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TWO CONDITIONS FOR SUBNORMALITY OF UNBOUNDED OPERATORS

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Abstract. We give two new sufficient conditions for unbounded Hilbert space operators to be subnormal. The first assumes that the sequence $\|T^n f\|^2$ on a suitable subset of the domain is completely monotonic, the second is similar to the one given by Lambert in [**3**] for bounded operators and involves the sequence of binomial expansion of the real part of the operator.

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Suppose that T is a closed, densely defined operator in a Hilbert space $\mathcal{H}$. T is said to be *subnormal* if there are another Hilbert space $\mathcal{K} \supset \mathcal{H}$ (isometrically) and a normal operator N in it such that $\mathcal{D}(S) \subset \mathcal{H} \cap \mathcal{D}(N)$ and $Sf = Nf$ for $f \in \mathcal{D}(S)$.

Unbounded subnormals play an important role in modern quantum physics. The most famous example of such an operator is the *creation operator* defined by

$$\frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right).$$

The theory of unbounded subnormal operators has been extensively studied in the last two decades and in particular in the series of papers [**4**],[**5**],[**6**]. The unbounded case is, as usual, more complicated than that of bounded operators. Except for shifts the known conditions for subnormality like those of Halmos and Bram are not sufficient in themselves (there are known counterexamples). One needs to add some other requirements, usually density of some class of $\mathcal{C}^\infty$-vectors. This can also be done by relating the subnormality to the problem of moments (as in the recent papers [**7**] and [**5**]); these two problems are closely related. The solutions of one give solutions for the other.

Usually such additional conditions happen not to be necessary conditions. There are subnormal operators not fulfilling them; examples can be found in [**5**]. This justifies the continuous effort to search for suitable requirements. We shall follow that line of procedure in this paper.

We now continue with some notation commonly used when dealing with unbounded subnormal operators.

A subset $\mathcal{E}$ is a *core* for a closed operator A if and only if A is equal to the closure of its restriction to $\mathcal{E}$; that is $A = (A|_{\mathcal{E}})^-$.

By $\mathcal{D}^\infty(T)$ we understand the intersection of domains of all powers of T. The vectors from $\mathcal{D}^\infty(T)$ are customarily called the $\mathcal{C}^\infty$-*vectors* for T.

Among the $\mathcal{C}^\infty$-vectors are distinguished other subclasses of *bounded, analytic* and *quasi-analytic vectors*.

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A *bounded vector* for T is a vector $f \in \mathcal{D}^\infty(T)$ for which there are $c, M \geq 0$ such that $\|T^n f\| \leq cM^n$ for any $n = 0, 1, 2 \ldots$. Denote the set of all bounded vectors for T by $\mathcal{B}(T)$.

A vector $f \in \mathcal{D}^\infty(T)$ is said to be *analytic* for T if there is a $t > 0$ such that

$$\sum_{n=1}^{\infty} \frac{\|S^n f\|}{n!} t^n < +\infty.$$

We denote the set of analytic vectors for T by $\mathcal{A}(T)$.

The set of *quasi-analytic vectors* for T, denoted by $\mathcal{Q}(T)$, is the set

$$\mathcal{Q}(T) = \left\{ f \in \mathcal{D}^\infty(T) : \sum_{n=1}^{\infty} \|T^n f\|^{-1/n} = +\infty \right\}.$$

Any bounded vector for T is obviously analytic for T and any analytic vector can be shown to be quasi-analytic.

In what follows we will need two more, less common conditions on $\mathcal{C}^\infty$-vectors. The first, stating the existence of a representing measure for a certain moment sequence, appeared in [**5**]. The other one is a requirement that this sequence is completely monotonic.

We say that *T fulfils the (S) condition at f* if and only if $f \in \mathcal{D}^\infty(T)$ and there exists a finite non-negative Radon measure μ on $[0, \infty)$ such that

$$\|T^n f\|^2 = \int_0^\infty t^n \mu(dt) \text{ for any } n \geq 0.$$

Now define a *cm-vector* for T as a vector f in $\mathcal{D}^\infty(T)$ such that there is a constant $a_f > 0$ which for any $n, m \geq 0$ gives

$$\sum_{k=0}^{n} (-1)^k a_f^k \binom{n}{k} \|T^{k+m} f\|^2 \geq 0.$$

The various subclasses of $\mathcal{C}^\infty$-vectors play a special role in sufficiency conditions for an unbounded operator to be subnormal. Usually such conditions take the form of some positive-definiteness requirement, analogous to the bounded case, plus a requirement for a certain subclass of $\mathcal{C}^\infty$-vectors to be dense in the Hilbert space $\mathcal{H}$. Our theorems will follow this pattern.

