Then (3.12) follows. ∎

4. **Infinite-finite range inequalities.** The basic result is due to Mhaskar and Saff [16]. Throughout, we assume that $W \in \mathcal{W}$. We also adopt the usual notation for L_p norms on real intervals:

THEOREM 4.1. *For $n \ge 1$ and $P \in \mathcal{P}_n$,*

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-a_n,a_n]}. \tag{4.1}$$

PROOF. *See [16].* ∎

LEMMA 4.2 A CRUDE NIKOLSKII INEQUALITY. *Let $0 < p < \infty$. $\exists C$ such that for $n \ge 1$ and $P \in \mathcal{P}_n$,*

$$\|PW\|_{L_\infty(\mathbb{R})} \le Cn^{2/\min\{p,1\}}\|PW\|_{L_p(\mathbb{R})}. \tag{4.2}$$

PROOF. *See [13, p. 53]. Note that in [13], (7.15) should read*

$$\lim_{|x|\to\infty} Q(x)/\log|x| = \infty,$$

rather than

$$\lim_{|x|\to\infty} \log Q(x)/\log|x| = \infty,$$

as is obvious from the proof. ∎

https://doi.org/10.4153/CJM-1992-036-2 Published online by Cambridge University Press

We shall need the following generalized L_p infinite-finite range inequality. Other L_p versions appear in [10, 13, 16, 20].

THEOREM 4.3. *Let $\alpha \geq 0$ and $\ell = 0, 1$ or 2. Let $0 < p \leq \infty$ and $\sigma > 0$. Then $\exists n_0 = n_0(\sigma)$ such that for $n \geq n_0$ and $P \in \mathcal{P}_n$,*

$$\|PW\left[1 + |Q^{(\ell)}|\right]^{\alpha}\|_{L_p(\mathbb{R})} \leq (1 + n^{-\sigma})\|PW\left[1 + |Q^{(\ell)}|\right]^{\alpha}\|_{L_p[-a_{4n}, a_{4n}]}. \tag{4.3}$$

PROOF. Let $P \in \mathcal{P}_n$ and $j \geq 1$. Then

$$t^{(2^j-1)n}P(t) \in \mathcal{P}_{2^j n},$$

so by Theorem 4.1, for $t \in \mathbb{R}$,

$$\begin{aligned} |t^{(2^j-1)n}P(t)W(t)| &\leq \|x^{(2^j-1)n}(PW)(x)\|_{L_\infty[-a_{2^j n}, a_{2^j n}]} \\ &\leq a_{2^j n}^{(2^j-1)n}\|PW\|_{L_\infty(\mathbb{R})}. \end{aligned}$$

Then for $|t| \geq a_{2^{j+1}n}$,

$$|PW|(t) \leq (a_{2^j n}/a_{2^{j+1}n})^{(2^j-1)n}\|PW\|_{L_\infty(\mathbb{R})}. \tag{4.4}$$

Here, from (3.12), uniformly for $n \geq n_1$ and $j \geq 1$,

$$\begin{aligned} a_{2^j n}/a_{2^{j+1}n} &\leq (1 + C_3/\{\log[2^j n]\}^2)^{-1} \\ &\leq \exp\left(-C_4/(j^2 + [\log n]^2)\right), \end{aligned}$$

so that

$$\begin{aligned} (a_{2^j n}/a_{2^{j+1}n})^{(2^j-1)n} &\leq \exp\left(-C_4(2^j - 1)n/(j^2 + [\log n]^2)\right) \\ &\leq \exp(-C_5 2^{j/2}n^{1/2}), \end{aligned} \tag{4.5}$$

$j \geq 1$, $n \geq n_0$, some $n_0 > n_1$. Furthermore, if

$$\mathcal{J}_{j,n} := \{t : a_{2^{j+1}n} \leq |t| \leq a_{2^{j+2}n}\},$$

we obtain from (3.4), (3.11) and (2.3) that whenever $t \in \mathcal{J}_{j,n}$,

$$\begin{aligned} |Q^{(\ell)}(t)| &\leq C_6(2^{j+2}n)(\log[2^{j+2}n])^3 \\ &\leq C_7 4^j n^2, \quad n \geq n_0, j \geq 1. \end{aligned}$$

Then (4.4), (4.5) and this last inequality yield

$$\begin{aligned} &\int_{\mathcal{J}_{j,n}} |PW|^p(t)\left[1+|Q^{(\ell)}(t)|\right]^{\alpha p}\,dt \\ &\qquad \leq C_9 \exp(-C_8 2^{j/2}n^{1/2})\|PW\|^p_{L_\infty(\mathbb{R})}(4^j n^2)^{\alpha p}a_{2^{j+2}n} \\ &\qquad \leq C_{10}n^{C_{11}}\|PW\|^p_{L_p(\mathbb{R})}4^{j[\alpha p+1]}\exp(-C_8 2^{j/2}n^{1/2}), \end{aligned}$$

where we have used the Nikolskii inequality Lemma 4.2, and the bound (3.7). Summing for $j = 1$ to ∞ yields

$$\int_{|t|\geq a_{4n}} |PW|^p(t)\big[1+|Q^{(\ell)}(t)|\big]^{\alpha p}\,dt$$
$$\leq \|PW\|^p_{L_p(\mathbb{R})} C_{10} n^{C_{11}} \sum_{j=1}^{\infty} 4^{j[\alpha p+1]} \exp(-C_8 2^{j/2} n^{1/2})$$
$$\leq \|PW\|^p_{L_p(\mathbb{R})} n^{-2\sigma p},$$

if $n \geq n_2$, which depends only on σ, α, p and not on P or n. Then

$$\|PW\big[1+|Q^{(\ell)}|\big]^{\alpha}\|_{L_p(|t|\geq a_{4n})} \leq n^{-2\sigma}\|PW\big[1+|Q^{(\ell)}|\big]^{\alpha}\|_{L_p(\mathbb{R})},$$

and (4.3) follows. ∎

We remark that we need a_{4n}, not $a_n(1+\epsilon)$, in the above lemma, since especially in the Erdös case, $Q'\big(a_n(1+\epsilon)\big)$ may grow much faster than $Q'(a_{4n})$. We can now obtain some standard estimates:

COROLLARY 4.4. *$\exists n_1$ such that*
(a) $\rho_n := \gamma_{n-1}/\gamma_n$ satisfies for $n \geq n_1$,

(4.6) $$\rho_n \leq a_{5n}.$$

(b) $x_{1n} = x_{1n}(W^2)$ satisfies for $n \geq n_1$,

(4.7) $$x_{1n} \leq a_{5n}.$$

PROOF. (a) We use Theorem 4.3 in the following identity:

$$\rho_n = \gamma_{n-1}/\gamma_n = \int_{-\infty}^{\infty} x p_{n-1}(x) p_n(x) W^2(x)\,dx$$
$$\leq \Big\{\int_{-\infty}^{\infty} x^2 p_{n-1}^2(x) W^2(x)\,dx\Big\}^{1/2}\Big\{\int_{-\infty}^{\infty} p_n^2(x) W^2(x)\,dx\Big\}^{1/2}$$
$$\leq (1+n^{-100})\Big\{\int_{-a_{4n}}^{a_{4n}} x^2 p_{n-1}^2(x) W^2(x)\,dx\Big\}^{1/2}\cdot 1$$
$$\leq (1+n^{-100}) a_{4n} \leq a_{5n},$$

by (3.12), if n is large enough.

