Preface

This essay is a work of historical fiction – the "What if Eleanor Roosevelt could fly?" kind.[1] The Borel conjecture is a central problem in topology: it asserts the topological rigidity of aspherical manifolds (definitions below!). Borel made his conjecture in a letter to Serre some 65 years ago,[2] after learning of some work of Mostow on the rigidity of solvmanifolds.

We shall reimagine Borel's conjecture as being made after Mostow had proved the more famous rigidity theorem that bears his name – the rigidity of hyperbolic manifolds of dimension at least three – as the geometric rigidity of hyperbolic manifolds is stronger than what is true of solvmanifolds, and the geometric picture is clearer.

I will consider various related problems in a completely ahistorical order. My motive in all this is to highlight and explain various ideas, especially recurring ideas, that illuminate our (or at least my own) current understanding of this area.

Based on the analogy between geometry and topology imagined by Borel, one can make many other conjectures: variations on Borel's theme. Many, but perhaps not all, of these variants are false and one cannot blame them on Borel. (On several occasions he described feeling lucky that he ducked the bullet and had not conjectured smooth rigidity – a phenomenon indistinguishable to the mathematics of the time from the statement that he *did* conjecture.)

However, even the false variants are false for good reasons, and studying these can be quite fun (and edifying); all of the problems we consider enrich our understanding of the geometric and analytic properties of manifolds. *Verum ex erroris.*

1 See *Saturday Night Live*, season 4, episode 4.
2 May 2, 1953.

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The tale I shall tell moves between topology and geometry, Lie groups, arithmetic, and operator theory, algebraic K-theory, and topics in Banach space geometry that are also of interest in theoretical computer science. The goal is to develop an appreciation for this landscape – not to explain the most recent or important results on the conjecture itself.[3]

The extent of the canvas that forms the natural backdrop to this problem is both a joy and a challenge. I cannot explain all the detail or even sketch all action going on about this canvas, but I will try to tell some good stories[4] – simplifying enough to explain the key ideas, and providing references as best I can to papers that have the missing parts, trying to do a bit more than that when the results have not appeared elsewhere, but hopefully not overdoing it[5] and making anything unnecessarily complicated. The goal is to give a feeling for what we understand rather than to give the most precise or complete statements – a moving target that, even if hit at the moment of writing, quickly turns into a miss.

While there is some overlap between this book and various other surveys, almost always their treatments are superior. In particular, I recommend the varied surveys (Farrell and Jones, 1991b; Ferry *et al.*, 1995; Gromov, 1996; Farrell, 2002; Farrell *et al.*, 2002; Valette, 2002; Roe, 2003; Higson and Guentner, 2004; Kreck and Lück, 2005; Lück,in preparation). My hope is that the current treatment will at the very least be useful to my own students as a response to their FAQs and that the brevity of the discussion will be stimulating to some.

The astute reader should be able to figure out what's in this book from its table of contents, and the knowledgeable reader will be able to figure out what's missing.

This book grew out of two lecture series given in 2013, the "Frontiers of Mathematics" lectures at Texas A&M University, and a mini-course two weeks later at "Noncommutative Geometry and Operator Algebras XIII" at Vanderbilt University, followed by another lecture series in Bloomington in 2014. It probably had its genesis in a lecture series I gave in memory of Borel at ETH Zürich in 2005, although much of the material presented here reflects developments that have occurred since then. I reworked the exposition somewhat in the succeeding years, and finally gave up at the point when I felt that my edits were ruining whatever sense of freshness and excitement the original showed. Given the choice between two evils, I chose the one that involved less work for me.

[3] Although the book would feel incomplete without some discussion of this.
[4] More O'Henry than Homer.
[5] I told myself that I didn't want this to be more than 250 pages long!

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