Proceedings of the Edinburgh Mathematical Society (1999) **42**, 559–574 ©

FAITHFUL, IRREDUCIBLE $*$-REPRESENTATIONS FOR GROUP ALGEBRAS OF FREE PRODUCTS

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(Received 23rd October 1997)

Let G be the free product of groups A and B, where $|A| \geq 3$ and $|B| \geq 2$. We construct faithful, irreducible $*$-representations for the group algebras $\mathbb{C}[G]$ and $\ell^1(G)$. The construction gives a faithful, irreducible representation for $\mathbf{F}[G]$ when the field $\mathbf{F}$ does not have characteristic 2.

1991 *Mathematics subject classification:* 22D10, 22D20.

1. Introduction

Let G be the free product of groups A and B, where $|A| \geq 3$ and $|B| \geq 2$. Let $\mathbb{F}$ be any field and let $\mathbb{C}$ be the complex field. Denote by $\mathbb{F}[G]$ the group algebra over $\mathbb{F}$ consisting of the finite sums of the form $\sum_1^n \alpha_i g_i$ ($n \in \mathbb{N}, \alpha_i \in \mathbb{F}, g_i \in G$), and denote by $\ell^1(G)$ the group algebra over $\mathbb{C}$ consisting of the infinite sums of the form $\sum_1^\infty \alpha_i g_i$ ($\alpha_i \in \mathbb{C}, g_i \in G, \sum_1^\infty |\alpha_i| < \infty$).

The primitivity of $\mathbb{F}[G]$ was established by Formanek in **[1]** using comaximal ideals. In certain special cases, faithful, irreducible representations have been constructed: in **[5]** for $\mathbb{C}[G]$ and $\ell^1(G)$ where G is the free group on a countable set with at least two elements; and by Irving in **[2]** and **[3]** for $\mathbb{F}[G]$ in the cases where A is infinite and residually $< |B|$, and where A and B are finite with B cyclic. A related topic was dealt with by Paschke and Salinas in **[6]** where it was shown that the C^*-algebra $C_r^*(G)$ generated by the regular representation of G is simple and therefore primitive.

Here we construct faithful, irreducible $*$-representations π of $\mathbb{C}[G]$ and $\ell^1(G)$ on inner product spaces L. In particular, this guarantees the primitivity of $\ell^1(G)$ which was not covered in **[1]**. The method also gives explicit faithful, irreducible representations of $\mathbb{F}[G]$ when $\mathbb{F}$ is any field not of characteristic 2.

Our construction technique has two variations which we refer to as the *character method* and the *identification method*. The essential difference lies in the way they yield irreducibility. We use the first in Sections 2 and 3 to deal with $\mathbb{C}[G]$ in two cases when $d^2 = 1$ for all $d \in B$. Then B is abelian and so admits a non-trivial homomorphism into $\mathbb{C}$, i.e. a character. The second method, which is applied in Sections 4 and 5 to the two remaining $\mathbb{C}[G]$ cases, involves identifying certain pairs of basis vectors from the earlier cases. The constructions for $\ell^1(G)$ are essentially the same as those for $\mathbb{C}[G]$ but the proofs require modification. These are dealt with in Section 6.

For $g = d_1 d_2 \dots d_m \in G\backslash\{1\}$, where $d_i \in A\backslash\{1\}$ (i odd) and $d_i \in B\backslash\{1\}$ (i even), or *vice versa*, define

$$\beta(g) = d_1, \quad \varepsilon(g) = d_m, \quad \lambda(g) = m, \quad \beta(1) = \varepsilon(1) = 1, \quad \lambda(1) = 0.$$

We call $d_1 d_2 \dots d_m$ the *reduced form* of g. A subset H of G, containing 1, is *right closed* if *right segments* of $g \in H$ are also in H, i.e. $g \in H$ with reduced form $d_1 \dots d_m$ implies $d_k \dots d_m \in H$ ($k = 1, \dots, m$).

In each case we construct a subset W of G which acts as an orthonormal basis for L. So expressions of the form $\sum \alpha_i v_i$ where $\alpha_i \in \mathbb{C}$ and $v_i \in G$ may represent vectors in L or elements of the algebra $\mathbb{C}[G]$ or $\ell^1(G)$. For $x = \sum \alpha_i w_i$ and $y = \sum \beta_i w_i$ in L, where $\alpha_i, \beta_i \in \mathbb{C}$ and w_i are distinct in W, the inner product on L is given by

$$\langle x, y \rangle = \sum \alpha_i \overline{\beta_i}.$$

For $f = \sum \alpha_i g_i$ in $\mathbb{C}[G]$ or $\ell^1(G)$, where $\alpha_i \in \mathbb{C}$ and $g_i \in G$, define

$$f^* = \sum \overline{\alpha_i} g_i^{-1}.$$

It is with respect to this involution that π is a $*$-representation, i.e. for $x, y \in L$,

$$\langle \pi(f)x, y \rangle = \langle x, \pi(f^*)y \rangle.$$

For a subset X of the vector space L, write $-X$ for $\{-x : x \in X\}$, $\pm X$ for $X \cup (-X)$, $\operatorname{lin}(X)$ for the linear span of X, and $\operatorname{supp}(X)$ for the *support* of X, i.e. the set of $w \in W$ such that, for some $x \in X$, $\langle x, w \rangle \neq 0$.

2. Character method – first case

Assume that $d^2 = 1$ for all $d \in B$ and fix $b \in B\backslash\{1\}$. Assume also that there exists $a \in A$ with $a^2 \neq 1$. Fix this a and let $c = ba$. Construct the inner product space L consisting of sums of the form $\sum_1^n \alpha_i w_i$ ($n \in \mathbb{N}, \alpha_i \in \mathbb{C}, w_i \in W$) with orthonormal basis

$$W = T \cup U_A \cup U_B \cup V_A \cup V_B \subseteq G,$$

where

$$\begin{aligned}
T &= \{g(ba^2)^{n(n-1)/2} : n \in \mathbb{N}, g \in G, \lambda(g) \leq 2n, \varepsilon(g) \in A\},\\
S &= \{g(ba^2)^{n(n-1)/2} : n \in \mathbb{N}, g \in G, \lambda(g) = 2n, \varepsilon(g) \in A, g \neq (ba^2)^n\},\\
U_B &= \{c^k v : v \in S, k \geq 0\}, \qquad U_A = aU_B,\\
V_A &= \{a'u : a' \in A\backslash\{1, a\}, u \in U_B\}, \qquad V_B = \{b'u : b' \in B\backslash\{1, b\}, u \in U_A\}.
\end{aligned}$$

Observe that $1 \in T$ (take $n = 1$ and $g = 1$). Also, $\beta(w) \in B$ for all $w \in S$, and so for all $w \in U_B$. It follows that $\beta(w) \in A$ for $w \in U_A \cup V_A$ and $\beta(w) \in B$ for $w \in V_B$. The sets T, U_A, U_B, V_A and V_B are pairwise disjoint except that $T \cap U_B = S$. Also, $U_B \backslash S = bU_A$. Since $c \in S$ (take $n = 1$ and $g = c$) it follows that $c^k \in U_B \backslash S$ for $k > 1$. We further define $U_B^* \subseteq U_B$ and $V_A^* \subseteq V_A$ by

$$U_B^* = \{c^{2^n} : n \in \mathbb{N}\} \quad \text{and} \quad V_A^* = a^2 U_B^*.$$