(b) This follows similarly from the well known identity [29]

$$x_{1n} = \max_{P \in \mathcal{P}_{n-1}} \int_{-\infty}^{\infty} xP^2(x)W^2(x)\,dx \Big/ \int_{-\infty}^{\infty} P^2(x)W^2(x)\,dx. \qquad \blacksquare$$

5. **Differential equation estimates.** Differential equations play a crucial role in analysing orthogonal polynomials. In recent times, work of Shohat, Nevai, Bonan, Bauldry and Mhaskar has had an influence. In this section, we shall use some recent work of Mhaskar [15].

Throughout, we assume that $W := e^{-Q} \in \mathcal{W}$, and we set

$$\bar{Q}(x,t) := \frac{Q'(t) - Q'(t)}{t - x}, (x,t) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{5.1}$$

Further, we define for $n \geq 1$,

$$\rho_n := \rho_n(W^2) := \gamma_{n-1}(W^2)/\gamma_n(W^2), \tag{5.2}$$

where $\gamma_n := \gamma_n(W^2)$ is as in (1.2), and we define

$$A_n(x) := 2\rho_n \int_{-\infty}^{\infty} p_n^2(t)W^2(t)\bar{Q}(x,t)\,dt. \tag{5.3}$$

As shown below, $A_n(x_{kn})$ and $A_n'(x_{kn})$ exist and play an important role in relating $p_n^{(j)}(x_{kn})$, $j = 0, 1, 2$, to each other. Throughout x_{kn}, λ_{kn} (see (1.23)), $\ell_{kn}(x)$, and so on, have the meaning assigned to them in Section 1.

LEMMA 5.1. *For* $1 \leq k \leq n$,

$$p_n'(x_{kn}) = A_n(x_{kn})p_{n-1}(x_{kn}), \tag{5.4}$$

and

$$p_n''(x_{kn}) = \left[2Q'(x_{kn}) + A_n'(x_{kn})/A_n(x_{kn})\right]p_n'(x_{kn}), \tag{5.5}$$

where

$$A_n(x_{kn}) = 2\lambda_{kn}^{-1}p_{n-1}(x_{kn})^{-1} \int_{-\infty}^{\infty} (\ell_{kn}p_nQ'W^2)(t)\,dt, \tag{5.6}$$

and

$$\begin{aligned} A_n'(x_{kn})/A_n(x_{kn}) &= \frac{2}{\lambda_{kn}} \int_{-\infty}^{\infty} \ell_{kn}^2(t)W^2(t)Q'(t)\,dt - 2Q'(x_{kn}) \\ &= \frac{2}{\lambda_{kn}} \int_{-\infty}^{\infty} \ell_{kn}^2(t)W^2(t)\left[Q'(t) - Q'(x_{kn})\right]dt. \end{aligned} \tag{5.7}$$

PROOF. *Assuming (for example) that Q''' is continuous in $\mathbb{R}$, and that W^2 has all finite power moments, Mhaskar established the relations [15, Theorem 3.2]*

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x),$$

and [15, Theorem 3.4],

$$p_n''(x) + M_n(x)p_n'(x) + N_n(x)p_n(x) = 0,$$

where [15, Theorem 3.5, Proposition 3.5],

$$M_n(x) = -2Q'(x) - A_n'(x)/A_n(x),$$

and $B_n(x)$ and $N_n(x)$ are certain continuous functions of x. In this case, (5.4) and (5.5) then follow. We proceed to derive (5.6) and (5.7) in this case. Now [5, pp. 23–34]

$$\ell_{kn}(x) = \lambda_{kn}\rho_n p_{n-1}(x_{kn})\frac{p_n(x)}{x - x_{kn}}, \tag{5.8}$$

and so setting $x = x_{kn}$ and using (5.4), we obtain

$$\begin{aligned} 1 &= \lambda_{kn}\rho_n p_{n-1}(x_{kn})p_n'(x_{kn}) \\ &= \lambda_{kn}\rho_n p_{n-1}^2(x_{kn})A_n(x_{kn}). \end{aligned}$$

Then

$$\lambda_{kn}\rho_n p_{n-1}^2(x_{kn}) = 1/A_n(x_{kn}), \tag{5.9}$$

and hence

$$|\ell_{kn}(x)| = \{\lambda_{kn}\rho_n/A_n(x_{kn})\}^{1/2}\left|\frac{p_n(x)}{x - x_{kn}}\right|. \tag{5.10}$$

We have

$$\begin{aligned} A_n(x_{kn}) &= 2\rho_n\int_{-\infty}^{\infty}\frac{p_n(t)}{t - x_{kn}}p_n(t)\big[Q'(t) - Q'(x_{kn})\big]W^2(t)\,dt \\ &= 2\rho_n\int_{-\infty}^{\infty}\frac{p_n(t)}{t - x_{kn}}p_n(t)Q'(t)W^2(t)\,dt \\ &\qquad \text{(by orthogonality of } p_nW^2 \text{ to } \mathcal{P}_{n-1}) \\ &= 2\{\lambda_{kn}p_{n-1}(x_{kn})\}^{-1}\int_{-\infty}^{\infty}\ell_{kn}(t)p_n(t)Q'(t)W^2(t)\,dt. \end{aligned}$$

Hence (5.6). Furthermore,

$$\begin{aligned} A_n'(x_{kn}) &= 2\rho_n\int_{-\infty}^{\infty}p_n^2(t)\frac{-Q''(x_{kn})(t - x_{kn}) + \big(Q'(t) - Q'(k_{kn})\big)}{(t - x_{kn})^2}W^2(t)\,dt \\ &= -2\rho_nQ''(x_{kn})\int_{-\infty}^{\infty}\frac{p_n(t)}{t - x_{kn}}p_n(t)W^2(t)\,dt \\ &\quad + 2\rho_n\int_{-\infty}^{\infty}\Big(\frac{p_n(t)}{t - x_{kn}}\Big)^2W^2(t)Q'(t)\,dt \\ &\quad - 2\rho_nQ'(x_{kn})\int_{-\infty}^{\infty}\Big(\frac{p_n(t)}{t - x_{kn}}\Big)^2W^2(t)\,dt \\ &= 0 + 2A_n(x_{kn})\lambda_{kn}^{-1}\int_{-\infty}^{\infty}\ell_{kn}^2(t)W^2(t)Q'(t)\,dt \\ &\quad - 2Q'(x_{kn})A_n(x_{kn})\lambda_{kn}^{-1}\int_{-\infty}^{\infty}\ell_{kn}^2(t)W^2(t)\,dt, \end{aligned} \tag{5.11}$$

by (5.10). Taking account of (1.23) yields (5.7) in this case.

