Cosmic Magnetic Fields:
From Planets, to Stars and Galaxies
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Planetary dynamos: from equipartition to asymptopia

Paul H. Roberts

IGPP University of California, Los Angeles, CA 90095, USA
email: roberts@math.ucla.edu

Abstract. This review focuses on three topics relevant to naturally-occurring dynamos. The first considers how a common belief, that states of equipartition of magnetic and kinetic energy are preferred in nonrotating systems, is modified when Coriolis forces are influential, as in the Earth's core. The second reviews current difficulties faced by planetary and stellar dynamo theories, particularly in representing the sub-grid scales. The third discusses recent attempts to extract scaling laws from numerical integrations of the Boussinesq dynamo equations.

Keywords. Convection – turbulence – magnetic fields – instabilities

1. Introduction

Stellar and planetary dynamo theory has exploded in the last 15 years, and the rate of expansion of this universe seems to be accelerating! It is obviously impossible to do more than touch on the two fields here, and my aim is modest: to draw attention to a few issues that seem (to me) to have relevance to both fields.

2. Equipartition

In applying magnetohydrodynamics (MHD) to cosmic contexts, order of magnitude arguments frequently appeal to equipartition: $M = K$, where $M = B^2/2\mu$ and $K = \rho V^2/2$ are the magnetic and kinetic energy densities, $\mathbf{B}$ being magnetic field, $\mathbf{V}$ fluid velocity, ρ mass density and μ magnetic permeability (here $4\pi \times 10^{-7}$H m^{-1}; SI units). This equality seems to be based on the idea that two of the principal nonlinearities in the equation of motion, the inertial force $\rho\mathbf{V}\cdot\nabla\mathbf{V}$ and the Lorentz force $\mathbf{J}\times\mathbf{B}$, should approximately balance, where $\mathbf{J}$, the electric current density, is given by Ampère's law, $\mu\mathbf{J} = \nabla\times\mathbf{B}$. If one now writes $|\rho\mathbf{V}\cdot\nabla\mathbf{V}| \approx \rho V^2/L$ and $|\mathbf{J}\times\mathbf{B}| \approx JB$ with $\mu J \approx B/L$, one at once finds that $M \approx K$.

Little thought is needed to identify weaknesses in this argument. Near a stellar surface where ρ is small, JB is a poor estimate of $|\mathbf{J}\times\mathbf{B}|$ because $\mathbf{J}$ and $\mathbf{B}$ tend to be parallel, so that $|\mathbf{J}\times\mathbf{B}| \ll JB$ and $K \ll M$. Also, deep in most stars, M is much less than K mainly because $\mathbf{V} \approx \mathbf{\Omega}_0\times\mathbf{r}$ where $\mathbf{\Omega}_0$ is the star's angular velocity and $\mathbf{r}$ is the radius vector from its center of mass. Then the largest part of $\mathbf{V}\cdot\nabla\mathbf{V}$ is $-\nabla\frac{1}{2}|\mathbf{\Omega}_0\times\mathbf{r}|^2$, which combines with the gradient of the gravitational potential Ψ in the approximate hydrostatic balance of the star, leaving only smaller terms. The kinetic energy density relative to $\mathcal{F}$ is $K^r = \rho U^2/2$, where $\mathbf{U} = \mathbf{V} - \mathbf{\Omega}_0\times\mathbf{r}$ is the velocity in frame $\mathcal{F}$. When $|\mathbf{U}| \ll |\mathbf{\Omega}_0\times\mathbf{r}|$, i.e., when the Rossby number $R_o = U/2\Omega_0 r_s$ is small where r_s is the radius of the body, the inertial force in $\mathcal{F}$ is the Coriolis force, $2\rho\mathbf{\Omega}_0\times\mathbf{U}$, which we assume is $O(2\Omega_0\rho U)$. This balances the Lorentz force of order $JB = O(B^2/\mu r_s)$ when $K^r/M = O(R_o) \ll 1$.