Lemma 2.1. *The sets W and T are right closed.*

Proof. Let $w \in W$ have reduced form $d_1 \dots d_m$. It is enough to consider the right segment $w' = d_2 \dots d_m$. This is straightforward for $w \in U_A \cup (U_B \backslash S) \cup V_A \cup V_B$ and for $w = g(ba^2)^{n(n-1)/2} \in T$ with $\lambda(g) > 0$. If $w = (ba^2)^{n(n-1)/2} = (ba^2)^{n-1}(ba^2)^{(n-1)(n-2)/2}$, where $n \geq 2$, then $w' = g'(ba^2)^{(n-1)(n-2)/2}$, where $\lambda(g') = 2n - 3$. Hence $w' \in T$. □

Lemma 2.2. *Let $w \in W$. Then*

$$\text{either } (A\backslash\{1\})w \subseteq W \text{ or } (A\backslash\{1\})w \subseteq G\backslash W, \text{ and}$$
$$\text{either } (B\backslash\{1\})w \subseteq W \text{ or } (B\backslash\{1\})w \subseteq G\backslash W.$$

Proof. Let $w = g(ba^2)^{n(n-1)/2} \in T$. Without loss suppose that $g \neq (ba^2)^n$. For $d \in A \cup B$, $|\lambda(dg) - \lambda(g)| \leq 1$, and if $\lambda(dg) = 2n + 1$ then $dw \in U_A \cup V_A$. Hence $dw \in W$ if $\lambda(g) > 0$. If $\lambda(g) = 0$ (i.e. $g = 1$) and $n > 1$ then $(A\backslash\{1\})w \subseteq T$, and writing $w = (ba^2)^{n-1}(ba^2)^{(n-1)(n-2)/2}$ we see that $(B\backslash\{1\})w \subseteq T$. If $g = 1$ and $n = 1$, then $w = 1$, so that $(A\backslash\{1\})1 = A\backslash\{1\} \subseteq T$ and $(B\backslash\{1\})1 = B\backslash\{1\} \subseteq G\backslash W$.

Let $w \in U_A \cup V_A$, so that $w = a'u$ with $a' \in A\backslash\{1\}$ and $u \in U_B$. If $d \in A$ then $dw = da'u \in U_A \cup V_A \cup U_B$ with the last case if $d^{-1} = a'$. Similarly, $dw \in W$ if $w \in (U_B\backslash S) \cup V_B$ and $d \in B$.

If $w \in U_A$ and $d \in B\backslash\{1\}$ then $dw \in U_B \cup V_B$. If $w \in U_B$ and $d \in A\backslash\{1\}$ then $dw \in U_A \cup V_A$.

Let $w \in V_A$ and $d \in B\backslash\{1\}$. So $w = a'u$ with $a' \in A\backslash\{1, a\}$ and $u \in U_B$. Now $dw = da'u \notin T$ since its right segment $a'u \in V_A$. Since $a'u \notin U_A$, $dw \notin U_B \cup V_B$. Since $\beta(dw) \in B$, $dw \notin U_A \cup V_A$. Hence $dw \notin W$.

Similarly, if $w \in V_B$ and $d \in A\backslash\{1\}$ then $dw \notin W$. □

Since $d^2 = 1$ for all $d \in B$, we can fix a homomorphism $\chi : B \to \{-1, 1\} \subseteq \mathbb{C}$ with $\chi(b) = -1$. For $d \in A \cup B$, let $\pi(d) : L \to L$ be the linear mapping defined, for $w \in W$, by

$$\pi(d)w = \begin{cases} dw & \text{if } dw \in W, \\ \chi(d)w & \text{if } w \in V_A^* \text{ and } d \in B\backslash\{1\}, \\ w & \text{in other cases when } dw \notin W. \end{cases}$$

Note that $\pi(1)$ is the identity operator on L. For $w \in V_A^*$ and $d_1, d_2 \in B$ we have

$$\pi(d_1)\pi(d_2)w = \pi(d_1)\chi(d_2)w = \chi(d_1)\chi(d_2)w = \chi(d_1d_2)w = \pi(d_1d_2)w.$$

It is straightforward to verify that $\pi(d_1)\pi(d_2) = \pi(d_1d_2)$ if d_1 and d_2 are both in A or both in B. So we may extend π to G and thence to a representation of $\mathbb{C}[G]$.

Let $w \in W$ with $w = d_m \dots d_2 d_1$ in reduced form. Since $d_1 \in W$, $\pi(d_1)1 = d_1$ and, similarly, $\pi(d_2d_1)1 = d_2d_1$. Continuing, it follows that $\pi(w)1 = w$. Also,

$$\pi(w^{-1})w = \pi(w^{-1})\pi(w)1 = \pi(1)1 = 1 \quad (w \in W). \tag{2.1}$$

From the proof of Lemma 2.2 we have, for $d \in A \cup B$ and $w \in W$,

$$\pi(d)w = \begin{cases} dw & \text{if } dw \in W, \\ w & \text{if } d \in A \text{ and } w \in V_B, \\ w & \text{if } d \in B \text{ and } w \in (V_A \backslash V_A^*) \cup \{1\}, \\ \chi(d)w & \text{if } d \in B \text{ and } w \in V_A^*, \end{cases} \tag{2.2}$$

which separates the cases when $dw \notin W$.

For $d \in A \cup B$, $\pi(d)\pi(d^{-1}) = \pi(1)$ and $\pi(d)$ acts on W as a permutation followed by scalar multiplication by ± 1. Therefore $\pi(d)$ is a unitary operator on L, $\pi(d^{-1}) = \pi(d)^*$ and π is a $*$-representation of $\mathbb{C}[G]$.

Proposition 2.3. *The representation π is faithful.*

Proof. Let $f \in \mathbb{C}[G]\backslash\{0\}$. Then $f = \sum_1^n \alpha_i g_i$ where $\alpha_i \in \mathbb{C}$ and $g_i \in G$. Choose an integer $k > \max\{\lambda(g_i) : i = 1, \dots, n\}$. Consider $w = g(ba^2)^{k(k-1)/2} \in T$ where $g \in G$ is such that $\lambda(g) = k$ and $\varepsilon(g) \in A$. For $i = 1, \dots, n$, we have $0 < \lambda(g_i g) < 2k$ and $\varepsilon(g_i g) = \varepsilon(g)$. So $g_i w \in T$ and each right segment of $g_i w$ is in T. Hence $\pi(g_i)w = g_i w$ and $\pi(f)w = fw \neq 0$ since $fww^{-1} = f \neq 0$. Thus $\pi(f) \neq 0$. □

Notes 2.4. (1) With the notation of the above proof, if g_i are distinct in G then $g_i w$ are distinct in T. This will be used in the proof of Proposition 6.1.

(2) In the proof, the restrictions imposed on A and B at the beginning of Section 2 are not invoked. Hence, the proposition holds for the cases in Sections 3, 4 and 5. The same is true of the next lemma.

Lemma 2.5. (i) *Let $w \in U_B$. Then $\pi(c)w \in U_B \backslash S$.*

(ii) *Let $w \in T$. Then $\pi(c^n)w \in U_B$ for all sufficiently large n.*

Proof. (i) This follows directly from the definitions which give $\pi(c)w = cw$.

