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EPIMORPHISMS FROM $S(X)$ ONTO $S(Y)$

K. D. MAGILL, JR., P. R. MISRA AND U. B. TEWARI

1. Introduction. In this paper, the expression *topological space* will always mean *generated space*, that is any T_1 space X for which

$$\{f^{-1}(x):x \in X, f \text{ a continuous selfmap of } X\}$$

forms a subbasis for the closed subsets of X. This is not at all a severe restriction since generated spaces include all completely regular Hausdorff spaces which contain an arc as well as all 0-dimensional Hausdorff spaces [**3**, pp. 198-201], [**4**].

The symbol $S(X)$ denotes the semigroup, under composition, of all continuous selfmaps of the topological space X. This paper really grew out of our efforts to determine all those congruences σ on $S(X)$ such that $S(X)/\sigma$ is isomorphic to $S(Y)$ for some space Y. Such a congruence will be referred to as a *congeneric congruence*. We have been able to determine all congeneric congruences on $S(X)$ for a large number of spaces X but in the course of doing so, the emphasis naturally took a somewhat different direction. The two main results of the paper are Theorems A and B in Section 2. Between them they describe completely how to obtain for a great many spaces X and any space Y, all epimorphisms from $S(X)$ onto $S(Y)$. Section 3 is devoted to various consequences of the main results and this includes determining all congeneric congruences on $S(X)$. We will see that in a large number of instances there are exactly three. For any space X, we always have the two trivial congeneric congruences δ and ν on $S(X)$ which are defined by

$$\delta = \{(f, g) \in S(X) \times S(X):f = g\}$$

and

$$\nu = S(X) \times S(X).$$

Of course, $S(X)/\delta$ is isomorphic to $S(X)$ while $S(X)/\nu$ is isomorphic to $S(Y)$ where Y is the one point space. We will see that for many spaces X, the one remaining congeneric congruence on $S(X)$ is the congruence γ consisting of all pairs $(f, g) \in S(X) \times S(X)$ such that any time one of them carries a component A of X into a component B of X, then the other function does the same. Section 4 consists of a discussion of a type of congruence whose definition is motivated by the congruence γ and, finally, some concluding remarks.

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2. Homomorphisms from $S(X)$ onto $S(Y)$. In any topological space X, the symbol $\mathscr{C}_X$ will denote the collection of all components of X and when no confusion can occur we will write more simply $\mathscr{C}$ in place of $\mathscr{C}_X$. The component of a point x in X will be denoted by C_x.

Definition (2.1). A topological space X is *admissible* if it satisfies the following conditions

(2.1.1) X is completely regular and Hausdorff.

(2.1.2) The arc components of X coincide with the components of X.

(2.1.3) Suppose $\mathscr{A} \subset \mathscr{C}$, $\cup\mathscr{A}$ is open and $x \in \cup\mathscr{A}$. Then there exists $\mathscr{B} \subset \mathscr{A}$ such that $x \in \cup\mathscr{B}$ and $\cup\mathscr{B}$ is clopen (i.e., both closed and open).

(2.1.4) X contains a subset H such that $H \cap C$ is a singleton for each $C \in \mathscr{C}$ and for each open subset V of X, $\cup\{C_a:a \in V \cap H\}$ is also open.

It is evident that both (2.1.3) and (2.1.4) are both satisfied by any locally connected space (in fact, any space in which components are open). Examples which are not locally connected but are nevertheless admissible are abundant. The following example not only fails at being locally connected, but no component is open. Specifically, let $X = Q \times I$ where Q denotes the rationals and $I = [0, 1]$. It is a straight forward matter to show that X is admissible. For the set H in (2.1.4) one may choose any number $b \in I$ and take

$$H = \{(r, b):r \in Q\}.$$

Before we state and prove our two main results, it will be convenient to verify a sequence of lemmas. The first of these is somewhat different than those that follow in that it does not involve a homomorphism from one function semigroup onto another. The symbol $X/\mathscr{C}$ will denote the quotient space obtained from X by identifying each component of X to a point.

LEMMA (2.2). *Let X be an admissible space. Then $X/\mathscr{C}$ is a* 0*-dimensional Hausdorff space. Furthermore, let H be the set described in* (2.1.4) *and for each $C \in \mathscr{C}$, let $t(C)$ be the unique point in $H \cap C$. Then t is a continuous map from $X/\mathscr{C}$ into X and has the property that $t(C) \in C$ for each $C \in \mathscr{C}$.*

Proof. $X/\mathscr{C}$ is T_1 since components are closed and (2.1.3) guarantees that it is 0-dimensional (i.e., has a basis of clopen sets). Thus, X must actually be Hausdorff. Continuity of t is an immediate consequence of (2.1.4).