Irrespective of the size of R_o, the statement $K^r = R_o M$ can also be written as

$$E_\ell \approx R_m, \qquad \text{where} \qquad R_m = \frac{Ur_s}{\eta}, \qquad E_\ell = \frac{B^2}{2\Omega_0 \eta\mu\rho} \tag{2.1a,b,c}$$

are the magnetic Reynolds number and Elsasser number, $\eta = 1/\mu\sigma$ being magnetic diffusivity and σ electrical conductivity. The condition for dynamo action is $R_m \geqslant R_m^c$ where R_m^c, the critical or marginal magnetic Reynolds number, is O(1). Here "O(1)" hides the fact that numerically the model–dependent R_m^c is usually of order 100. Obviously (2.1a) is inappropriate at and near the marginal state, since $R_m = \mathrm{O}(1)$ implies $E_\ell = \mathrm{O}(1)$. The statement $E_\ell = \mathrm{O}(1)$ expresses a balance between the Coriolis force $2\Omega_0\rho U$ and the Lorentz force JB when Ohm's law $\mathbf{J} = \sigma(\mathbf{E}+\mathbf{U}\times\mathbf{B})$ is used to estimate J not as $B/\mu r_s$, but as σUB, where $\mathbf{E}$ is the electric field in $\mathcal{F}$. Often E_ℓ is written as $V_A^2/2\Omega_0\eta$, where $V_A = B/\sqrt{(\mu\rho)}$ is the Alfvén velocity. In the geophysical literature E_ℓ is normally denoted by Λ, but we have a different use for Λ below.

If η and ρ are known, an estimate for U also gives R_o, R_m and K^r, so that $E_\ell = \mathrm{O}(R_m)$ and/or $K^r = \mathrm{O}(R_o M)$ provide an approximate B. Its reliability can, as for all order of magnitude arguments, be questioned. The possibility that JB overestimates $|\mathbf{J}\times\mathbf{B}|$ has already been mentioned, and $UB \approx |\mathbf{U}\times\mathbf{B}|$ and $2\Omega_0 U \approx |2\boldsymbol{\Omega}_0\times\mathbf{U}|$ are open to the same criticism; similarly B/r_s is likely to underestimate $|\boldsymbol{\nabla}\times\mathbf{B}|$. But the main objection (Christensen & Aubert 2006) is a physical one, that the magnitude of B is decided not by a force balance but by power availability. Fig. 7 of Christensen & Aubert (2006) indicates that E_ℓ/R_m is not constant but increases with R_m, roughly as $R_m^{1/2}$. See also §5 below.

3. Some geomagnitudes

The sole aim of this Section is to consider §2 in relation to the Earth. Uninterested readers are advised to skip to §4.

The observed $\mathbf{B}$ at the Earth's surface is a potential field. Because the Earth's mantle is a poor electrical conductor, spherical harmonic components of $\mathbf{B}$ of harmonic number $\ell \lesssim 13$ can be extrapolated downwards to the core surface $r = r_s = 3.480\times10^6$ m. Sources of permanent magnetism in the Earth's crust prevent extrapolation of harmonics with $\ell > 13$. Extrapolation gives $B(r_s) \approx 0.39$ mT (Bloxham & Jackson 1992). As the power spectrum at $r = r_s$ is nearly flat [$P_\ell \propto \exp(-0.1\ell)$, e.g., Roberts *et al.* 2003], the missing harmonics $\ell > 13$ add significantly to the total rms B giving $B(r_s) \approx 0.46$ mT.