(ii) Write $w = g(ba^2)^{m(m-1)/2}$ with $\lambda(g) \leq 2m$, $\varepsilon(g) \in A$ and $g \neq (ba^2)^m$.

(a) Let $\lambda(g) = 2m$. Then $w \in S$ and $\pi(c^n)w = c^n w \in U_B$ for all $n \in \mathbb{N}$.

(b) Let $\lambda(g)$ be even and $< 2m$. Then $\beta(g) \in B$, and $k = m - \frac{1}{2}\lambda(g)$ gives $\pi(c^k)w =$

$c^k w \in S$ and $\pi(c^n)w \in U_B$ for all $n > k$, by (a).

(c) Now suppose that $\lambda(g)$ is odd. Then $\beta(g) \in A$ and we can write $w = c^{-p}w'$ where $p \geq 0$, $\beta(w') \in A$ and $w' \neq c^{-1}g'$ with $g' \in G$ and $\beta(g') \in A$. Since each right segment of w is in T, $\pi(c^p)w = c^p w = w'$ and $w' \in T$. Hence we can reduce to the case $w \in T$, $\beta(w) \in A$ and $w \neq c^{-1}g'$ with $g' \in G$ and $\beta(g') \in A$. If $\lambda(w) \geq 3$, c does not cancel completely into w and so $\beta(cw) \in B$. Since $\lambda(cw) \leq \lambda(w) + 1$, $cw \in T$ and so $\pi(c)w = cw$. Now cases (a) and (b) give $\pi(c^n)w \in U_B$ for all sufficiently large n.

Finally, if $\lambda(w) = 1$ then $w = a' \in A\backslash\{1\}$. If $a' \neq a^{-1}$ then $\pi(a)w = aa'$ and $\pi(c)w = baa' \in T$ which gives $\pi(c^n)w \in U_B$ for all sufficiently large n, by (a) and (b). Also, $\pi(a)a^{-1} = 1$ and $\pi(c)a^{-1} = \pi(b)1 = 1$. Then $\pi(c^r)1 = c^r \in U_B$ for all $r \in \mathbb{N}$ which gives the result in the last case. □

We now define

$$V_A^0 = \{a^{-s}u : s \in \mathbb{N}, a^{-s} \notin \{1, a\}, u \in U_B\},$$
$$V_A^\infty = \{a'u : a' \in A\backslash\{a^t : -\infty < t \leq 1\}, u \in U_B\}.$$

Thus $V_A = V_A^0 \cup V_A^\infty$ and $V_A^0 \cap V_A^\infty = \emptyset$. Note that V_A^∞ may be empty.

Lemma 2.6. *For $d \in \{a, b\}$,*

$$\pi(d)V_A^\infty \subseteq \pm V_A^\infty \quad \textit{and} \quad \pi(d)V_B \subseteq V_B.$$

Proof. Consider $w \in V_A^\infty$ so that $w = a'u$ where $a' \in A\backslash\{a^t : -\infty < t \leq 1\}$ and $u \in U_B$. Then $\pi(a)w = aa'u$ and $aa' \in A\backslash\{a^t : -\infty < t \leq 1\}$. Hence $aa'u \in V_A^\infty$. Also, $\pi(b)w = \pm w \in \pm V_A^\infty$.

The case of V_B follows directly from the definitions together with the assumption that $b^2 = 1$ which guarantees that, for $b' \in B\backslash\{b\}$, $bb' \neq 1$. □

Lemma 2.7. *Let $w \in U_A \cup V_A^0$. Then $\pi(c^n)w \in \pm(U_B \cup V_A^\infty)$ for all sufficiently large n.*

Proof. (i) Assume a has finite period p. Let $w \in U_A \cup V_A^0$. Write $w = a^r u$ with $1 \leq r < p$ and $u \in U_B$. Then $\pi(c)w = \pi(b)a^{r+1}u = \pm a^{r+1}u$ if $a^{r+1} \neq 1$. Continuing, we have

$$\pi(c^{p-r})w = \pm\pi(b)a^p u = \pm\pi(b)u = \pm bu.$$

Now, if $u \in S$ then $bu \in T$ and Lemma 2.5 gives $\pi(c^n)w \in \pm U_B$ for all sufficiently large n. If $u \in U_B\backslash S$ then $u = c^k v$ with $k \in \mathbb{N}$ and $v \in S$. Hence, when $w = a^r c^k v$, we have $\pi(c^{p-r})w = \pm bu = \pm ac^{k-1}v$. Continuing, we have, for some $m \in \mathbb{N}$, that $\pi(c^m)w = \pm av$. Then the first part gives $\pi(c^n)w \in \pm U_B$ for all sufficiently large n.

(ii) Assume a has infinite period. If $w = au \in U_A$, where $u \in U_B$, then

$$\pi(c)w = \pi(b)a^2u = \pm a^2u \in \pm V_A^\infty$$

and so $\pi(c^n)w \in \pm V_A^\infty$ for all $n \in \mathbb{N}$, by Lemma 2.6. If $w = a^{-s}u \in V_A^0$, where $s \in \mathbb{N}$ and $u \in U_B$, then $\pi(c)w = \pi(b)a^{1-s}u = a^{1-s}u$ if $s > 1$. Continuing, we have $\pi(c^s)w = \pi(b)u = bu$. If $u \in S$ then $bu \in T$, and if $u \in U_B \backslash S$ then $bu \in U_A$. Hence, the above gives $\pi(c^n)w \in \pm(U_B \cup V_A^\infty)$ for all sufficiently large n. □

Proposition 2.8. *Let X be a finite subset of W and let $x_0 \in X$. Then there exists $h \in \mathbb{C}[G]$ such that $\pi(h)x_0 = 1$ and $\pi(h)x = 0$ for all $x \in X\backslash\{x_0\}$.*

Proof. Consider first the case $X \subseteq U_B \cup V_A^\infty$ and $x_0 = c^{m_1} \in X$. Write the distinct elements of $X \cap U_B$ as $c^{m_1}, \dots, c^{m_\alpha}$, $c^{n_1}v_1, \dots, c^{n_\beta}v_\beta$, where $m_i \in \mathbb{N}$, $n_j \in \mathbb{N} \cup \{0\}$ and $v_j \in S\backslash\{c\}$. Consider $X_1 = \pi(c^p)X$, where $p \in \mathbb{N}$. Since $\pi(c)V_A^\infty \subseteq \pm V_A^\infty$ and $\pi(c)U_B \subseteq U_B$, we have $X_1 \subseteq U_B \cup (\pm V_A^\infty)$ and

$$X_1 \cap U_B = \{c^{p+m_1}, \dots, c^{p+m_\alpha}, c^{p+n_1}v_1, \dots, c^{p+n_\beta}v_\beta\}.$$