This proves the lemma when Q''' is continuous. Now we observe that the identities (5.4) to (5.7) involve only Q', not Q'' or Q'''. We can by Carleman's Theorem, approximate our given Q by Q_ϵ that is entire and such that for $j = 0$ and 1,

$$|Q_\epsilon^{(j)}(x) - Q^{(j)}(x)| < \epsilon, \quad x \in \mathbb{R}.$$

Then $W_\epsilon := \exp(-Q_\epsilon)$ satisfies (5.4) to (5.7) and for small enough ϵ,

$$|W_\epsilon(x) - W(x)| < W(x)2\epsilon, \quad x \in \mathbb{R}.$$

This is sufficient to guarantee convergence of the moments of W_ϵ to those of W as $\epsilon \longrightarrow 0+$, and hence that $p_n(W_\epsilon^2; x) \longrightarrow p_n(W^2; x)$, $\epsilon \longrightarrow 0+$, uniformly on compact sets. Continuity of the zeros of orthogonal polynomials then ensures that (5.4) to (5.7) hold for W^2. ∎

We next turn to estimation of $A_n(x_{kn})$ and $A_n'(x_{kn})/A_n(x_{kn})$. To estimate the former, we proceed as in [15].

LEMMA 5.2. *For $n \geq n_1$ and uniformly for $1 \leq k \leq n$,*

$$A_n(x_{kn}) \geq Cn/(a_n \log n), \tag{5.12}$$

and

$$\rho_n A_n(x_{kn}) \geq Cn/\log n. \tag{5.13}$$

PROOF. We first show that uniformly for $1 \leq k \leq n$,

$$A_n(x_{kn}) = 2\rho_n \int_{-a_{6n}}^{a_{6n}} (p_n W)^2(t)\bar{Q}(x_{kn}, t)\, dt + o(1). \tag{5.14}$$

For, since $x_{kn} \leq a_{5n}$ (by (4.7)), we have

$$\begin{aligned}
\tau_{kn} &:= \int_{|t|\geq a_{6n}} (p_n W)^2(t)\bar{Q}(x_{kn}, t)\, dt \\
&\leq \frac{Q'(a_{5n})}{a_{6n} - a_{5n}} \int_{|t|\geq a_{6n}} (p_n W)^2(t)\, dt + \frac{1}{a_{6n} - a_{5n}} \int_{|t|\geq a_{6n}} (p_n W)^2(t)|Q'(t)|\, dt \\
&\leq C_1 n(\log n)^3 a_{5n}^{-2} n^{-100} \int_{-a_{6n}}^{a_{6n}} (p_n W)^2(t)\, dt \\
&\quad + (\log n)^2 a_{5n}^{-1} n^{-100} Q'(a_{6n}) \int_{-a_{6n}}^{a_{6n}} (p_n W)^2(t)\, dt \\
&= o(1),
\end{aligned}$$

by Theorem 4.3 and (3.11) and (3.12). Also by Theorem 4.3,

$$\begin{aligned}
\sigma_{kn} &:= 2\rho_n \int_{-\infty}^{\infty} (p_n W)^2(t)\{Q'(t) - Q'(x_{kn})\}^2\, dt \\
&= 2\rho_n \int_{-a_{6n}}^{a_{6n}} (p_n W)^2(t)\{Q'(t) - Q'(x_{kn})\}^2\, dt + o(1),
\end{aligned} \tag{5.15}$$

uniformly for $1 \leq k \leq n$. Now by an elementary argument [15, Corollary 3.3], Mhaskar showed that

$$\sigma_{kn} \geq n^2/(2\rho_n).$$

Then from (5.14) and (5.15),

$$A_n(x_{kn}) \geq o(1) + 2\rho_n \frac{\int_{-a_{6n}}^{a_{6n}} (p_n W)^2(t)\{Q'(t) - Q'(x_{kn})\}^2\, dt}{\max_{|s|\leq a_{6n}} \{|Q'(s) - Q'(x_{kn})|\, |s - x_{kn}|\}}$$
$$\geq o(1) + \big(n^2/(2\rho_n) + o(1)\big)/Cn\log n),$$

by (3.11). Then (5.13) follows and the bound (4.6) then yields (5.12). ■

We remark that if

$$K_n(x,t) = \rho_n \frac{p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x)}{x - t},$$

is the usual kernel function, then using the idenitity

$$\ell_{kn}(t) = \lambda_{kn} K_n(t, x_{kn}),$$

we can re-express (5.7) in the form

$$\begin{aligned} A_n'(x_{kn})/A_n(x_{kn}) &= 2\Big[\lambda_{kn}\int_{-\infty}^{\infty} K_n^2(t,x_{kn})W^2(t)Q'(t)\,dt - Q'(x_{kn})\Big] \\ &= 2\big[G_n(W^2,Q',x_{kn}) - Q'(x_{kn})\big]. \end{aligned} \tag{5.16}$$

Here $G_n(W^2,\cdot,\cdot)$ is Nevai's operator [19, p. 74]. Unfortunately, this interesting representation does not facilitate estimation of $A_n'(x_{kn})/A_n(x_{kn})$. That is the purpose of the following lemma:

LEMMA 5.3. *If $W \in \mathcal{W}$, then uniformly for $n \geq 1$ and $1 \leq k \leq n$,*

$$|A_n'(x_{kn})/A_n(x_{kn})| \leq C_1\big[1 + |Q'(x_{kn})|\big]. \tag{5.17}$$

REMARK. If one assumes more, for example, that Q''' is continuous and admits certain estimates, then one can prove much better bounds [15]. However, (5.17) holds more generally and is sufficient for our purposes.