In the remaining lemmas of this section (that is, Lemmas (2.3) to (2.9) inclusive) the following assumptions will be made without explicit mention:

(H-1) X is an admissible space and Y is any space with more than one point.

(H-2) φ is an epimorphism from $S(X)$ onto $S(Y)$ which is not an isomorphism.

(H-3) There exists a function h from X into Y and a function k from Y into X such that

$$\varphi(f) = h \circ f \circ k \quad \text{for each } f \in S(X).$$

We will see later that (H-3) is, in fact, a consequence of (H-2) but we don't need this for Lemmas (2.3) to (2.9) which are concerned, for the most part, with properties that any pair of functions must have if they induce an epimorphism.

LEMMA (2.3). *h is surjective, k is injective and $h \circ k$ is the identity on Y.*

Proof. Let any $y \in Y$ be given and let $\langle y \rangle$ denote the constant function which maps everything to y. We will use this notation throughout the paper. We then have $\varphi(f) = \langle y \rangle$ for some $f \in S(X)$ which implies

$$y = \langle y \rangle(y) = \varphi(f)(y) = h(f(k(y))).$$

Thus, h is surjective. Since φ must carry the identity of $S(X)$ to the identity of $S(Y)$, it follows that $h \circ k$ is the identity on Y which, in turn, implies that k is injective.

LEMMA (2.4). *x and $k(h(x))$ belong to the same component of X for each $x \in X$.*

Proof. Suppose, to the contrary, that x and $k(h(x))$ belong to different components. Then

$$k(h(x)) \in \cup \mathscr{A} \quad \text{where } \mathscr{A} = \{C_y : y \neq x\}.$$

$\cup \mathscr{A} = X - C_x$ is open so that according to condition (2.1.3) there exists $\mathscr{B} \subset \mathscr{A}$ such that $k(h(x)) \in \cup \mathscr{B} \subset \cup \mathscr{A}$ and $\cup \mathscr{B}$ is clopen. Since Y has more than one point and h maps X onto Y, we can choose points p and q in X such that $h(p) \neq h(q)$. Define a continuous selfmap f of X by

$$f(z) = p \text{ for } z \in \cup \mathscr{B}$$

$$f(z) = q \text{ for } z \in X - \cup \mathscr{B}.$$

Then

$$h \circ f \circ k \circ h \circ \langle x \rangle \circ k = \langle h(p) \rangle$$

while

$$h \circ f \circ \langle x \rangle \circ k = \langle h(q) \rangle.$$

In other words, $\varphi(f) \circ \varphi\langle x \rangle \neq \varphi(f \circ \langle x \rangle)$ which contradicts the fact that φ is a homomorphism.

LEMMA (2.5). *There exists a nondegenerate component C of X such that $k \circ h$ is not the identity map on C.*

Proof. Deny the assertion. Then $k \circ h$ is the identity on all nondegenerate components and it follows from Lemma (2.4) that $k \circ h$ is also the identity on all degenerate components. Thus, $k \circ h$ is the identity on all of X. We already have from Lemma (2.3) that $h \circ k$ is the identity on Y. Consequently, both h and k are bijections and $k = h^{-1}$. This implies φ is bijective which is a contradiction.

LEMMA (2.6). *The function h is constant on each component of X.*

Proof. Suppose B is a component of X such that h is not constant on B. Choose $a, b \in B$ so that $h(a) \neq h(b)$ and choose a point p in the component C of the previous lemma so that $k \circ h(p) \neq p$. By (2.1.2), B is arcwise connected so that a and b are endpoints of some arc $A \subset B$. Since X is completely regular, it is immediate that there exists a continuous function f from X into A such that

$$f(k \circ h(p)) = a \quad \text{and} \quad f(p) = b.$$

It then follows that

$$h \circ f \circ k \circ h \circ \langle p \rangle \circ k = \langle h(a) \rangle$$

while

$$h \circ f \circ \langle p \rangle \circ k = \langle h(b) \rangle.$$

Thus, we have arrived at the contradiction

$$\varphi(f) \circ \varphi\langle p \rangle \neq \varphi(f \circ \langle p \rangle).$$

LEMMA (2.7). *$h(a) = h(b)$ if and only if a and b lie in the same component of X.*

Proof. Sufficiency is just the previous lemma. Conversely, suppose $h(a) = h(b)$. By Lemma (2.4), a and $k(h(a))$ belong to the same component as do b and $k(h(b))$. But $k(h(a)) = k(h(b))$ since $h(a) = h(b)$ so that a and b must belong to the same component.