For each j, $c^{p+n_j}v_j \neq c^k$ for any k and so $c^{p+n_j}v_j \notin U_B^*$. Choose p such that $p + m_1$ is the only power of 2 among the $p + m_i$. Then $X_1 \cap U_B^* = \{c^{p+m_1}\} = X_1 \cap (\pm U_B^*)$. Put $X_2 = \pi(a^2)X_1 = \pi(a^2c^p)X$. Then $X_2 \subseteq \pm V_A$ since $\pi(a^2)U_B \subseteq V_A$ and $\pi(a)V_A^\infty \subseteq V_A^\infty$. Also $x_0 = c^{m_1} \in X$ gives $a^2c^px_0 \in X_2 \cap V_A^*$. If $x \in X$ and $\pi(a^2c^p)x \in \pm V_A^*$ then $\pi(a^{-2})\pi(a^2c^p)x = \pi(c^p)x \in X_1 \cap (\pm U_B^*)$. Hence $\pi(c^p)x = c^px_0 = \pi(c^p)x_0$ which gives $x = x_0$. So, if $x \in X\backslash\{x_0\}$ then $\pi(a^2c^p)x \in \pm(V_A \backslash V_A^*)$, and hence, by (2.2), $\pi(1-b)\pi(a^2c^p)x = 0$. Also

$$\pi((1-b)a^2c^p)x_0 = \pi(1-b)a^2c^px_0 = 2a^2c^px_0 = 2a^2c^{p+m_1},$$

since $a^2c^px_0 \in V_A^*$. Hence, if we take $h = \frac{1}{2}(a^2c^{p+m_1})^{-1}(1-b)a^2c^p$, we have $\pi(h)x_0 = 1$ and $\pi(h)x = 0$ for all $x \in X\backslash\{x_0\}$.

Now consider any finite $X \subseteq W$ and any $x_0 \in X$. By replacing X with $\pi(x_0^{-1})X$ we may assume, by (2.1), that $x_0 = 1$. By Lemmas 2.5, 2.6 and 2.7 we have, for some $p \in \mathbb{N}$,

$$X_3 = \pi(c^p)X \subseteq \pm(U_B \cup V_A^\infty \cup V_B).$$

Then

$$X_4 = \pi(1-a)X_3 \subseteq \operatorname{lin}(U_A \cup U_B \cup V_A^\infty)$$

since $\pi(a)U_B = U_A$ and $\pi(1-a)V_B = \{0\}$. By Lemmas 2.6 and 2.7, for some $q \in \mathbb{N}$,

$$X_5 = \pi(c^q)X_4 \subseteq \operatorname{lin}(U_B \cup V_A^\infty).$$

So, putting $h_1 = c^q(1-a)c^p$, we have

$$X_5 = \pi(h_1)X = \{\pi(c^{q+p})x - \pi(c^q ac^p)x : x \in X\}.$$

Suppose that $c^{q+p} \in \text{supp}(X_5)$. Then, for some $x \in X$, $\pi(c^{q+p})x = \pm c^{q+p}$ or $\pi(c^q ac^p)x = \pm c^{q+p}$. In the first case, applying $\pi(c^{-q-p})$ gives $x = 1$. In the second case, applying $\pi(a^{-1}c^{-p})$ gives $\pi(c^q)x = \pm a^{-1}c^p \in X_3 \cap (\pm V_A^0)$, which is impossible. Thus $x_1 = c^{q+p}$ appears in $\text{supp}(X_5)$ only as $\pi(c^{q+p})1$. The first part now gives $h_2 \in \mathbb{C}[G]$ such that $\pi(h_2)x_1 = 1$ and $\pi(h_2)x = 0$ for all $x \in \text{supp}(X_5)\backslash\{x_1\}$. Also, $\pi(h_1)1 = x_1 \pm x_2$ with $x_2 \in \text{supp}(X_5)\backslash\{x_1\}$ and $\pi(h_1)x \in \text{lin}(\text{supp}(X_5)\backslash\{x_1\})$ if $x \in X\backslash\{1\}$. Hence $\pi(h_2h_1)1 = 1$ and $\pi(h_2h_1)x = 0$ for all $x \in X\backslash\{1\}$. So $h = h_2h_1$ has the desired effect. □

Corollary 2.9. *The representation π is irreducible.*

Proof. Let $x = \sum_1^n \alpha_i w_i \in L$ with $\alpha_i \in \mathbb{C}$, $\alpha_1 \neq 0$ and w_i distinct in W. By Proposition 2.8, there exists $h \in \mathbb{C}[G]$ such that $\pi(h)w_1 = 1$ and $\pi(h)w_i = 0$ for $i \in \{2, \dots, n\}$. Hence $\pi(h)x = \alpha_1 1$. Since $\pi(w)1 = w$ for all $w \in W$, 1 in L is cyclic. Hence x is cyclic and it follows that π is irreducible. □

3. Character method – second case

Assume that $d^2 = 1$ for all $d \in A$ and all $d \in B$. Fix $a_1, a_2 \in A\backslash\{1\}$ with $a_1 \neq a_2$, and fix $b \in B\backslash\{1\}$. Let $a_3 = a_1a_2$ and observe that if $\{i, j, k\} = \{1, 2, 3\}$ then $a_ia_j = a_k$. Modify the construction of W in Section 2 as follows. In the definitions of $T, S, U_A, U_B, V_A, V_B, U_B^*$ and V_A^*, where a^2 appears replace it with a_2, replace the other occurrences of a with a_1, and take c to be ba_1. Thus, in this case, $U_A = a_1U_B$ and $V_A^* = a_2U_B^*$. For $i = 2, 3$, let $V_A^i = a_iU_B$ and $V_A' = V_A\backslash(V_A^2 \cup V_A^3)$. Define π as in Section 2. Then, arguing as before, we have that π is a $*$-representation of $\mathbb{C}[G]$.

Proposition 3.1. *The representation π is faithful.*

Proof. Follow the proof of Proposition 2.3, replacing a^2 with a_2. □

Lemma 3.2. *Let $w \in T \cup U_A$. Then $\pi(c^n)w \in U_B$ for all sufficiently large n.*

Proof. For $w \in T$, the proof follows that of Lemma 2.5.
Let $w = a_1c^kv \in U_A$ where $k \geq 0$ and $v \in S$. If $k > 0$ then $\pi(c)w = a_1c^{k-1}v$, which gives $\pi(c^k)w = a_1v$ and $\pi(c^{k+1})w = bv \in T$. Apply the first part. □

Lemma 3.3. *For $d \in \{a_1, b\}$ and $w \in V_A$,*

$$\pi(d)V_A \subseteq \pm V_A, \quad \pi(d)V_B \subseteq V_B, \quad \pi(c^2)w = \pm w.$$

Also,

$$\pi(c)V_A^2 \subseteq V_A^3, \quad \pi(c)V_A^3 \subseteq \pm V_A^2, \quad \pi(c)V_A' \subseteq V_A'.$$

Proof. These follow directly from the definitions. □

Proposition 3.4. *Let X be a finite subset of W and let $x_0 \in X$. Then there exists $h \in \mathbb{C}[G]$ such that $\pi(h)x_0 = 1$ and $\pi(h)x = 0$ for all $x \in X \setminus \{x_0\}$.*