This is very likely to be an underestimate of B deep in the core, where $\mathbf{B}$ has a toroidal part $\mathbf{B}_T$ that does not contribute to the inferred $B(r_s)$, which is purely poloidal. Fifteen years ago, it was widely believed that the axisymmetric toroidal flow $\overline{\mathbf{U}}_T$ would dominate $\mathbf{U}$ and create from $\mathbf{B}_P$ a much larger $\mathbf{B}_T$ through the $\Omega-$effect (usually called "the $\omega-$effect" in the geophysical literature). The very first fully three-dimensional MHD dynamo simulations showed however that, while B_T tends to be larger than B_P, it is comparable in strength. It tends to be larger partly because $\overline{\mathbf{U}}_T$ contains a geostrophic flow, $\mathbf{U}_G = U_G(s,t)\widehat{\boldsymbol{\phi}}$; here t is time and (s,ϕ,z) are cylindrical coordinates with Oz parallel to $\boldsymbol{\Omega}_0$. This flow is unopposed by the Coriolis force because $2\boldsymbol{\Omega}_0\times\mathbf{U}_G = \boldsymbol{\nabla}\psi$, where $\psi = -2\Omega_0\int U_G\,ds$ can be absorbed into the gravitational potential Ψ. Because it is unopposed, it tends to be larger than the ageostrophic flow $\mathbf{U}-\mathbf{U}_G$. In simulations that have a conducting inner core, there is often a large zonal shear at the tangent cylinder (the imaginary cylinder that touches the inner core on its equator). This shear also tends to enhance the toroidal field by the $\Omega-$effect. Some notion of the overall effect of these processes can be gauged from the early simulation of Glatzmaier & Roberts (1996) for which $B(r_s)$ is about $\frac{1}{3}$ that of the Earth's, but for which the maximum B in the core is

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approximately 20 mT. If we take $B = 10$ mT as the mean, this exceeds $B(r_s)$ by a factor of over 60. If the same factor applies to the Earth. $B = 30$ mT would be a reasonable guestimate.

Torsional oscillations provide an indirect way of finding the rms strength of B_s. It may however underestimate B, because only $\mathbf{B}_P$ and the nonaxisymmetric part of $\mathbf{B}_T$ contribute to B_s; the zonal field $\overline{\mathbf{B}}_T$, which plausibly exceeds both, does not. This may be one reason why Zatman & Bloxham (1997) obtained their rather small mean value for B_s, approximately 0.4 mT. Another may be connected with the argument of Braginsky (1975) that the lines of force of $\overline{\mathbf{B}}_P$ in the core should tend to be parallel to the polar axis. The Alfvén velocity $\overline{V}_{As}$ for $B_s = 0.4$ mT is 3.6 mm s^{-1}, so that torsional waves cross the core in a time $2r_s/\overline{V}_{As}$ of about 60 years. There is evidence for a 60 year period in the geomagnetic secular variation and in the length of day, e.g., Roberts *et al.* (2007).

Traditionally, U is estimated from the speed of the westward drift of $\mathbf{B}(r_s)$. This is irregular and dependent on latitude, but has typically been taken as 0.2°yr^{-1}so that $\overline{U}_\phi(r_s) \approx 0.4$ mm s^{-1} at the core equator. More detailed analysis, described in detail by Holme (2007), gives a maximum $U(r_s)$, of order 1.2 mm s^{-1}. Of course, this tells nothing about U deeper in the core. It is also unclear how well Alfvén's frozen flux theorem applies and how closely the inferred motion of $\mathbf{B}$ betrays the magnitude of $\mathbf{U}$; conceivably it might be partly a wave motion (Braginsky 1964a; Hide 1966), as was seen clearly in the simulation of Glatzmaier & Roberts (1996). We take $U = 0.4$ mm s^{-1}. This gives $R_o = K^r/M = 7.9 \times 10^{-7}$, $K^r = 0.8$ mJ m^{-3} (taking $\rho = 10^4$ kg m^{-3}), $M = K^r/R_o = 1$ kJ m^{-3}, $R_m = E_\ell = 700$ (taking $\eta = 2$ m^{2}s^{-1}). Therefore V_A=0.45 m s^{-1} and $B = 50$ mT, which is about 50 times greater than the estimate of Christensen & Aubert (2006). Can this discrepancy be reduced? The following arguments help a little.

The Earth is cooling, currently radiating about 42TW into space. The fluid outer core (FOC) is known to be in a nearly isentropic state, implying that it is homogenized by convection that contributes $\mathcal{Q}_c$ to the outward heat flow $\mathcal{Q}(r_s)$ at the core surface. The remainder is the adiabatic heat flow of the isentropic state. The adiabatic temperature gradient $g\widetilde{\alpha}T/C_p$ is about 0.5 K km^{-1} at the core surface (from $g = 10.68$ m s^{-2} as gravitational acceleration, $\widetilde{\alpha} = 10^{-5}$ K^{-1} as thermal expansivity, $T = 4000$ K as temperature, and $C_p = 830$J kg^{-1}K^{-1} as specific heat, all at $r = r_s$). Taking the thermal conductivity as 40W m^{-1}K^{-1}, the adiabatic heat flow at the core surface is 2.8TW. Estimates of $\mathcal{Q}(r_s)$ range from 5TW to 15TW, so that 2TW $\lesssim \mathcal{Q}_c \lesssim$ 12TW.