PROOF. From (5.11) we see that

$$A_n'(x_{kn}) = 2\rho_n \int_{-\infty}^{\infty} (p_n W)^2(t) \frac{\bar{Q}(x_{kn},t)}{t - x_{kn}}\, dt,$$

so

$$\begin{aligned} |A_n'(x_{kn})| &\leq 2\rho_n \int_{|t-x_{kn}|\geq(1+|Q'(x_{kn})|)^{-1}} \big(p_n W)^2(t)|\bar{Q}(x_{kn},t)|(1 + |Q'(x_{kn})|\big)\, dt \\ &\quad + 2\rho_n \int_{|t-x_{kn}|\leq(1+|Q'(x_{kn})|)^{-1}} (p_n W)^2(t) \frac{|\bar{Q}(x_{kn},t)|}{|t - x_{kn}|}\, dt \\ &=: I_1 + I_2. \end{aligned} \tag{5.18}$$

Now, as Q is convex, $\bar{Q}(x,t)$ is non-negative, so

$$I_1 \le 2\big(1+|Q'(x_{kn})|\big)\rho_n \int_{-\infty}^{\infty} (p_nW)^2(t)\bar{Q}(x_{kn},t)\,dt = \big(1+|Q'(x_{kn})|\big)A_n(x_{kn}). \tag{5.19}$$

Next, we estimate I_2. Suppose first $x_{kn} \ne 0$. Writing $|x_{kn}| = a_u$, some $u > 0$, we have if $|t - x_{kn}| \le \big(1+|Q'(x_{kn})|\big)^{-1}$, then $|t| \le a_u + \big(1+Q'(a_u)\big)^{-1}$, which implies that

$$|t/x_{kn}| \le 1 + \big(a_uQ'(a_u)\big)^{-1} \le 1 + u^{-1},$$

by (3.11), so if $u \ge u_0$, (3.12) yields

$$|t| \le a_u(1+u^{-1}) \le a_{2u}.$$

Also then if $|t| < |x_{kn}|$, we have $|Q'(t)| \le |Q'(x_{kn})|$, while if $|t| \ge |x_{kn}|$, (3.10) yields

$$\begin{aligned}|Q'(t)/Q'(x_{kn})| &\le |t/x_{kn}|^{\max\{T(s):s\in[|x_{kn}|,|t|]\}-1}\\ &\le (1+u^{-1})^{\max\{T(s):s\in[a_u,a_{2u}]\}}\\ &\le (1+u^{-1})^{C(\log u)^2},\end{aligned}$$

by (3.2) if $u \ge u_1$, say. It follows that

$$|Q'(t)| \le C_1|Q'(x_{kn})|, \quad |t-x_{kn}| \le \big(1+|Q'(x_{kn})|\big)^{-1},$$

if at least $|x_{kn}| \ge C_2$. Since Q' is continuous in $\mathbb{R}$, we obtain

$$|Q'(t)| \le C_3\big[1+|Q'(x_{kn})\big], \quad |t-x_{kn}| \le \big(1+|Q'(x_{kn})|\big)^{-1},$$

uniformly for $1 \le k \le n$. Then for $1 \le k \le n$,

$$\begin{aligned}I_2 &\le C_4\big[1+|Q'(x_{kn})|\big]\rho_n \int_{-\infty}^{\infty}\Big(\frac{p_n(t)}{t-x_{kn}}\Big)^2 W^2(t)\,dt\\ &= C_4\big[1+|Q'(x_{kn})|\big]\rho_n\big(\lambda_{kn}\rho_n p_{n-1}(x_{kn})\big)^{-2}\int_{-\infty}^{\infty}\ell_{kn}^2(t)W^2(t)\,dt\\ &\quad \text{(by (5.8))}\\ &= C_4\big[1+|Q'(x_{kn})|\big]A_n(x_{kn}),\end{aligned}$$

by (5.9) and (1.23). Together with (5.18) and (5.19), this last inequality yields (5.17). ∎

6. **Christoffel function estimates.** The important role played by *Christoffel function* estimates in approximation theory is well known [20]. Recall that one can define the nth Christoffel function $\lambda_n(W^2,x)$ by

$$\lambda_n(W^2,x) := \inf_{P\in\mathcal{P}_{n-1}} \int_{-\infty}^{\infty}(PW)^2(t)\,dt/P^2(x), \tag{6.1}$$

$$= 1\Big/\sum_{j=0}^{n-1} p_j^2(x). \tag{6.2}$$

Here we derive bounds for $\lambda_n(W^2,x)$ using fairly standard methods. We include full proofs as the results hold simultaneously for Freud and Erdös weights, so are formulated a little differently from those in the literature. For stronger results in different cases, see [9, 10, 20].

LEMMA 6.1. *Let $W \in \mathcal{W}$ and $0 < \epsilon < 1$. $\exists n_1$ such that for $n \geq n_1$ and $|x| \leq a_{n^\epsilon}$,*

$$\lambda_n(W^2, x) \leq C\frac{a_n}{n}W^2(x). \tag{6.3}$$

PROOF. We adopt the method of Freud [6]. Using Theorem 4.3,

$$\lambda_n(W^2, x) \leq 2 \inf_{P \in \mathcal{P}_{n-1}} \int_{-a_{4n}}^{a_{4n}} (PW)^2(t)\, dt / P^2(x), \tag{6.4}$$

$n \geq n_1$, $x \in \mathbb{R}$. Define a linear polynomial in t,

$$\psi_x(t) := Q(x) + (t - x)Q'(x).$$

If Q'' exists throughout $\mathbb{R}$, we see that by convexity,

$$\psi_x(t) - Q(t) = -\frac{1}{2}Q''(\xi)(t - x)^2 \leq 0,$$

$t \in \mathbb{R}$, some ξ between t and x. As the left side does not involve Q'', a continuity argument establishes this even when $Q''(0)$ does not exist. Hence

$$\exp\big(\psi_x(t)\big)W(t) \leq 1, \quad t \in \mathbb{R}. \tag{6.5}$$

Next, if $n \geq n_1$, for $|x| \leq a_{n^\epsilon}$ and $|t| \leq a_{4n}$,

$$|\psi_x(t)| \leq Q(a_{n^\epsilon}) + 2a_{4n}Q'(a_{n^\epsilon}).$$

Here, by (3.8), for n large enough,

$$a_{4n}/a_{n^\epsilon} \leq C(n^{1-\epsilon})^{1/(1+\eta)},$$

while by (3.4) and (3.11),

$$Q(a_{n^\epsilon}) \leq Cn^\epsilon;$$

$$a_{n^\epsilon}Q'(a_{n^\epsilon}) \leq Cn^\epsilon \log n.$$

Then combining these estimates,

$$|\psi_x(t)| \leq C_1(n^\epsilon + n^{\frac{1+\epsilon\eta}{1+\eta}} \log n) = o(n), \tag{6.6}$$

uniformly for $|t| \leq a_{4n}$, $|x| \leq a_{n^\epsilon}$. Now let

$$s_m(u) := \sum_{j=0}^{m} u^j / j!.$$

It is well known and easy to see that for some $C_2 > 0$,

$$\frac{1}{2} \leq s_m(u)e^{-u} \leq 2, \quad |u| \leq C_2 m.$$

Define

$$\tau_{x,n}(t) := s_{\langle n/2\rangle}\big(\psi_x(t)\big),$$

where $\langle x\rangle$ denotes the greatest integer $\leq x$. We see that $\tau_{x,n}(t)$ has degree at most $n/2$ in t, and in view of (6.6), for $n \geq n_1$,

$$\frac{1}{2} \leq \tau_{x,n}(t)\exp\big(-\psi_x(t)\big) \leq 2, \quad |t| \leq a_{4n}, \quad |x| \leq a_{n^\epsilon}.$$