Proof. Without loss, take $x_0 = 1$. By Lemmas 3.2 and 3.3, for some $p \in \mathbb{N}$,

$$X_1 = \pi(c^p)X \subseteq \pm(U_B \cup V_A \cup V_B).$$

As in Proposition 2.8, choose p such that $\operatorname{supp}(X_1) \cap U_B^* = \{c^p\}$. Then

$$X_2 = \pi(a_2)X_1 \subseteq \pm(U_A \cup U_B \cup V_A^2 \cup V_A' \cup V_B).$$

In view of Lemmas 3.2 and 3.3, again as in Proposition 2.8 we can choose q sufficiently large, and here odd, such that

$$X_3 = \pi(c^q)X_2 \subseteq \pm(U_B \cup V_A^3 \cup V_A' \cup V_B)$$

and such that $\operatorname{supp}(X_3) \cap U_B^* = \emptyset$. Then

$$X_4 = \pi(a_2)X_3 \subseteq \pm(U_A \cup V_A^2 \cup V_A' \cup V_B)$$

and

$$X_5 = \pi(1-b)X_4 \subseteq \operatorname{lin}(U_A \cup U_B \cup V_B),$$

since $\operatorname{supp}(X_4) \cap V_A^* = \emptyset$. Then

$$X_6 = \pi(a_3)X_5 \subseteq \operatorname{lin}(V_A^2 \cup V_A^3 \cup V_B).$$

Suppose $n \in \mathbb{N}$ and $a_2c^n \in \operatorname{supp}(X_6)$. Then one of $\pi(a_3)a_2c^n = a_1c^n$ and $\pi(ba_3)a_2c^n = c^{n+1}$ is in $\operatorname{supp}(X_4)$. Since $\operatorname{supp}(X_4) \cap U_B = \emptyset$, we have $a_1c^n \in \operatorname{supp}(X_4)$. Hence $\pi(a_2c^{-q}a_2)a_1c^n \in \pm\operatorname{supp}(X_1)$, i.e. $c^n \in \operatorname{supp}(X_1)$. For $m \in \mathbb{N}$, if $c^m \in X_1$ then its image in X_6 is $\pm a_2c^m \mp a_3c^{m+1}$. Hence

$$c^m \in \operatorname{supp}(X_1) \Leftrightarrow a_2c^m \in \operatorname{supp}(X_6).$$

Since $\operatorname{supp}(X_1) \cap U_B^* = \{c^p\}$, we have $\operatorname{supp}(X_6) \cap V_A^* = \{a_2c^p\}$, and a_2c^p in $\operatorname{supp}(X_6)$ appears only in the image of 1 in X. Let $v = a_2c^p$. By Lemma 3.3,

$$X_7 = \pi(1-b)X_6 \subseteq \operatorname{lin}(\{v\} \cup V_B),$$
$$X_8 = \pi(1-a_1)X_7 \subseteq \operatorname{lin}(\{v - a_1v\}).$$

It is straightforward to verify that 1 in X maps to $v - a_3c^{p+1}$ in X_6, then to $2v$ in X_7 and to $2(v - a_1v)$ in X_8. Other elements of X map into $\mathrm{lin}((V_A\setminus V_A^*) \cup V_B)$ in X_6, then to 0 in X_8. We have $\pi(1-b)(v-a_1v) = 2v$ since $a_1v = a_3c^p \in V_A\setminus V_A^*$. Hence

$$h = \tfrac{1}{4}v^{-1}(1-b)(1-a_1)(1-b)a_3(1-b)a_2c^qa_2c^p$$

gives $\pi(h)1 = 1$ and $\pi(h)x = 0$ for all $x \in X\setminus\{1\}$. □

Corollary 3.5. *The representation π is irreducible.*

Proof. Follow the proof of Corollary 2.9. □

4. Identification method – first case

Assume there exists $a \in A$ with $a^2 \neq 1$ and $b \in B$ with infinite period. Fix such a and b and let $c = ba$. Define sets T, S, U_A, U_B, V_A, V_B and U_B^* exactly as in Section 2. Define also

$$V_{AB} = \{a^2u : u \in U_B\} \subseteq V_A \quad \text{and} \quad V_{BA} = \{b^2au : u \in U_B\} \subseteq V_B.$$

We identify each element of V_{AB} with an element of $\pm V_{BA}$. Specifically, for $u \in U_B$, we identify a^2u with $\lambda_u b^2au$, and write

$$a^2u \equiv \lambda_u b^2au,$$

where $\lambda_u \in \mathbb{R}$ is given by

$$\lambda_u = \begin{cases} 1 & \text{if } u \in U_B\setminus U_B^*, \\ -1 & \text{if } u \in U_B^*. \end{cases}$$

Thus, for all $u \in U_B$, $a^2u \equiv \pm b^2au$. Here we take L to be the inner product space with orthonormal basis

$$W = T \cup U_A \cup U_B \cup (V_A\setminus V_{AB}) \cup (V_B\setminus V_{BA}) \cup V_{AB}.$$

For $d \in A \cup B$, let $\pi(d) : L \to L$ be the linear mapping defined, for $w \in W\setminus V_{AB}$, by

$$\pi(d)w = \begin{cases} dw & \text{if } dw \in W, \\ w & \text{if } dw \notin W, \end{cases}$$

and, for $w = a^2u \equiv \lambda_u b^2au \in V_{AB}$ where $u \in U_B$, by

$$\pi(d)w = \begin{cases} da^2u & \text{if } d \in A, \\ \lambda_u db^2au & \text{if } d \in B. \end{cases}$$

Lemma 2.2 remains valid, so we may extend π to G and thence to a representation of $\mathbb{C}[G]$. As in Section 2, for $d \in A \cup B$, $\pi(d)$ is a unitary operator on L, and π is a $*$-representation.

Proposition 4.1. *The representation π is faithful.*

Proof. Follow the proof of Proposition 2.3. □

It follows, directly from the definitions, that

$$\pi(1-a')w = 0 \quad (a' \in A, w \in V_B \backslash V_{BA}),$$
$$\pi(1-b')w = 0 \quad (b' \in B, w \in V_A \backslash V_{AB}),$$
$$\pi(a^2 - b^2 a)w = \begin{cases} 0 & \text{if } w \in U_B \backslash U_B^*, \\ 2a^2 w & \text{if } w \in U_B^*. \end{cases} \tag{4.1}$$

Lemma 2.5 remains true since vectors in V_{AB} do not appear in the calculation, and the proof is the same.

Define subsets of V_A and V_B as follows.

$$V_A^0 = \{a^{-s}u : s \in \mathbb{N}, a^{-s} \notin \{1, a, a^2\}, u \in U_B\},$$
$$V_A^\infty = \{a'u : a' \in A \backslash \{a^s : -\infty < s \le 2\}, u \in U_B\},$$
$$V_B^0 = \{b^{-s}u : s \in \mathbb{N}, u \in U_A\},$$
$$V_B^\infty = \{b'u : b' \in B \backslash \{b^s : -\infty < s \le 2\}, u \in U_A\}.$$

These four sets are disjoint. Their union is $(V_A \backslash V_{AB}) \cup (V_B \backslash V_{BA})$. Note that V_A^0 and V_A^∞ differ slightly from the corresponding sets in Section 2.