LEMMA (2.8). *The function h is continuous.*

Proof. Let y be any point in Y, let g be any function in $S(Y)$ and let f be any function in $S(X)$ such that $\varphi(f) = g$. We assert that

(2.8.1) $\quad h^{-1}[g^{-1}(y)] = f^{-1}[C_{k(y)}].$

Let x be any point in $h^{-1}[g^{-1}(y)]$. Then

(2.8.2) $\quad h \circ f \circ k \circ h(x) = g(h(x)) = y$

which implies

(2.8.3) $\quad k \circ h \circ f \circ k \circ h(x) = k(y).$

To conclude that $x \in f^{-1}[C_{k(y)}]$, we must show that $f(x)$ and $k(y)$ belong to the same component in X. To begin with, Lemma (2.4) tells us that

(2.8.4) $\quad k \circ h(x)$ and x belong to the same component.

Continuity of f alone allows us to conclude

(2.8.5) $\quad f \circ k \circ h(x)$ and $f(x)$ belong to the same component.

Again we apply Lemma (2.4) to get the fact that

(2.8.6) $\quad k \circ h \circ f \circ k \circ h(x)$ and $f \circ k \circ h(x)$ belong to the same component.

It now follows from (2.8.3) and (2.8.5) that $f(x)$ and $k(y)$ belong to the same component.

Now suppose $x \in f^{-1}[C_{k(y)}]$. Then $f(x)$ and $k(y)$ must belong to the same component and Lemmas (2.3) and (2.6) together allow us to conclude that

(2.8.7) $\quad h \circ f(x) = h \circ k(y) = y.$

The continuity of f, in conjunction with Lemma (2.4), tells us as in (2.8.5) that $f \circ k \circ h(x)$ and $f(x)$ belong to the same component so that another application of Lemma (2.6) yields

(2.8.8) $\quad h \circ f \circ k \circ h(x) = h \circ f(x).$

But $g = \varphi(f) = h \circ f \circ k$ so that (2.8.7) and (2.8.8) together imply

$$x \in h^{-1}[g^{-1}(y)].$$

This verifies (2.8.1). To complete the proof of the lemma, recall our blanket assumption that spaces are all generated. Thus, $g^{-1}(y)$ is a typical subbasic closed set of Y. Since f is continuous and components are closed, we see from (2.8.1) that $h^{-1}[g^{-1}(y)]$ is closed. Thus, h is indeed continuous.

One might think it tempting at this point to try to prove that the function k is also continuous but this cannot be done. One can produce counter-examples. We say more about this later. The trick is to replace k by another function which is not only continuous but, in fact, is a homeomorphism. For this, we need the next, and final, lemma of this section but before we state it, we need to introduce some notation. We use our mapping h to define a mapping $\hat{h}$ from $X/\mathscr{C}$ into Y. Specifically, for any $C \in \mathscr{C}$, we define

(2.8.9) $\quad \hat{h}(C) = h(a)$ where a is any element of C.

In view of Lemma (2.6), the definition of $\hat{h}(C)$ does not depend on the

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point a in C so that $\hat{h}$ is indeed a function. For the function $\hat{h}$, we have the following

LEMMA (2.9). *The function $\hat{h}$ is a homeomorphism from $X/\mathscr{C}$ onto Y.*

Proof. It follows from Lemmas (2.3) and (2.7) that $\hat{h}$ is both injective and surjective. The continuity of $\hat{h}$ follows from that of h and it remains for us to verify that $\hat{h}^{-1}$ is continuous. Take any $f \in S(X/\mathscr{C})$ and define

(2.9.1) $\quad g = h \circ t \circ f \circ \pi \circ k$

where π is the canonical map from X onto $X/\mathscr{C}$ and t is the mapping of Lemma (2.2). We then take any $x \in X$ and verify that

(2.9.2) $\quad (\hat{h}^{-1})^{-1}[f^{-1}(C_x)] = g^{-1}(h(x))$.