Then from (6.5),

$$0 \leq \tau_{x,n}(t)W(t) \leq 2, \quad |t| \leq a_{4n}, \quad |x| \leq a_{n^\epsilon}. \tag{6.7}$$

Furthermore, for this range of x,

$$\tau_{x,n}(x)W(x) = \tau_{x,n}(x)\exp\big(-\psi_x(x)\big) \geq 1/2. \tag{6.8}$$

Substituting $P(t) := \tau_{x,n}(t)R(t)$ in (6.4), where $R \in \mathcal{P}_{\langle n/2\rangle -1}$ is arbitrary, yields for $|x| \leq a_{n^\epsilon}$,

$$\begin{aligned}\lambda_n(W^2,x)W^{-2}(x) &\leq 2 \inf_{R\in\mathcal{P}_{\langle n/2\rangle-1}} \int_{-a_{4n}}^{a_{4n}} R^2(t)(\tau_{x,n}W)^2(t)\,dt \Big/ \{R^2(x)(\tau_{x,n}W)^2(x)\} \\ &\leq 32 \inf_{R\in\mathcal{P}_{\langle n/2\rangle-1}} \int_{-a_{4n}}^{a_{4n}} R^2(t)\,dt/R^2(x) \quad \text{(by (6.7) and (6.8))} \\ &\leq 32a_{4n}\lambda_{\langle n/2\rangle}(w; x/a_{4n}),\end{aligned}$$

where $w \equiv 1$ in $[-1,1]$ is the Legendre weight. Using (3.8), we have $a_{4n} \leq Ca_n$, and then standard bounds for the Christoffel functions for the Legendre weight yield (6.3). ■

We shall also need an estimate for $\lambda_n(W^2; x)$ for $|x| \geq a_{n^\epsilon}$. This is based on:

LEMMA 6.2. *For $n \geq 1$*

$$\sum_{j=1}^{n} \lambda_{jn}W^{-2}(x_{jn})(2+x_{jn}^2)^{-1/2}\big(\log(2+x_{jn}^2)\big)^{-2} \leq C. \tag{6.9}$$

PROOF. Define for $n \geq 1$,

$$\phi(x) := \exp\Big[2Q(\sqrt{x}) - \frac{1}{2}\log(2+x) - 2\log\log(2+x)\Big].$$

Then

$$\psi(x) := \frac{x\phi'(x)}{\phi(x)} = \sqrt{x}Q'(\sqrt{x}) - \frac{x}{2(2+x)} - \frac{2x}{\{\log(2+x)\}(2+x)}.$$

It is easy to see that $\psi(x)$ is increasing for x large, since

$$\psi'(x) = T(\sqrt{x})Q'(\sqrt{x})/(2\sqrt{x}) + O(1/x^2) \geq C_2x^{(\eta-1)/2},$$

by (3.5). (Recall here that T is given by (3.1)). It is then also easy to see that for any fixed $L > 1$, and x large enough,

$$\psi(Lx) - \psi(x) \geq 1.$$

By modifying Q for small x, we may assume this inequality holds for $x \geq 1$. A theorem of Clunie and Kovari [4, p. 19, Theorem 4], then asserts the existence of entire

$$H(z) = \sum_{j=0}^{\infty} h_j z^j, \quad h_j > 0, \quad j \geq 1,$$

such that

$$C_1 \leq H(x)/\phi(x) \leq C_2, \quad x \geq 1.$$

We can obviously assume this is also true for $x \in [0,1]$. Then setting

$$G(x) := H(x^2) = \sum_{j=0}^{\infty} h_j x^{2j},$$

we have for $x \in \mathbb{R}$,

$$C_1 \leq G(x)/\phi(x^2) = G(x)W^2(x)(2+x^2)^{1/2}\big(\log(2+x^2)\big)^2 \leq C_2. \tag{6.10}$$

The generalized Markov-Stieltjes inequality [5, p. 92] yields

$$\sum_{j=1}^{n} \lambda_{jn} G(x_{jn}) \leq \int_{-\infty}^{\infty} G(x)W^2(x)\,dx.$$

The last inequality and (6.10) yield

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})(2+x_{jn}^2)^{-1/2}\big(\log(2+x_{jn}^2)\big)^{-2}$$
$$\leq C_3 \int_{-\infty}^{\infty} (2+x^2)^{-1/2}\big(\log(2+x^2)\big)^{-2}\,dx < \infty. \qquad \blacksquare$$

We can now deduce:

LEMMA 6.3. *Let $0 < \epsilon < 1$. Then for $n \geq n_1$ and $|x_{jn}| \geq a_{n^\epsilon}$,*

$$\lambda_{jn} \leq C\exp(-n^{\epsilon/2}). \tag{6.11}$$

PROOF. From (3.7), (4.7), and the previous lemma,

$$\lambda_{jn} \leq C_1 W^2(x_{jn}) a_{5n} (\log a_{5n})^2$$
$$\leq C_2 (n \log n) \exp\big(-2Q(x_{jn})\big).$$

Now for $|x_{jn}| \geq a_{n^\epsilon}$,

$$Q(x_{jn}) \geq Q(a_{n^\epsilon/2}) + \int_{a_{n^\epsilon/2}}^{a_{n^\epsilon}} Q'(t)\,dt$$
$$\geq Q'(a_{n^\epsilon/2})[a_{n^\epsilon} - a_{n^\epsilon/2}]$$
$$\geq C_4 n^\epsilon/(\log n)^2,$$

by (3.12) and (3.11). Then we obtain (6.11). $\blacksquare$

7. **Proof of the theorems.**

PROOF OF THEOREM 2.2. Now [21, p. 44] if $R_n \in \mathcal{P}_n$, there is the identity

$$\begin{aligned} R_n(x) - H_n(W^2, R_n, x) &= \sum_{j=1}^{n} R_n'(x_{jn})(x - x_{jn})\ell_{jn}^2(x) \\ &= \rho_n p_n(x) \sum_{j=1}^{n} R_n'(x_{jn}) p_{n-1}(x_{jn}) \lambda_{jn} \ell_{jn}(x) \\ &= \rho_n p_n(x) L_n(x), \end{aligned}$$

where we have used (5.8), and $L_n \in \mathcal{P}_{n-1}$ is the Lagrange interpolation polynomial satisfying

$$L_n(x_{jn}) = R_n'(x_{jn}) p_{n-1}(x_{jn}) \lambda_{jn}, \quad 1 \leq j \leq n.$$