Lemma 4.2. *For $d \in \{a, b\}$,*

$$\pi(d)V_A^\infty \subseteq V_A^\infty \quad \text{and} \quad \pi(d)V_B^\infty \subseteq V_B^\infty.$$

Proof. Consider $w = a'u \in V_A^\infty$ where $a' \in A \backslash \{a^s : -\infty < s \le 2\}$ and $u \in U_B$. Then $\pi(a)w = aa'u \in V_A^\infty$ since $aa' \in A \backslash \{a^s : -\infty < s \le 2\}$. Also, $\pi(b)w = w$. The case $w \in V_B^\infty$ is similar. □

Lemma 4.3. (i) *Let $w \in U_A \cup V_B^0$. Then $\pi(c^n)w \in \pm V_B^\infty$ for all sufficiently large n.*
(ii) *Let $w \in V_A^0 \cup V_{AB}$. Then $\pi(c^n)w \in \pm(U_B \cup V_A^\infty)$ for all sufficiently large n.*

Proof. (i) Consider $w = au \in U_A$ where $u \in U_B$. Then

https://doi.org/10.1017/S0013091500020526 Published online by Cambridge University Press

$$\pi(c)w = \pi(b)a^2u \equiv \pm\pi(b)b^2au = \pm b^3au \in \pm V_B^\infty$$

since $b^3 \in B\backslash\{b^s : s \leq 2\}$. Hence $\pi(c^n)w \in \pm V_B^\infty$ for all $n \in \mathbb{N}$.

Consider $w = b^{-s}u \in V_B^0$ where $s \in \mathbb{N}$ and $u \in U_A$. Then $\pi(a)w = w$ and $\pi(c)w = \pi(b)w = b^{1-s}u$. Continuing, we get $\pi(c^s)w = u$. By the first part, $\pi(c^n)w \in \pm V_B^\infty$ for all $n \geq s+1$.

(ii) Consider $w = a^{-s}u$ where $s \in \mathbb{N}$, $u \in U_B$ and, if a has finite period p, $s \leq p-2$. This includes any $w \in V_A^0$, together with $w \in V_{AB}$ in the case of a having finite period.

Take first the case $u \in S$. If $s > 1$ then $\pi(c)w = \pi(b)a^{1-s}u = a^{1-s}u$, and continuing we get $\pi(c^s)w = \pi(b)u = bu$, which also holds if $s = 1$. Since $bu \in T$, this gives $\pi(c^n)w \in U_B$ for all sufficiently large n.

Now suppose that $u \in U_B\backslash S$, so that $w = a^{-s}u = a^{-s}c^kv$ where $k \in \mathbb{N}$ and $v \in S$. As above,

$$\pi(c^s)w = bu = b^2ac^{k-1}v \equiv \pm a^2c^{k-1}v \in \pm V_{AB}.$$

Now, $\pi(c)a^2c^{k-1}v = \pi(b)a^3c^{k-1}v$. If a has infinite period then $\pi(b)a^3c^{k-1}v = a^3c^{k-1}v \in V_A^\infty$, and so $\pi(c^n)w \in \pm V_A^\infty$ for all $n > s$. If a has finite period then $\pi(c^s)w$ is of the same form as $\pm w$, with $k-1$ in place of k. Repetition gives some $m \in \mathbb{N}$ such that $\pi(c^m)w = \pm a^2v$. Now the first part gives $\pi(c^n)w \in \pm U_B$ for all sufficiently large n.

Finally, if a has infinite period and $w \in V_{AB}$ then $w = a^2u$ with $u \in U_B$. Hence, $\pi(c)w = a^3u \in V_A^\infty$ and $\pi(c^n)w \in V_A^\infty$ for all $n \in \mathbb{N}$. □

Proposition 4.4. *Let X be a finite subset of W and let $x_0 \in X$. Then there exists $h \in \mathbb{C}[G]$ such that $\pi(h)x_0 = 1$ and $\pi(h)x = 0$ for all $x \in X\backslash\{x_0\}$.*

Proof. Without loss, take $x_0 = 1$. By Lemmas 4.2 and 4.3, and results corresponding to Lemma 2.5, for all sufficiently large $p \in \mathbb{N}$,

$$X_1 = \pi(c^p)X \subseteq \pm(U_B \cup V_A^\infty \cup V_B^\infty).$$

The vector 1 in X has image c^p in X_1. As in Proposition 2.8, we choose p such that $\operatorname{supp}(X_1) \cap U_B^* = \{c^p\}$. Let $h_1 = \frac{1}{2}(a^2 - b^2a)c^p$. Then $\pi(h_1)1 = a^2c^p \equiv -b^2ac^p$. Consider any $x \in X\backslash\{1\}$. Since $c^p \in \operatorname{supp}(X_1)$ can arise only from $1 \in X$, we have that $\pi(c^p)x \in \pm((U_B\backslash U_B^*) \cup V_A^\infty \cup V_B^\infty)$. Since $\pi(a^2 - b^2a)w = 0$ if $w \in U_B\backslash U_B^*$, and using Lemma 4.2, we deduce that $\pi(h_1)x \in \operatorname{lin}(V_A^\infty \cup V_B^\infty)$. Let $h_2 = (1-a)(1-b)$. If $w \in V_A^\infty$ then $\pi(1-b)w = 0$. If $w \in V_B^\infty$ then $\pi(1-b)w \in \operatorname{lin}(V_B^\infty)$ and $\pi(h_2)w = 0$. Hence, $\pi(h_2)w = 0$ if $w \in V_A^\infty \cup V_B^\infty$. Also, since $a^2c^p \equiv -b^2ac^p$, $\pi(b)a^2c^p = -b^3ac^p$ and hence

$$\pi(h_2)a^2c^p = \pi(1-a)(a^2c^p + b^3ac^p) = a^2c^p - a^3c^p.$$

Let $h_3 = h_2h_1$. Then, by above, $\pi(h_3)1 = a^2c^p - a^3c^p$ and $\pi(h_3)x = 0$ for all $x \in X\backslash\{1\}$. Also, $\pi(a^{-2})(a^2c^p - a^3c^p) = c^p - ac^p$. Next,

$$\pi(c^{-1})ac^p = \pi(a^{-1}b^{-1})ac^p = \pi(a^{-1})b^{-1}ac^p = b^{-1}ac^p \in V_B^0$$

and

$$\pi(c^{-1})b^{-1}ac^p = \pi(a^{-1}b^{-1})b^{-1}ac^p = \pi(a^{-1})b^{-2}ac^p = b^{-2}ac^p.$$

Continuing, we get $\pi(c^{-p})ac^p = b^{-p}ac^p$. Since $\pi(c^{-p})c^p = 1$, we have $\pi(c^{-p})(c^p - ac^p) = 1 - b^{-p}ac^p$. Then $\pi(1-a)(1-b^{-p}ac^p) = 1 - a$, and $\pi(b)(1-a) = 1 - c$ since $\pi(b)1 = 1$. Further, since $a^2c \equiv b^2ac$ and $a^2c^2 \equiv -b^2ac^2$,

$$\pi((a^2 - b^2a)c)(1-c) = \pi(a^2 - b^2a)(c - c^2) = -2a^2c^2.$$

Let $h_4 = (a^2 - b^2a)cb(1-a)c^{-p}a^{-2}h_3$. Then $\pi(h_4)1 = -2a^2c^2$ and $\pi(h_4)x = 0$ for all $x \in X\backslash\{1\}$. Thus, if we take $h = -\frac{1}{2}(a^2c^2)^{-1}h_4$, we get $\pi(h)1 = 1$ and $\pi(h)x = 0$ for all $x \in X\backslash\{1\}$. □

Corollary 4.5. *The representation π is irreducible.*

Proof. Follow the proof of Corollary 2.9. □

5. Identification method – second case

Assume that there exist $a \in A$ and $b \in B$ with finite periods $p_a > 2$ and $p_b > 2$, respectively. Construct L and define π as in Section 4. Then, arguing as before, we have that π is a $*$-representation of $\mathbb{C}[G]$.