The source of $\mathcal{Q}_c$ is partly thermal and partly gravitational. Radioactivity (^{40}K) has been estimated as providing at most 1TW. As the core cools, it becomes increasingly centrally condensed. Lighter constituents of the FOC, particularly oxygen, are preferentially rejected as the fluid freezes to form the solid inner core (SIC), These light constituents are buoyant, rising and mixing with the overlying fluid. This gravitational source of energy is much larger than the gravitational energy released in a mere contraction of the core, which is less than 1TW. Moreover it is a buoyancy source that feeds energy into the convective motions. The latent heat released in the freezing is a further (thermal) buoyancy source. Both these sources are proportional to the rate of advance, dr_i/dt, of the inner core boundary. Currently $r_i = 1.2215 \times 10^6$m and estimates of dr_i/dt are about 10^{-11}m s^{-1}, making the age of the SIC less than $\frac{1}{3}$ of the age of the Earth. Before the birth of the SIC, only primordial heat and radioactive sources (stronger then!) would provide buoyancy and one would expect the dynamo would produce a weaker field (or no field at all!). Except during brief polarity reversals, the geodipole moment has had however the same strength, within a factor of 2–3, for more than 3.4Gyr (Kono & Tanaka 1995, Fig. 6). This paradox is not faced by Christensen & Aubert (2006), since the power requirement when $B = 1$TW is so small that dr_i/dt is less than 10^{-12}m s^{-1} and the

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SIC is as old as the Earth. Presumably the FOC would be maintained on its adiabat by compositionally-driven convection, and heat would be pumped downward from the mantle (Loper 1978). This scenario might meet some geophysical opposition!

Since the Earth's core is a ferrous alloy, its magnetic Prandtl number

$$P_m = \nu/\eta \tag{3.1}$$

is very small, perhaps about 10^{-6}, where ν is the kinematic viscosity. Because $P_m \ll 1$, the total Joule loss,

$$\mathcal{Q}^\eta = \mu \int_{\rm core} \eta J^2 dv\,, \tag{3.2}$$

is by far the larger part of the total dissipation $\mathcal{Q} = \mathcal{Q}^\eta + \mathcal{Q}^\nu$. Values of order 10^{-3} may be typical for P_m in a stellar plasma so that it is again plausible that $\mathcal{Q} \approx \mathcal{Q}^\eta$.

If we estimate the average value of J by equating $\mathcal{Q}^\eta$ and $\mathcal{Q}_c$ we obtain 0.06A m^{-2} $\lesssim J \lesssim$ 0.16A m^{-2}. An estimate of the mean strength B of the magnetic field $\mathbf{B}$ follows from $B = \mu JL$, again obtained from $\mu\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{B}$. To use this, an estimate of L is required, but what should this be? Under plausible conditions, the scales mainly responsible for generating $\mathbf{B}$ are of order $R_m^{-1/2} r_{\rm s} = \sqrt{(\eta r_{\rm s}/U)}$, according to Christensen & Tilgner (2004) and Tobias & Cattaneo (2008). This gives 11 mT $\lesssim B \lesssim$ 28 mT.