In what follows we shall demand that a certain set is a core for the considered operator. Such requirement is necessary as the regarded conditions make sense only on a subspace of $\mathcal{C}^\infty$-vectors, and thus any result obtained is valid only for the closure of the part of our operator acting on this subspace. When the subspace is a core, the closure equals the original operator, as required.

THEOREM 1. *Let T be a closed Hilbert space operator. Suppose that there exists a dense linear subspace $\mathcal{E}$ that is a core for T and consists of cm-vectors for T. Then T is subnormal.*

Proof. Take any *cm*-vector f for T. Then from the defining condition on f it easily follows that the sequence $\{a_f^n \|T^n f\|^2\}_{n=0}^\infty$ is completely monotonic.

(Recall that a sequence $\{c_n\}_{n=0}^{\infty}$ is called *completely monotonic* if and only if $c_n \geq 0$ and

$$(-1)^k \Delta^k c_n \geq 0 \text{ for any } n, k = 0, 1, 2, \ldots,$$

where

$$\Delta^k c_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} c_{n+k-m}.$$

See the definitions in [**8**].)

We use [**8**, Theorem 4a, p. 108] to infer that our completely monotonic sequence constitutes a moment sequence on the interval [0, 1]. By changing variable we can get rid of a_f and obtain a nonnegative regular Radon measure μ_f, with compact support (equal to $[0, a_f^{-1}]$), such that for any $n \geq 0$ there is

$$\|T^n f\|^2 = \int_0^\infty t^n \mu_f(dt);$$

i.e. T fulfils the (S) condition at f.

The set $\mathcal{E}$ we assumed to be a dense linear subspace such that any $f \in \mathcal{E}$ is a *cm*-vector for T. Each $f \in \mathcal{E}$ is also a bounded vector. Indeed $\|T^n f\| \leq cM^n$, where M is such that the support of μ_f is contained in $[0, M]$ and $c = \mu_f([0, M])$. The definition of a *cm*-vector implies that $\mathcal{E}$ is invariant for T (take $m+1$ instead of m in the definition of $\mathcal{E}$ to get the required inequality for Tf). The operator $S = T|_{\mathcal{E}}$ has an invariant domain consisting of analytic (even bounded) vectors, each of them fulfilling the (S) condition. Theorem 7 from [**5**] states that this is equivalent to S being subnormal. As the set $\mathcal{E}$ is a core for T the latter is subnormal too. □

Theorem 2. *Let T be a closed densely defined Hilbert space operator. Suppose that the linear span (denote it by $\mathcal{E}$) of the set $\mathcal{Q}(T)$ of quasi-analytic vectors for T is a core for T and that*

$$\sum_{k=0}^{n} \langle T^k f, T^{n-k} f \rangle \geq 0 \textit{ for any } x \in \mathcal{E}, n \geq 0.$$

Then T is subnormal.

This theorem is an analogue of the one in the paper of Lambert [**3**] given for the bounded case. Thanks to the characterisation of subnormals by the condition (S) from [**6**] the proof is shorter, although follows similar lines to the original. We needed an additional assumption about the set $\mathcal{E}$ to deal with the unbounded case and thus have lost the necessity part of the theorem.

Proof. The condition put on T gives, for $n = 1$

$$\langle Tx, x \rangle + \langle x, Tx \rangle \geq 0 \text{ for any } x \in \mathcal{E}.$$

As T is a closure of T restricted to $\mathcal{E}$ the inequality above can be extended onto all $x \in \mathcal{D}(T)$. This means that $-T$ is dissipative. We assumed that it has a total set of quasi-analytic vectors and so, by the result of Hasegawa from [**1**], we infer that $-T$ is a generator of some strongly continuous contraction semigroup, denoted by $\mathcal{G}(t)$.

The set $\mathcal{E}$ is invariant under each $\mathcal{G}(t)$. Indeed the $\mathcal{G}(t)$ are contractive and commute with T. $\|T^n\mathcal{G}(t)f\| = \|\mathcal{G}(t)T^nf\| \le \|T^nf\|$ so that

$$\sum_{n=1}^{\infty} \|T^n\mathcal{G}(t)f\|^{-1/n} \ge \sum_{n=1}^{\infty} \|T^nf\|^{-1/n} = +\infty.$$

Thus, the quasi-analytic vectors for T are preserved by the $\mathcal{G}(t)$.