Here we define

$$V_A^0 = \{a^s u : 2 < s < p_a, u \in U_B\},$$
$$V_B^0 = \{b^s au : 2 < s < p_b, u \in U_B\},$$
$$V_A^\infty = \{a'u : a' \in A\backslash\{a^s : s \in \mathbb{Z}\}, u \in U_B\},$$
$$V_B^\infty = \{b'u : b' \in B\backslash\{b^s : s \in \mathbb{Z}\}, u \in U_A\},$$

so that, as in Section 4, $V_A\backslash V_{AB} = V_A^0 \cup V_A^\infty$ and $V_B\backslash V_{BA} = V_B^0 \cup V_B^\infty$. Note that Lemma 4.2 holds in the present case.

Proposition 5.1. *The representation π is faithful.*

Proof. Follow the proof of Proposition 2.3. □

Lemma 5.2. (i) *Let $w \in V_A^0 \cup V_{AB}$. Then $\pi(c^n)w \in \pm U_B$ for all sufficiently large n.*
(ii) *Let $w \in U_A \cup V_B^0$. Then $\pi(c^{2p_b - 4})w = w$.*

Proof. (i) Follow the proof of Lemma 4.3 (ii) in the finite period case, where the definitions of V_A^0 and V_A^∞ agree with those in this section.

(ii) First, consider $w = au \in U_A$ where $u \in U_B$. Then $\pi(a)w = a^2u \equiv \pm b^2au$, and so $\pi(c)w = \pm\pi(b)b^2au = \pm b^3au$, and $b^3au \in V_B^0$ if $p_b > 3$. If $2 < s < p_b$ then $\pi(c)b^sau = \pi(b)\pi(a)b^sau = \pi(b)b^sau = b^{s+1}au$. It follows that $\pi(c^{p_b-2})w = \pm b^{p_b}au = \pm au = \pm w$, and this also holds if $p_b = 3$. Hence $\pi(c^{2p_b-4})w = w$.

Secondly, if $w \in V_B^0$ so that $w = b^sau$ where $2 < s < p_b$ and $u \in U_B$, then as above, $\pi(c^{p_b-s})w = b^{p_b}au = au \in U_A$. So $\pi(c^{2p_b-4})w = \pi(c^{2p_b-4})\pi(c^{s-p_b})au = \pi(c^{s-p_b})\pi(c^{2p_b-4})au = \pi(c^{s-p_b})au = w$. □

Proposition 5.3. *Let X be a finite subset of W and let $x_0 \in X$. Then there exists $h \in \mathbb{C}[G]$ such that $\pi(h)x_0 = 1$ and $\pi(h)x = 0$ for all $x \in X\backslash\{x_0\}$.*

Proof. Without loss, take $x_0 = 1$. By Lemma 5.2, for all sufficiently large $p \in \mathbb{N}$,

$$X_1 = \pi(c^p)X \subseteq \pm(U_A \cup U_B \cup V_B^0 \cup V_A^\infty \cup V_B^\infty).$$

If we replace p by $p + k(2p_b - 4)$ where $k \in \mathbb{N}$, then each element of $\operatorname{supp}(X_1) \cap (U_A \cup V_B^0)$ is unaltered while the elements of $\operatorname{supp}(X_1) \cap (U_B \cup V_A^\infty \cup V_B^\infty)$ remain in $U_B \cup V_A^\infty \cup V_B^\infty$. Since $ac^p \in U_A$, we may choose p such that $ac^p \notin \operatorname{supp}(X_1)$. Then

$$X_2 = \pi(1-a)X_1 \subseteq \operatorname{lin}(U_A \cup U_B \cup V_A),$$

since $\pi(1-a)w = 0$ if $w \in V_B^0 \cup V_B^\infty$, and $\pi(a)w \in V_{AB} \subseteq V_A$ if $w \in U_A$. By Lemma 5.2, we may choose q to be a sufficiently large multiple of $2p_b - 4$ that

$$X_3 = \pi(c^q)X_2 \subseteq \operatorname{lin}(U_A \cup (U_B\backslash S) \cup V_A^\infty)$$

since U_B is mapped into $U_B\backslash S$ and each element of U_A is mapped to itself. Then

$$X_4 = \pi(b)X_3 \subseteq \operatorname{lin}(U_B \cup V_A)$$

since $\pi(b)(U_B\backslash S) \subseteq \pm V_{AB}$. By Lemma 5.2, we may choose $r \in \mathbb{N}$ such that

$$X_5 = \pi(c^r)X_4 \subseteq \operatorname{lin}((U_B\backslash S) \cup V_A^\infty),$$

and then

$$X_6 = \pi(1-b)X_5 \subseteq \operatorname{lin}(U_B \cup V_{AB})$$

since, again here, $\pi(b)(U_B\backslash S) \subseteq \pm V_{AB}$. For all sufficiently large $s \in \mathbb{N}$,

$$X_7 = \pi(c^s)X_6 \subseteq \operatorname{lin}(U_B).$$

Thus $X_7 = \pi(h_1)X$ where

$$h_1 = c^s(1-b)c^rbc^q(1-a)c^p.$$

Let $u = c^{p+r+s+1}$ and suppose that $u \in \operatorname{supp}(X_7)$. This arises only from $\pi(c^{-s})u = c^{p+r+1} \in \operatorname{supp}(X_6)$ which, in turn, can come from either c^{p+r+1} or $\pi(b^{-1})c^{p+r+1}$ equalling ac^{p+r} in $\operatorname{supp}(X_5)$. Since $\operatorname{supp}(X_5) \cap U_A = \emptyset$, only $c^{p+r+1} \in \operatorname{supp}(X_5)$ is possible. Then this is the image only of $\pi(c^{-r})c^{p+r+1} = c^{p+1}$ in $\operatorname{supp}(X_4)$, then of $\pi(b^{-1})c^{p+1} = ac^p$ in $\operatorname{supp}(X_3)$, and so of $\pi(c^{-q})ac^p = ac^p$ in $\operatorname{supp}(X_2)$, since $\pi(c^q)$ fixes the elements of U_A. Now $ac^p \in \operatorname{supp}(X_2)$ comes from ac^p or $\pi(a^{-1})ac^p$ equalling c^p in $\operatorname{supp}(X_1)$. Since $ac^p \notin \operatorname{supp}(X_1)$, c^p is the only precursor in X_1, and this arises only from $\pi(c^{-p})c^p = 1$ in X.

For $x \in W$, $\pi(h_1)x$ is a linear combination of four basis vectors, one of which is $\pi(h_2)x$ where $h_2 = -c^{s+r}bc^qac^p$. The above shows that if $x \in X\backslash\{1\}$ then $u \notin \operatorname{supp}(\{\pi(h_1)x\})$ while $\pi(h_2)1 = -u$, as may be easily verified, and the other vectors in $\pi(h_1)1$ involve elements of $U_B\backslash\{u\}$.