Let

$$y \in (\hat{h}^{-1})^{-1}[f^{-1}(C_x)].$$

By Lemma (2.3), we have

$$\hat{h}(C_{k(y)}) = h(k((y)) = y$$

which means

$$\hat{h}^{-1}(y) = C_{k(y)}.$$

Since $\hat{h}^{-1}(y) \in f^{-1}(C_x)$, this implies

(2.9.3) $\quad f(C_{k(y)}) = C_x$.

From this, we get

(2.9.4) $\quad t \circ f \circ \pi \circ k(y) = t(C_x)$.

By Lemma (2.2),

$$t \circ f \circ \pi \circ k(y) \in C_x$$

and it follows from Lemma (2.7) that

(2.9.5) $\quad h \circ t \circ f \circ \pi \circ k(y) = h(x)$.

This, together with (2.9.1) implies $y \in g^{-1}(h(x))$ and one inclusion of (2.9.2) has been verified. To verify the other, let $y \in g^{-1}(h(x))$. From (2.9.1), we get

(2.9.6) $\quad h \circ t \circ f \circ \pi \circ k(y) = h(x)$.

This, together with Lemma (2.7), implies

(2.9.7) $\quad t \circ f \circ \pi \circ k(y) \in C_x$

and we then appeal to Lemma (2.2) to get

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(2.9.8) $\quad f \circ \pi \circ k(y) = C_x.$

But this can be rewritten as $f(C_{k(y)}) = C_x$ and since, as we noticed previously, $\hat{h}^{-1}(y) = C_{k(y)}$ we have

$$f(\hat{h}^{-1}(y)) = C_x.$$

Thus,

$$y \in (\hat{h}^{-1})^{-1}[f^{-1}(C_x)]$$

and (2.9.2) has now been verified.

Next, we note that $t \circ f \circ \pi$ is a continuous selfmap of X and thus g must be a continuous selfmap of Y since

$$\varphi(t \circ f \circ \pi) = g.$$

This means that $g^{-1}(h(x))$ is a closed subset of Y and the continuity of $\hat{h}^{-1}$ now follows from this, statement (2.9.2) and the fact $X/\mathscr{C}$ is a 0-dimensional Hausdorff space (Lemma (2.2)) and is therefore generated [**4**], [**3**, p. 200].

We are now in a position to state and prove the two main results. We state both before completing any proofs. We emphasize that in the first of the two theorems, the only additional requirement on the epimorphism is that it not be an isomorphism.

THEOREM A. *Let X be an admissible space, let Y be any space with more than one point and let φ be a noninjective epimorphism from $S(X)$ onto $S(Y)$. Then there exists a continuous function h from X onto Y and a homeomorphism k from Y into X such that the following conditions are satisfied*:

(A-1) $\varphi(f) = h \circ f \circ k$ *for each* $f \in S(X)$.

(A-2) $h \circ k$ *is the identity mapping on* Y.

(A-3) x *and* $k(h(x))$ *lie in the same component of* X *for each* $x \in X$.

(A-4) $h(a) = h(b)$ *if and only if* a *and* b *both belong to the same component of* X. *In particular*, h *is constant on components of* X.

Theorem A essentially states that the existence of an epimorphism implies the existence of two functions which satisfy a number of conditions. The question now is, "Does the existence of two functions satisfying those conditions imply the existence of an epimorphism?" The answer is yes. In fact, the conditions can be stated in such a manner that they give the appearance of being somewhat less stringent than those listed in Theorem A. This is the content of

THEOREM B. *Let X be an admissible space, let Y be any space and let h and k be continuous maps from X into Y and Y into X respectively which satisfy the following conditions*:

(B-1) *$h \circ k$ is the identity mapping on Y.*

(B-2) *x and $k(h(x))$ lie in the same component of x for each $x \in X$.*

(B-3) *h is constant on components of X. Then the mapping φ defined by $\varphi(f) = h \circ f \circ k$ for each $f \in S(X)$ is an epimorphism from $S(X)$ onto $S(Y)$.*

Proof of Theorem A. Theorem (2.2) of [**1**] guarantees the existence of two functions h and w from X into Y and Y into X respectively such that

(A-6) $\quad \varphi(f) = h \circ f \circ w$ for each $f \in S(X)$.

Lemmas (2.3) to (2.9) inclusive now apply to the functions h and w. In particular, Lemma (2.9) tells us that $\hat{h}$ is a homeomorphism from $X/\mathscr{C}$ onto Y. Let t be the function in Lemma (2.2) and define a function k by

(A-7) $\quad k = t \circ \hat{h}^{-1}$.