For $x \in \mathcal{H}$ define $f_x(t) = \|\mathcal{G}(t)x\|^2$. Then for any $x \in \mathcal{E}$, $n \ge 0$ and $0 \le t < \infty$ we have

$$\frac{d^n}{dt^n} f_x(t) = (-1)^n \sum_{k=0}^{n} \langle T^k\mathcal{G}(t)x, T^{n-k}\mathcal{G}(t)x\rangle.$$

Taking into account the condition we put on T, we see that for each f_x, $n \ge 0$ and $0 \le t < \infty$ we have

$$(-1)^n \frac{d^n}{dt^n} f_x(t) \ge 0.$$

This is the defining condition for f_x to be a *completely monotonic* function on $[0, \infty)$, in the sense of Widder. (See [**8**, p. 145].) From the Bernstein theorem (see [**8**, p. 160]) it follows that for any $x \in \mathcal{E}$ there exists a positive measure $\widetilde{\mu_x}$ on $[0, \infty)$ such that

$$f_x(t) = \int_0^{\infty} e^{-ts}\widetilde{\mu_x}(ds).$$

Take now nt instead of t and fix it. Substitute the measure $\widetilde{\mu_x}$ by $\mu_{x,t}$ replacing $u = e^{-ts}$ in the integral above. As $\mathcal{G}(nt) = \mathcal{G}(t)^n$ we get, for any x in $\mathcal{E}$, $u \in [0, \infty)$ and $n \ge 0$, the following representation:

$$\|\mathcal{G}(t)^n x\|^2 = \int_0^1 u^n \mu_{x,t}(du),$$

with positive Radon measures $\mu_{x,t}$ on $[0, 1]$. Once more, we use Theorem 7 of [**5**]. We can do that as $\mathcal{G}(t)$, being bounded, has the set of bounded vectors equal to $\mathcal{H}$, and the set $\mathcal{E}$ is invariant under $\mathcal{G}(t)$. By the theorem mentioned $\mathcal{G}(t)|_{\mathcal{E}}$ (and then also $\mathcal{G}(t)$) is subnormal, for any $t \in [0, \infty)$.

By the work of Itô [**2**] the semigroup $\mathcal{G}(t)$ has a normal semigroup extension, say $\mathcal{N}(t)$. Let N be the generator of $\mathcal{N}(t)$. It is normal and extends $-T$, the generator of $\mathcal{G}(t)$. Thus $-T$ is subnormal, which concludes the proof. □

In the third paper of a series on subnormal operators [**6**] the authors introduced the notion of *minimal normal extension* for unbounded subnormal operators. The concept seems to be well understood in general for bounded operators. As it happens, not everything is that obvious in the case of unbounded operators. There exist two types of minimal normal extensions: of *spectral type* and of *cyclic type*. In both cases the space $\mathcal{K}$ in which the normal extension acts is somehow generated by the original space and the subnormal operator S. The first type involves the spectral measure of the normal extension, the second powers of its adjoint taken on vectors from the original space.

From the two the cyclic type is more important, as then we have the uniqueness of the minimal normal extension, as one would expect. (There can exist several minimal normal extensions of the spectral type, not unitarily equivalent.) Moreover, each minimal extension of cyclic type is also of spectral type. On the other hand, the spectral type minimal extensions always exist, while the cyclic type ones need not. Hence the positive answer to the question of existence of the latter is of some importance.

The definitions we speak of are as follows.

N is a *minimal normal extension of spectral type* of the operator S if and only if N is a normal extension of S and the only closed subspace containing $\mathcal{H}$ and reducing N is the whole space $\mathcal{K}$, in which acts N.

N is a *minimal normal extension of cyclic type* of the operator S (we assume here that S has an invariant domain) if and only if the linear span (in the space $\mathcal{K}$ in which acts N) of the set

$$\{N^{*n}f : f \in \mathcal{D}(S), n \geq 0\}$$

is the core for N.

It is easily seen that the normal extension, which exists by our theorems, happens to be *minimal of cyclic type*.

Corollary 3. *The subnormal operators considered in Theorem 1 and Theorem 2 have minimal normal extensions of cyclic type.*

Proof. Luckily, Theorem 4 of [**6**] is applicable in our case. It says that a formally subnormal operator (and the subnormal operator is formally subnormal) with invariant domain has a minimal normal extension of cyclic type if its domain is linearly spanned by the quasi-analytic vectors.

The set $\mathcal{E}$ appearing in both our theorems was invariant under the operator T considered and was linearly spanned by quasi-analytic (even bounded in the first theorem) vectors for T. We apply the cited theorem to $T|_{\mathcal{E}}$ and notice that any normal extension of $T|_{\mathcal{E}}$ is necessarily an extension of T. The set $\mathcal{E}$ is assumed to be a core for T. □

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