Using the Cauchy-Schwarz inequality and the Gauss quadrature formula yields:

$$\begin{aligned} \Delta &:= \int_{-\infty}^{\infty} \left| R_n(x) - H_n(W^2, R_n, x) \right| W^2(x)\, dx \\ &\leq \rho_n \left\{ \int_{-\infty}^{\infty} (p_n W)^2(x)\, dx \right\}^{1/2} \left\{ \int_{-\infty}^{\infty} (L_n W)^2(x)\, dx \right\}^{1/2} \\ &= \rho_n \left\{ \sum_{j=1}^{n} \lambda_{jn} \Big(R_n'(x_{jn}) p_{n-1}(x_{jn}) \lambda_{jn} \Big)^2 \right\}^{1/2} \\ &\leq \rho_n \max_{1 \leq j \leq n} \lambda_{jn}^{1/2} \left\{ \sum_{j=1}^{n} \lambda_{jn} R_n'(x_{jn})^2 \Big/ \{ \rho_n A_n(x_{jn}) \} \right\}^{1/2}, \end{aligned}$$

by (5.9). Here by Lemmas 6.1 and 6.3 and the boundedness of W^2,

$$\max_{1 \leq j \leq n} \lambda_{jn} \leq C a_n / n.$$

Also, applying (5.13) and (4.6) yields

$$\begin{aligned} \Delta &\leq C_1 a_{5n}^{1/2} (a_n/n)^{1/2} (a_n \log n / n)^{1/2} \left\{ \sum_{j=1}^{n} \lambda_{jn} R_n'(x_{jn})^2 \right\}^{1/2} \\ &\leq C_2 a_n^{3/2} (\log n)^{1/2} \Big/ n \left\{ \int_{-\infty}^{\infty} (R_n' W)(x)^2\, dx \right\}^{1/2}, \end{aligned}$$

by the Gauss quadrature formula. This establishes (2.9). Finally, (2.11) follows easily from (2.10) and the bound (3.4): Together they imply that

$$a_n^{3/2+\epsilon} = O(n),$$

and hence that

$$a_n^{3/2} = O(n^{1/(1+2\epsilon/3)}) = o\Big(n(\log n)^{-1/2}\Big). \qquad \blacksquare$$

In proving the other results of Section 2, we shall need two lemmas:

https://doi.org/10.4153/CJM-1992-036-2 Published online by Cambridge University Press

LEMMA 7.1. *Assume the hypotheses of Theorem 2.3. Then there exists C such that*

$$\sum_{j=1}^{n} \lambda_{jn}^2 W^{-4}(x_{jn})\big[1+|Q'(x_{jn})|\big]^{-2\alpha} \le Ca_n/n, \quad n \ge 1, \tag{7.1}$$

PROOF. *Let σ be as in (2.13). Choose $\epsilon \in (0,1)$ so close to 1 that*

$$(1+\sigma)\epsilon > 1+\sigma/2. \tag{7.2}$$

Now for $n \ge n_1$ and $|x_{jn}| \ge a_{n^\epsilon}$, we can write $|x_{jn}| = a_m$, with $m \ge n^\epsilon$. Then

$$|Q'(x_{jn})| = Q'(a_m) \ge m/a_m,$$

by (3.11), so if n_1 is large enough,

$$\big[1+|Q'(x_{jn})|\big]^{-\alpha} \le (m/a_m)^{-\alpha} \le C_m - 1 - \sigma \le n^{-1-\sigma/2},$$

by (2.13), (7.2) and since $m \ge n^\epsilon$. Then for $n \ge n_1$ and $|x_{jn}| \ge a_{n^\epsilon}$, Lemma 6.2, (4.7) and (3.8) yield

$$\begin{aligned}\lambda_{jn} W^{-2}(x_{jn})\big[1+|Q'(x_{jn})|\big]^{-\alpha} &\le C_1|x_{jn}|(\log|x_{jn}|)^2 n^{-1-\sigma/2}\\ &\le C_2 a_n(\log n)^2 n^{-1-\sigma/2}\\ &\le C_3 a_n/n.\end{aligned}$$

Also for $|x_{jn}| \le a_{n^\epsilon}$, Lemma 6.1 shows that this last inequality persists. Hence

$$\sum_{j=1}^{n} \lambda_{jn}^2 W^{-4}(x_{jn})\big[1+|Q'(x_{jn})|\big]^{-2\alpha} \le C_3(a_n/n)\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})\big[1+|Q'(x_{jn})|\big]^{-\alpha}. \tag{7.3}$$

To estimate this last sum, let us suppose that $|x_{jn}| \ge 1$, and write $|x_{jn}| = a_m$. As Q' is increasing and positive in $(0,\infty)$*,*

$$|x_{jn}| = a_m \le a_m Q'(a_m)/Q'(1) \le Cm\log m,$$

by (3.11). Then by (3.11),

$$\begin{aligned}|Q'(x_{jn})|^{-\alpha} &= Q'(a_m)^{-\alpha} \le (a_m/m)^{\alpha}\\ &\le Cm^{-1-\sigma} \le C_4(m\log m)^{-1-\sigma/2} \le C_5|x_{jn}|^{-1-\sigma/2}\\ &\quad \textit{(by (2.13))}\\ &\le C_6(2+x_{jn}^2)^{-1/2}\big(\log(2+x_{jn}^2)\big)^{-2}.\end{aligned} \tag{7.4}$$

We deduce that

$$\begin{aligned}&\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})\big[1+|Q(x_{jn})|\big]^{-\alpha}\\ &\qquad \le C_7\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})(2+x_{jn}^2)^{-1/2}\big(\log(2+x_{jn}^2)\big)^{-2}\\ &\qquad \le C_8,\end{aligned}$$

by Lemma 6.2. Now (7.3) yields the result. ∎

LEMMA 7.2. *Assume the hypotheses of Theorem 2.3. For $n \geq 1$, let $\{s_{jn}\}_{j=1}^{n} \subset \mathbb{R}$ and set*

(7.5) $$\chi_n := \max_{1\leq j\leq n} |s_{jn}|\, W^2(x_{jn})\big[1+|Q'(x_{jn})|\big]^{\alpha}, \quad n \geq 1,$$

and

(7.6) $$\xi_n := \max_{1\leq j\leq n} |s_{jn}|\, W^2(x_{jn})\big[1+|Q'(x_{jn})|\big]^{\alpha+1}, \quad n \geq 1,$$

Then for $n \geq 1$,

(7.7) $$\int_{-\infty}^{\infty}\Big|\sum_{j=1}^{n} s_{jn}(x-x_{jn})\ell_{jn}^2(x)\Big|\, W^2(x)\,dx \leq Ca_n^{3/2}(\log n)^{1/2}n^{-1}\chi_n,$$

and

(7.8) $$\int_{-\infty}^{\infty}\Big|\sum_{j=1}^{n} s_{jn}\frac{p_n''(x_{jn})}{p_n'(x_{jn})}(x-x_{jn})\ell_{jn}^2(x)\Big|W^2(x)\,dx \leq Ca_n^{3/2}(\log n)^{1/2}n^{-1}\xi_n.$$