The function k is certainly continuous and we assert that

(A-8) $\quad \varphi(f) = h \circ f \circ k \quad$ for each $f \in S(X)$.

For any $x \in X$, Lemma (2.2) assures us that $t(C_x) \in C_x$ so that $f \circ t(C_x)$ and $f(x)$ must belong to the same component for any $f \in S(X)$. Since h is constant on components, we then have

(A-9) $\quad h \circ f \circ t(C_x) = h \circ f(x)$.

But we also have

$$\begin{aligned} h \circ f \circ k \circ h(x) &= h \circ f \circ t \circ \hat{h}^{-1}(h(x)) \\ &= h \circ f \circ t(C_x) \end{aligned}$$

and this with (A-9) implies

(A-10) $\quad h \circ f \circ k \circ h(x) = h \circ f(x)$.

Now x and $w \circ h(x)$ belong to the same component by Lemma (2.4) and hence both $f \circ w \circ h(x)$ and $f(x)$ must also belong to the same component. Again, we appeal to the fact that h is constant on components to conclude that

(A-11) $\quad h \circ f \circ w \circ h(x) = h \circ f(x)$.

From (A-10) and (A-11) we have

(A-12) $\quad h \circ f \circ k \circ h(x) = h \circ f \circ w \circ h(x)$.

Since the map h is surjective, it follows that

$$h \circ f \circ k(y) = h \circ f \circ w(y) \quad \text{for each } y \in Y$$

and this together with (A-6) implies (A-8).

Now (A-8) together with the previous lemmas imply the validity of all the conclusions of Theorem A with the exception that k is a homeomorphism. But we know that $h \circ k$ is the identity on Y and this implies that $k \circ h$ is the identity on $k[Y]$. Since both h and k are continuous, it follows that k is a homeomorphism from Y into X.

Proof of Theorem B. This is really accomplished quite easily. We first show that

(B-4) $\quad \varphi(f \circ g) = \varphi(f) \circ \varphi(g) \quad$ for all $f, g \in S(X)$.

Let any $y \in Y$ be given. By (B-2), $g \circ k(y)$ and $k \circ h \circ g \circ k(y)$ lie in the same component of X. Consequently, $f \circ g \circ k(y)$ and $f \circ k \circ h \circ g \circ k(y)$ also lie in the same component. Condition (B-3) then implies

(B-5) $\quad h \circ f \circ g \circ k(y) = h \circ f \circ k \circ h \circ g \circ k(y)$

which means (B-4) holds. For any $g \in S(Y)$,

$$k \circ g \circ h \in S(X)$$

and (B-1) implies

$$\varphi(k \circ g \circ h) = g.$$

Thus, φ is an epimorphism from $S(X)$ onto $S(Y)$.

3. Applications of theorems A **and** B. Our first application is to determine all congeneric congruences on $S(X)$ where X is any admissible space. This is really the problem that motivated this paper. Recall from the introduction that a congruence σ on $S(X)$ is congeneric if $S(X)/\sigma$ is isomorphic to $S(Y)$ for some space Y. Recall also the three congruences δ, ν and γ where δ is the identity congruence, ν is the universal congruence and γ consists of all pairs (f, g) with the property that if one of the functions carries a component A of X into a component B of X then the other must also carry A into B. Let us now consider the map ψ from $S(X)$ to $S(X/\mathscr{C})$ which is defined by

$$(\psi(f))(C_x) = C_{f(x)}$$

for each component $C_x \in X/\mathscr{C}$. The map $\psi(f)$ does indeed belong to $S(X/\mathscr{C})$ since

$$(\psi(f)) \circ \pi = \pi \circ f.$$

It is easily seen that ψ is a homomorphism. To see that it is surjective, take any $g \in S(X/\mathscr{C})$. Then

$$t \circ g \circ \pi \in S(X)$$

where t is the map of Lemma (2.2) and it readily follows that

$$\psi(t \circ g \circ \pi) = g.$$

Consequently, ψ is an epimorphism from $S(X)$ onto $S(X/\mathscr{C})$. It is immediate that the congruence induced by ψ is none other than γ. Thus, $S(X)/\gamma$ is isomorphic to $S(X/\mathscr{C})$ which, according to Lemma (2.2), is a 0-dimensional Hausdorff space. Since such spaces are generated [**4**], [**3**, p. 200] it follows that γ is a congeneric congruence on $S(X)$ when X is admissible. Of course, it is immediate that both δ and ν are congeneric congruences. Our next result says that these exhaust the possibilities.