As in Proposition 2.8, choose s sufficiently large that $\operatorname{supp}(X_7) \cap U_B^* = \{u\}$. Then $\pi((a^2 - b^2a)h_1)x = 0$ if $x \in X\backslash\{1\}$, since then $\operatorname{supp}(\{\pi(h_1)x\}) \subseteq U_B\backslash U_B^*$, while $\pi((a^2 - b^2a)h_1)1 = -\pi(a^2 - b^2a)u = -2a^2u$ by (4.1). Thus, taking

$$h = -\tfrac{1}{2}(a^2u)^{-1}(a^2 - b^2a)h_1$$

gives $\pi(h)1 = 1$ and $\pi(h)x = 0$ for all $x \in X\backslash\{1\}$. □

Corollary 5.4. *The representation π is irreducible.*

Proof. Follow the proof of Corollary 2.9. □

6. Extending to $\ell^1(G)$

Here we consider the algebra $\ell^1(G)$ which consists of the sums $\sum_1^\infty \alpha_i g_i$ ($\alpha_i \in \mathbb{C}$, $g_i \in G$, $\sum_1^\infty |\alpha_i| < \infty$). The four cases for G dealt with in Sections 2 to 5 still apply but, for the most part, they can be treated together. The construction of L and the definition of π are the same as in these earlier sections except that L now consists of the sums of the form $x = \sum_1^\infty \alpha_i w_i$ ($\alpha_i \in \mathbb{C}$, $w_i \in W$, $\sum_1^\infty |\alpha_i| < \infty$) with $\|x\| = \sum_1^\infty |\alpha_i|$ when w_i are distinct.

Since, for $d \in A \cup B$, $\pi(d)$ acts on W as a permutation followed by multiplication by ± 1, it follows that

$$\|\pi(g)w\| = 1, \quad \|\pi(g)x\| = \|x\| \quad (g \in G, w \in W, x \in L). \tag{6.1}$$

Then arguing as before, we have that π is a $*$-representation of $\ell^1(G)$.

Proposition 6.1. *The representation π is faithful.*

Proof. Let $f = \sum_1^\infty \alpha_i g_i \in \ell^1(G)$ with $\alpha_i \in \mathbb{C}$, $\alpha_1 \neq 0$ and g_i distinct in G. Choose $n \in \mathbb{N}$ such that $\sum_{n+1}^\infty |\alpha_i| < |\alpha_1|$. Let w be as in the proof of Proposition 2.3 with corresponding modifications for the other cases. Then

$$\left\| \sum_{n+1}^{\infty} \alpha_i \pi(g_i) w \right\| \le \sum_{n+1}^{\infty} |\alpha_i| \, \|\pi(g_i) w\| \le \sum_{n+1}^{\infty} |\alpha_i| < |\alpha_1|$$

and, in view of Note 2.4 (1),

$$\left\| \sum_{1}^{n} \alpha_i \pi(g_i) w \right\| = \left\| \sum_{1}^{n} \alpha_i g_i w \right\| = \sum_{1}^{n} |\alpha_i| \ge |\alpha_1|.$$

Hence

$$\|\pi(f) w\| \ge \left\| \sum_{1}^{n} \alpha_i \pi(g_i) w \right\| - \left\| \sum_{n+1}^{\infty} \alpha_i \pi(g_i) w \right\| > |\alpha_1| - |\alpha_1| = 0.$$

Thus $\pi(f) \neq 0$. □

The elements h defined in the proofs of Propositions 2.8, 3.4, 4.4 and 5.3, when regarded as elements of $\ell^1(G)$, have the property that

$$\|\pi(h) w\| \le 8 \quad (w \in W). \tag{6.2}$$

[In some cases, 8 can be replaced by a smaller integer.] For example, in the case of Proposition 4.4, we have

$$\begin{aligned} h &= -\tfrac{1}{2}(a^2c^2)^{-1}(a^2 - b^2a)cb(1-a)c^{-p}a^{-2}(1-a)(1-b)\tfrac{1}{2}(a^2 - b^2a)c^p \\ &= \tfrac{1}{4}(\alpha_1 g_1 + \cdots + \alpha_{32} g_{32}), \end{aligned}$$

where $\alpha_i = \pm 1 \in \mathbb{C}$ and $g_i \in G$. Hence, using (6.1), for $w \in W$,

$$\|\pi(h) w\| \le \tfrac{1}{4}(\|\pi(g_1) w\| + \cdots + \|\pi(g_{32}) w\|) \le 8.$$

The assumption in the proof of Proposition 4.4 that $x_0 = 1$ does not affect this argument.

Proposition 6.2. *The representation π is irreducible.*

Proof. Let $x = \sum_1^\infty \alpha_i w_i \in L$ with $\alpha_i \in \mathbb{C}$, $\alpha_1 \neq 0$ and w_i distinct in W. Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $x = \sum_{n+1}^\infty |\alpha_i| < \varepsilon |\alpha_1|/8$. Let h be as in the proof of Corollary 2.9 with corresponding modifications for the other cases. Then

$$\pi(h)x = \sum_{1}^{n} \alpha_i \pi(h) w_i + \sum_{n+1}^{\infty} \alpha_i \pi(h) w_i = \alpha_1 1 + \sum_{n+1}^{\infty} \alpha_i \pi(h) w_i.$$

Hence, using (6.2),

$$\left\|\frac{1}{\alpha_1}\pi(h)x - 1\right\| = \left\|\frac{1}{\alpha_1}\sum_{n+1}^{\infty}\alpha_i\pi(h)w_i\right\| \le \frac{1}{|\alpha_1|}\sum_{n+1}^{\infty}|\alpha_i|\,\|\pi(h)w_i\| \le \frac{8}{|\alpha_1|}\sum_{n+1}^{\infty}|\alpha_i| < \varepsilon.$$

Since $\pi(w)1 = w$ for all $w \in W$, it follows that 1 in L is cyclic. This together with the above gives x topologically cyclic. Hence, since L is complete, every non-zero vector in L is cyclic (see, for example, McGregor [4, proof of Theorem 5]) and π is irreducible. □

7. Concluding remarks

(1) Combining the results of Sections 2 to 6, we have constructed faithful, irreducible $*$-representations of $\mathbb{C}[G]$ and $\ell^1(G)$ for all cases of G the free product of groups A and B with $|A| \ge 3$ and $|B| \ge 2$. The cases not dealt with explicitly all have $|B| \ge 3$ and are covered by interchanging the roles of A and B.

(2) The constructions and conclusions obtained for $\mathbb{C}[G]$ and $\ell^1(G)$ hold for $\mathbb{R}[G]$ and the corresponding ℓ^1-algebra over $\mathbb{R}$. They also hold, apart from the $*$-condition, for $\mathbb{F}[G]$ where $\mathbb{F}$ is any field not of characteristic 2.

(3) The character method can be used for some cases covered by the identification method. The essential requirement for the character method is the existence of a non-trivial homomorphism from B to $\mathbb{C}$, and this does not require $d^2 = 1$ for all $d \in B$.

(4) In all cases, the space L is a subspace of the Hilbert space $\ell^2(W)$ consisting of the sums of the form $\sum_1^\infty \alpha_i w_i$ ($\alpha_i \in \mathbb{C}$, $w_i \in W$, $\sum_1^\infty |\alpha_i|^2 < \infty$).

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