PROOF. Let us denote the left-hand side of (7.7) by Δ_n. Then, using (5.8), we see that

$$\Delta_n = \int_{-\infty}^{\infty}\Big|\rho_n p_n(x)\sum_{j=1}^{n} s_{jn}\lambda_{jn}p_{n-1}(x_{jn})\ell_{jn}(x)\Big|\, W^2(x)\,dx$$
$$= \int_{-\infty}^{\infty}|\rho_n p_n(x)L_n(x)|\, W^2(x)\,dx,$$

where $L_n \in \mathcal{P}_{n-1}$ is the Lagrange interpolation polynomial satisfying

$$L_n(x_{jn}) = s_{jn}\lambda_{jn}p_{n-1}(x_{jn}), \quad 1 \leq j \leq n.$$

Then using the Cauchy-Schwarz inequality and the Gauss quadrature formula, we see that

$$\Delta_n \leq \rho_n\Big\{\int_{-\infty}^{\infty}(p_nW)^2(x)\,dx\Big\}^{1/2}\Big\{\int_{-\infty}^{\infty}(L_nW)^2(x)\,dx\Big\}^{1/2}$$
$$= \rho_n\Big\{\sum_{j=1}^{n}\lambda_{jn}\big(s_{jn}\lambda_{jn}p_{n-1}(x_{jn})\big)^2\Big\}^{1/2}$$
$$= \rho_n\Big\{\sum_{j=1}^{n}\lambda_{jn}^2 s_{jn}^2\big(\rho_nA_n(x_{jn})\big)^{-1}\Big\}^{1/2} \quad \text{(by (5.9))}$$
$$\leq \max_{1\leq j\leq n}\{\rho_n/A_n(x_{jn})\}^{1/2}\max_{1\leq j\leq n}|s_{jn}|\, W^2(x_{jn})\big[1+|Q'(x_{jn})|\big]^{\alpha}$$
$$\times\Big\{\sum_{j=1}^{n}\lambda_{jn}^2W^{-4}(x_{jn})\big[1+|Q'(x_{jn})|\big]^{-2\alpha}\Big\}^{1/2}$$
$$\leq Ca_n^{1/2}(a_n\log n/n)^{1/2}\chi_n(a_n/n)^{1/2},$$

by (4.6), (3.8), (5.12) and (7.1). So we have (7.7). To obtain (7.8) from (7.7), it suffices to show that uniformly for $n \geq 1$ and $1 \leq k \leq n$,

(7.9) $$|p_n''(x_{kn})/p_n'(x_{kn})| \leq C\big[1+|Q'(x_{kn})|\big].$$

https://doi.org/10.4153/CJM-1992-036-2 Published online by Cambridge University Press

This inequality follows directly from (5.5) and (5.17). ∎

PROOF OF THEOREM 2.3. From (1.8), (1.10) and (1.15), we see that

$$\begin{aligned}\Delta_n &:= \int_{-\infty}^{\infty} |H_n(W^2,f,x) - Y_n(W^2,f,x)|\, W^2(x)\,dx \\ &= \int_{-\infty}^{\infty} \Big|\sum_{j=1}^{n} f(x_{jn}) \frac{p_n''(x_{jn})}{p_n'(x_{jn})}(x - x_{jn})\ell_{jn}^2(x)\Big| W^2(x)\,dx \\ &\le Ca_n^{3/2}(\log n)^{1/2} n^{-1}\xi_n,\end{aligned}$$

where by Lemma 7.2,

$$\begin{aligned}\xi_n &:= \max_{1\le j\le n} |f(x_{jn})|\, W^2(x_{jn})\big[1 + |Q'(x_{jn})|\big]^{\alpha+1} \\ &\le \big\|fW^2\big[1+|Q'|\big]^{\alpha+1}\big\|_{L_\infty(\mathbb{R})}, \quad n \ge 1.\end{aligned}$$

Hence, (2.12) and (2.14) guarantee that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |H_n(W^2,f,x) - Y_n(W^2,f,x)|\, W^2(x)\,dx = 0. \tag{7.10}$$

We proceed to establish (2.15). First, we note that the argument used at (7.4) also shows that

$$\big[1 + |Q'(x)|\big]^{-\alpha} = o\Big((2+x^2)^{-1/2}\big(\log(2+x^2)\big)^{-2}\Big), \quad |x| \to \infty,$$

and hence, for any polynomial P, as $|x| \to \infty$,

$$|f - P|(x)W^2(x)(2+x^2)^{1/2}\big(\log(2+x^2)\big)^2 = o(1)\big[1 + \|fW^2\big[1+|Q'|\big]^{\alpha}\|_{L_\infty(\mathbb{R})}\big].$$

Here, of course, the $o(1)$ depends on P. Then if G is the even entire function with non-negative Maclaurin series coefficients satisfying (6.10), we have

$$\lim_{|x|\to\infty} |f - P|(x)/\,G(x) = 0.$$

By a classical theorem on quadrature convergence [5, p. 94, Theorem 1.6(a)],

$$\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn}|f - P|(x_{jn}) = \int_{-\infty}^{\infty} |f(x) - P(x)|\, W^2(x)\,dx.$$

Then

$$\begin{aligned}&\int_{-\infty}^{\infty} |H_n(W^2,f,x) - f(x)|\, W^2(x)\,dx \\ &\le \int_{-\infty}^{\infty} |H_n(W^2,f-P,x)|\, W^2(x)\,dx \\ &\quad + \int_{-\infty}^{\infty} |H_n(W^2,P,x) - P(x)|\, W^2(x)\,dx + \int_{-\infty}^{\infty} |P(x) - f(x)|\, W^2(x)\,dx \\ &\le \int_{-\infty}^{\infty} |Y_n(W^2,f-P,x)|\, W^2(x)\,dx + o(1) + o(1) \\ &\quad + \int_{-\infty}^{\infty} |P(x) - f(x)|\, W^2(x)\,dx\end{aligned}$$

(by (7.10) and Theorem 2.2)

$$\le \sum_{j=1}^{n} \lambda_{jn}|f - P|(x_{jn}) + o(1) + o(1) + \int_{-\infty}^{\infty} |P(x) - f(x)|\, W^2(x)\,dx,$$

https://doi.org/10.4153/CJM-1992-036-2 Published online by Cambridge University Press

by (1.15) and (1.23). We deduce that

$$\limsup_{n\to\infty} \int_{-\infty}^{\infty} |H_n(W^2,f,x) - f(x)|\, W^2(x)\,dx \le 2\int_{-\infty}^{\infty} |P(x) - f(x)|\, W^2(x)\,dx,$$

for any polynomial P. A classical theorem of M. Riesz [5, p 73, Theorem 3.3] asserts (in a somewhat stronger "one-sided" form) that this last right-hand side can be made arbitrarily small. Although stated there for functions of polynomial growth at infinity, by considering "truncations" of f to finite intervals, we see that this persists for the present f. Thus we have (2.15). Then (7.10) yields (2.16). ∎