THEOREM (3.1). *Let X be admissible. Then the only congeneric congruences on $S(X)$ are δ, ν and γ.*

Proof. Let β be a congeneric congruence on $S(X)$ which is neither δ nor ν. Then there exists a space Y and an isomorphism θ from $S(X)/\beta$ onto $S(Y)$. Let $\hat{\beta}$ be the canonical homomorphism from $S(X)$ onto $S(X)/\beta$. Then $\theta \circ \hat{\beta}$ is an epimorphism from $S(X)$ onto $S(Y)$. Moreover, $\theta \circ \hat{\beta}$ is not an isomorphism since $\beta \neq \delta$ and Y must have more than one point since $\beta \neq \nu$. Thus, according to Theorem A there exists a continuous map h from X onto Y and a homeomorphism k from Y into X such that

(3.1.1) $\quad \theta \circ \hat{\beta}(f) = h \circ f \circ k \quad$ for each $f \in S(X)$

and the maps h and k satisfy conditions (A-2) to (A-4) inclusive. Now suppose $(f, g) \in \beta$ and $f[A] \subset B$ where A and B are two components of X. We want to show that $g[A] \subset B$ also. Since $(f, g) \in \beta$, we have $\hat{\beta}(f) = \hat{\beta}(g)$ which implies

(3.1.2) $\quad h \circ f \circ k = h \circ g \circ k.$

Take any $a \in A$. Then $k(h(a)) \in A$ by (A-3) and (3.1.2) implies

(3.1.3) $\quad h \circ f \circ k \circ h(a) = h \circ g \circ k \circ h(a).$

Then $f \circ k \circ h(a)$ and $g \circ k \circ h(a)$ belong to the same component by (A-4). But $f \circ k \circ h(a) \in B$ and it follows that g maps A into B. This shows that $\beta \subset \gamma$.

Now suppose $(f, g) \in \gamma$. Then if one of the functions carries a component A of X into a component B, the other function must do the same. Since h is constant on components, it readily follows that $h \circ f \circ k = h \circ g \circ k$. Thus,

$$\theta \circ \hat{\beta}(f) = \theta \circ \hat{\beta}(g)$$

which implies $\hat{\beta}(f) = \hat{\beta}(g)$ since θ is injective. This means $(f, g) \in \beta$ and we have verified that $\beta = \gamma$. This completes the proof.

Our next result, although not specifically stated, has essentially been proved in the previous section.

THEOREM (3.2). *Let X be an admissible space and let Y be a space with more than one point which is not homeomorphic to X. Then the following statements are equivalent*:

(3.2.1) *$S(Y)$ is a homomorphic image of $S(X)$.*

(3.2.2) *Y is homeomorphic to $X/\mathscr{C}$.*

(3.2.3) *There exists a continuous function h from X onto Y and a homeomorphism k from Y into X such that the following conditions are satisfied*:

(i) *$h \circ k$ is the identity mapping on Y*

(ii) *x and $k(h(x))$ lie in the same component of X for each $x \in X$*

(iii) *h is constant on components of X.*

Proof. Assume (3.2.1) and let φ be an epimorphism from $S(X)$ onto $S(Y)$. Since X is not homeomorphic to Y, Theorem (2.3) of [**2**, p. 198] assures us that φ cannot be an isomorphism. Thus Theorem A applies and we have (3.2.3). Of course (3.2.3) implies (3.2.1) because of Theorem B. Statement (3.2.1) implies (3.2.2) because of Lemma (2.9). To complete the proof we must show that (3.2.2) implies either (3.2.1) or (3.2.3). The latter can be shown by defining $h = v \circ \pi$ and $k = t \circ v^{-1}$ where π is the canonical map from X onto $X/\mathscr{C}$, v is any homeomorphism from $X/\mathscr{C}$ onto Y and t is the mapping of Lemma (2.2). However, we have also essentially shown that (3.2.2) implies (3.2.1) for in the first paragraph of this section we constructed an epimorphism from $S(X)$ onto $S(X/\mathscr{C})$ and when $X/\mathscr{C}$ and Y are homeomorphic, $S(X/\mathscr{C})$ and $S(Y)$ must be isomorphic.

The next result is an immediate consequence of the previous theorem.

COROLLARY (3.3). *Let X be an admissible space and let Y be any space whatsoever. Then $S(Y)$ is a homomorphic image of $S(X)$ if and only if Y is homeomorphic to either X, $X/\mathscr{C}$ or the one-point space.*

Recall that a semigroup is *Hopfian* if every epimorphism of the semigroup onto itself is an automorphism.

THEOREM (3.4). *Let X be any admissible space. Then $S(X)$ is Hopfian.*

Proof. The conclusion is immediate if X consists of only one point so we assume otherwise. Let φ be an epimorphism of $S(X)$ onto itself and suppose that it is not an automorphism. It then follows from Lemma (2.9) that X and $X/\mathscr{C}$ are homeomorphic. But $X/\mathscr{C}$ is a 0-dimensional Hausdorff space by Lemma (2.2) and we now have a contradiction since Theorem (3.6) and Corollary (4.2) of [**1**] assure us that every epimorphism of $S(X)$ must be an automorphism.

THEOREM (3.5). *Let X be an admissible space and let Y be any space such that $S(Y)$ is a homomorphic image of $S(X)$. Then there exists an isomorphism from $S(Y)$ into $S(X)$.*

Proof. The conclusion is immediate if Y has only one point so assume otherwise and let φ be an epimorphism from $S(X)$ onto $S(Y)$. If φ happens to be an isomorphism then the conclusion follows so we need only consider the case where φ is not an isomorphism. Then according to Theorem A there exists a continuous map h from X onto Y and a homeomorphism k from Y into X such that (A-1) to (A-4) inclusive are satisfied. Define a map ψ from $S(Y)$ into $S(X)$ by

(3.5.1) $\quad \psi(f) = k \circ f \circ \mathrm{h} \quad$ for each $f \in S(Y)$.

ψ is a homomorphism because of (A-2) and it is injective because k is injective and h is surjective.

The symbol $\mathcal{T}_Y$ denotes the full transformation semigroup on the set Y.

THEOREM (3.6). *Let X be admissible. Then there exists a set Y with more than one point such that $\mathcal{J}y$ is a homomorphic image of $S(X)$ if and only if X is not connected and all of its components are open.*

Proof. Suppose first that X is not connected and all of its components are open. Then $X/\mathcal{C}$ is discrete and consists of more than one point. If X is homeomorphic to $X/\mathcal{C}$, then there is an isomorphism from $S(X)$ onto $S(X/\mathcal{C})$. If X is not homeomorphic to $X/\mathcal{C}$, then Theorem (3.2) applies and there exists a homomorphism from $S(X)$ onto $S(X/\mathcal{C})$. Since $X/\mathcal{C}$ is discrete, $S(X/\mathcal{C})$ is the full transformation semigroup on $X/\mathcal{C}$ and the conclusion follows.

Now suppose card $Y > 1$ and $\mathcal{T}_Y$ is a homomorphic image of $S(X)$. We regard Y as a discrete topological space and we then have a homomorphism $S(X)$ onto $S(Y)$. If X is homeomorphic to Y then it is immediate that it is not connected and its components are open. If X is not homeomorphic to Y then Theorem (3.2) applies once again and (3.2.2) tells us that $X/\mathcal{C}$ is discrete and consists of more than one point. Hence, in this case also, X is not connected and its components are open.

The situation for locally connected spaces is particularly simple and we state several results for such spaces.

THEOREM (3.7). *Let X be a locally connected completely regular Hausdorff space whose components and arc components coincide. Then there exists a set Y with more than one point such that $\mathcal{T}_Y$ is a homomorphic image of $S(X)$ if and only if X is not connected.*

Proof. Since X is locally connected, all of its components are open so that (2.1.3) and (2.1.4) are trivially satisfied. Consequently, X is admissible and the proof now follows from Theorem (3.6).

THEOREM (3.8). *Let X be a locally connected completely regular Hausdorff space whose components and arc components coincide and let Y be any space which is not discrete. Suppose there exists an epimorphism φ from $S(X)$ onto $S(Y)$. Then φ must necessarily be an isomorphism and X and Y are homeomorphic.*

Proof. As in the previous proof, X is admissible. If φ is not an isomorphism then Theorem (3.2) applies and Y is homeomorphic to $X/\mathscr{C}$ by (3.2.2). But $X/\mathscr{C}$ is discrete since X is locally connected. Thus, we have a contradiction so we conclude that φ must be an isomorphism. It follows from Theorem (2.3) of [**2**, p. 198] that X and Y are homeomorphic.