PROOF OF THEOREM 2.4. Since

$$\int_{-\infty}^{\infty} |H_n^*(W^2,f,\{d_{jn}\},x) - f(x)|\, W^2(x)\,dx \le \int_{-\infty}^{\infty} |H_n(W^2,f,x) - f(x)|\, W^2(x)\,dx + \int_{-\infty}^{\infty} \Big|\sum_{j=1}^{n} d_{jn}(x - x_{jn})\ell_{jn}^2(x)\Big|\, W^2(x)\,dx,$$

it suffices to show that the second integral, which we denote by Δ_n, approaches 0 as $n \to \infty$. To do this, we use Lemma 7.2. By that lemma,

$$\Delta_n \le C a_n^{3/2}(\log n)^{1/2} n^{-1} \chi_n = o(\chi_n),$$

where by (7.5) and (2.20),

$$\chi_n := \max_{1\le j\le n} |d_{jn}|\, W^2(x_{jn})\big[1 + |Q'(x_{jn})|\big]^{\alpha} = O(1).$$

Hence

$$\lim_{n\to\infty} \Delta_n = 0.$$

Then (2.21) follows, and (2.23) follows directly. ∎

PROOF OF COROLLARY 2.5. Since

$$I_n[k;f] - I[k;f] = \int_{-\infty}^{\infty} \{H_n(W^2,f,x) - f(x)\}\, k(x)\,dx,$$

and

$$J_n[k;f] - I[k;f] = \int_{-\infty}^{\infty} \{Y_n(W^2,f,x) - f(x)\}\, k(x)\,dx,$$

we obtain the result directly from Theorem 2.3. ∎

PROOF OF COROLLARY 2.7. It can be shown as in[14] that the corresponding modified Gauss rule $J_n^*[W^2;f]$ converges to $I[W^2;f]$ as $n \longrightarrow \infty$. Then using (1.24), one proceeds as in [26] to complete the proof. ∎

REFERENCES

1. A. Alaylioglu and D. S. Lubinsky, *A Product Quadrature Algorithm by Hermite Interpolation*, J. Comp. Appl. Math., **17**(1987), 237–269.

2. W. C. Bauldry, *Estimates of Asymmetric Polynomials*, J. Approx. Theory, **63**(1990), 225–237.

3. S. S. Bonan and D. S. Clark, *Estimates of the Hermite and Freud Polynomials*, J. Approx. Theory **63**(1990), 210–224.

4. J. Clunie and T. Kovari, *On Integral Functions Having Prescribed Asymptotic Growth II*, Can. J. Math. **20**(1968), 7–20.

5. G. Freud, *Orthogonal Polynomials*, Pergamon Press/Akademiai Kiado, Budapest, 1970.

6. ———, *On Markov-Bernstein Type Inequalities and Their Applications*, J. Approx. Theory **19**(1977), 22–37.

7. G. Grünwald, *On the Theory of Interpolation*, Acta Math. **75**(1942), 219–245.

8. A. Knopfmacher and D. S. Lubinsky, *Mean Convergence of Lagrange Interpolation for Freud's Weights with Application to Product Integration Rules*, J. Comp. Appl. Math. **17**(1987), 79–103.

9. A. L. Levin and D. S. Lubinsky, *Canonical Products and the Weights* $\exp(-|x|^\alpha)$, $\alpha > 1$, *with Applications*, J. Approx. Theory **49**(1987), 170–195.

10. D. S. Lubinsky, *Strong Asymptotics for Extremal Errors and Polynomials Associated with Erdős-Type Weights*, Pitman Research Notes in Math. **202**, Longmans, Harlow, (1989).

11. D. S. Lubinsky and T. Z. Mthembu, L_p *Markov-Bernstein Inequalities for Erdös Weights*, J. Approx. Theory **65**(1991), 301–321.

12. D. S. Lubinsky and P. Rabinowitz, *Rates of Convergence of Guassian Quadrature for Singular Integrands*, Math. Comp. **43**(1984), 219–242.

13. D. S. Lubinsky and E. B. Saff, *Strong Asymptotics for Extremal Errors and Extremal Polynomials Associated with Weights on* $(-\infty, \infty)$, Springer Lecture Notes in Math. **1305**, Springer, Berlin, (1988).

14. D. S. Lubinsky and A. Sidi, *Convergence of Product Integration Rules for Functions with Interior and Endpoint Singularities over Bounded and Unbounded Intervals*, Math. Comp. **46**(1986), 297–313.

15. H. N. Mhaskar, *Bounds for Certain Freud-Type Orthogonal Polynomials*, J. Approx. Theory **63**(1990), 238–254.

16. H. N. Mhaskar and E. B. Saff, *Where Does the Sup-Norm of a Weighted Polynomial Live?*, Constr. Approx. **1**(1985), 71–91.

17. ———, *Where Does the* L_p*-Norm of a Weighted Polynomial Live?*, Trans. Amer. Math. Soc. **303**(1987), 109–124. (Errata: **308**(1988), 431).

18. H. N. Mhaskar and Y. Xu, *Mean Convergence of Expansions in Freud-Type Orthogonal Polynomials*, SIAM J. Math. Anal. **22**(1991), 847–855.

19. P. Nevai, *Orthogonal Polynomials*, Memoirs of the Amer. Math. Soc. **213**(1979).

20. ———, *Geza Freud: Orthogonal Polynomials and Christoffel Functions, A Case Study*, J. Approx. Theory **48**(1986), 3–167.

21. P. Nevai and P. Vértesi, *Mean Convergence of Hermite-Fejér Interpolation*, J. Math. Anal. Appl. **105**(1985), 26–58.

22. ———, *Convergence of Hermite-Fejér Interpolation at Zeros of Generalized Jacobi Polynomials*, Acta Sci. Math. Szeged **53**(1989), 77–98.

23. P. Rabinowitz, *Ignoring the Singularity in Numerical Integration*, in Topics in Numerical Analysis III (J. J. A. Miller, ed.), Academic Press, London, 1977, 361–368.

24. ———, *Numerical Integration in the Presence of an Interior Singularity*, J. Comp. Appl. Math. **17**(1987), 31–41.

25. ———, *Product Integration Based on Hermite-Fejér Interpolation*, J. Comp. Appl. Math. **28**(1989), 85–101.

26. ———, *Product Integration of Singular Integrands using Hermite-Fejér Interpolation*, to appear in BIT.

27. P. Rabinowitz and P. Vértesi, *Hermite-Fejér-Related Interpolation and Product Integration*, in preparation.

28. W. E. Smith, I. H. Sloan and A. H. Opie, *Product Integration over Infinite Intervals I. Rules Based on the Zeros of Hermite Polynomials*, Math. Comp. **40**(1983), 519–536.

29. G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Pub. **23** Amer. Math. Soc., Providence, R.I., 1939, 4th Edn., 1975.

Department of Mathematics
University of the Witwatersrand
P.O. Wits 2050
Republic of South Africa

Department of Applied Mathematics and Computer Science
The Weizmann Institute of Science
Rehovot 76100
Israel