4. Some concluding remarks. We first expand on a remark made at the conclusion of Lemma (2.8). Most of the lemmas in Section 2 deal with two functions h and k which induce an epimorphism from $S(X)$ onto $S(Y)$. The function h must be continuous but, as we mentioned after the proof of Lemma (2.8), the function k need not be continuous. Of course, what we did in order to prove Theorem A was to replace k (we called it w in the proof of Theorem A) with a function we knew to be continuous (in fact, it turned out to be a homeomorphism) and we chose the function so that together with h, it induced the same epimorphism as did k. Examples in which k is not continuous are abundant. We give one. Let

$$Y = \{1/n\}_{n=1}^{\infty} \cup \{0\},$$

let $I = [0, 1]$ and let $X = Y \times I$. Define a mapping h from X onto Y by $h(a, b) = a$ and a mapping k from Y into X by $k(a) = (a, 0)$ for $a \neq 0$ and $k(0) = (0, 1/2)$. Then k is not continuous but nevertheless the mapping φ defined by

$$\varphi(f) = h \circ f \circ k$$

is an epimorphism from $S(X)$ onto $S(Y)$.

We next say a few words about a type of congruence on $S(X)$ whose definition was motivated by the congeneric congruence γ. Let X be any topological space and let $\mathscr{A}$ be any nonempty collection of nonempty subsets of X which satisfy the following two conditions:

(1) For any $f \in S(X)$ and any $A \in \mathscr{A}$ there exists a $B \in \mathscr{A}$ such that $f[A] \subset B$.

(2) For any $f \in S(X)$, $A \in \mathscr{A}$, $B \in \mathscr{A}$ and nonempty $E \subset A$, $f[E] \subset B$ implies $f[A] \subset B$.

One can associate with $\mathscr{A}$ a congruence $\sigma(\mathscr{A})$ on $S(X)$ by declaring two functions to be equivalent if whenever one of the functions carries A into $B (A, B \in \mathscr{A})$ then the other does also. It follows easily from (1) and (2) that $\sigma(\mathscr{A})$ is a congruence. If $\mathscr{A}$ is taken to be all components of X then $\sigma(\mathscr{A})$ is just the congruence γ (which, it should be mentioned, may not be a congeneric congruence if X is not admissible). Other examples of families $\mathscr{A}$ satisfying (1) and (2) are the collection of all arc components and the collection of all singletons. In the latter case, $\sigma(\mathscr{A})$ is just the identity congruence. That is, $(f, g) \in \sigma(\mathscr{A})$ if and only if $f = g$. One can obtain the universal congruence by taking $\mathscr{A} = \{X\}$. It may well turn out to be an interesting project to investigate families satisfying conditions (1) and (2) and the congruences arising therefrom.

In closing, we make a few remarks: About one possible generalization of Theorem A. We recall from [**2**, p. 146] that an I-subsemigroup of a semigroup T is any subsemigroup of the form

$$T_v = \{a \in T: av = va = a\}$$

where v is any idempotent of T. Of course, T_v coincides with T whenever v is the identity of T. Now suppose X is admissible and we have a noninjective homomorphism φ from $S(X)$ onto a I-subsemigroup $S(Y)_v$ of $S(Y)$. Let V denote the range of the idempotent v. It is a straight forward matter to check that the map α defined by $\alpha(f) = f/V$ is an isomorphism from $S(Y)_v$ onto $S(V)$ and that $\alpha^{-1}(g) = g \circ v$ for each $g \in S(V)$. Thus $\alpha \circ \varphi$ is a noninjective homomorphism from $S(X)$ onto $S(V)$ and thus, Theorem A applies. It follows that there exists a continuous function h from X onto V and a homeomorphism k from V into X such that (A-2), (A-3) and (A-4) are satisfied. Moreover, from (A-1), we get

$$(\alpha \circ \varphi)(f) = h \circ f \circ k$$

which implies that

$$\varphi(f) = \alpha^{-1}(h \circ f \circ k) = h \circ f \circ k \circ v$$

for each $f \in S(X)$. Thus, we see that Theorem A generalizes easily to a statement about homomorphisms onto I-subsemigroups.

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Added in proof. M. C. Thorton obtained results analogous to some of the results in this paper for a completely different class of spaces than those considered here. (*Semigroups of isotone selfmaps on partially ordered sets*, J. London Math. Soc. (2) *14* (1976), 545-553.)

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SUNY at Buffalo, Buffalo, New York;
College of Staten Island, Staten Island, New York;
IIT-Kanpur, Kanpur, India