4. Turbulent dynamos

Modeling planets and stars means coming to terms with turbulence. In the classic picture of turbulence, energy is injected on the largest scales and cascades to small scales through the inertial range, to be extracted as heat on the dissipation scale. Turbulence in the situations considered here is markedly different: energy is acquired at all scales through buoyancy. (For simplicity we consider only thermal buoyancy here.) There may be a significant sub-range of scales in which the energy cascade does not dominate, the dissipative losses being replenished directly by the buoyancy forces:

$$\rho g \widetilde{\alpha} v \theta \sim \mu \eta j^2\,. \tag{4.1}$$

Here $\mathbf{v}$ is the fluctuating part of $\mathbf{V}$, i.e., we are writing $\mathbf{V} = \overline{\mathbf{V}} + \mathbf{v}$, where $\overline{\mathbf{V}}$ is the ensemble or statistical mean; similarly $\mathbf{J} = \overline{\mathbf{J}} + \mathbf{j}$ and $T = \overline{T} + \theta$. Relation (4.1) also emphasizes another point: in MHD, the Lorentz force may be more important in the dynamical balance than the inertial force, especially when the Alfvén number, V/V_A, is small; also, ohmic dissipation may be much more significant than viscous dissipation, when $P_m \ll 1$ (see §3). This takes one even further away from the classic turbulence picture. In what follows. we shall no longer reserve $\mathbf{U}$ for velocity relative to $\mathcal{F}$, it being clear from the context whether $\mathbf{V}$ refers to the inertial or rotating frame.

The lack of an MHD turbulence theory is a void that cannot be filled by the computer, either now or in the foreseeable future. Advances in computer technology continually push back the GS/SGS frontier between the large *grid scales* (GS) of main interest and the small, numerically-unresolvable *sub-grid scales* (SGS), but the effect of the latter on the former cannot be ignored and must be represented in a physically plausible way. The first recourse is the classic Boussinesq–Reynolds ansatz (BRA) that draws an analogy between randomly moving molecules, carrying momentum and energy, and randomly moving SGS eddies or "blobs" of fluid that perform the same function for the GS, and much more effectively in strong turbulence. In this classic picture of turbulence in an incompressible fluid, the averages of the governing equations,

$$\partial_t \mathbf{V} + \mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{V} = -\boldsymbol{\nabla}\Pi + \nu\nabla^2\mathbf{V}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{V} = 0\,, \tag{4.2a,b}$$

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are

$$\partial_t \overline{\mathbf{V}} + \overline{\mathbf{V}}\boldsymbol{\cdot}\boldsymbol{\nabla}\overline{\mathbf{V}} = -\boldsymbol{\nabla}\overline{\Pi} + \nu\nabla^2\overline{\mathbf{V}} + \overline{\mathbf{M}}^v , \qquad \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{V}} = 0 , \tag{4.2c,d}$$

where $\partial_t = \partial/\partial t$, $\Pi = P/\rho$ and P is pressure; $\overline{\mathbf{M}}^v = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{Q}} = \overline{\mathbf{vv}}$ is the Reynolds stress tensor. The BRA represents $\overline{\mathbf{M}}^v$, as in molecular dynamics, by

$$\overline{\mathbf{M}}^v = \boldsymbol{\nabla}\boldsymbol{\cdot}(\nu^T \boldsymbol{\nabla}\overline{\mathbf{V}}) , \qquad \text{i.e.,} \qquad \overline{M}^v_i = \nabla_j(\nu^T \nabla_j \overline{V}_i) , \tag{4.2e,f}$$

where the turbulent viscosity $\nu^T \sim \frac{1}{3}\overline{u}\lambda$, in analogy with molecular dynamics; here $\overline{u} = \sqrt{\overline{v^2}}$ and λ is the correlation length. The relation (4.2e) is a godsend! The unmanageable (4.2a) is instantly converted (for constant ν^T) into the much more amenable

$$\partial_t \overline{\mathbf{V}} + \overline{\mathbf{V}}\boldsymbol{\cdot}\boldsymbol{\nabla}\overline{\mathbf{V}} = -\boldsymbol{\nabla}\overline{\Pi} + \overline{\nu}\nabla^2\overline{\mathbf{V}} , \tag{4.2g}$$

where the total viscosity $\overline{\nu} = \nu^T + \nu$ is dominated by ν^T. A cynic might also say that uncertainties in $\overline{u}$ and λ and therefore in $\overline{\nu}$ afford an irresistable opportunity of choosing the $\overline{\nu}$ that makes the simulated system mimic reality best!

The BRA is clearly convenient but there are now alternative ways of incorporating the effect of the SGS on the GS. These methods are not described here but are discussed in reviews such as Meneveau & Katz (2000) and Geurts *et al.* (2008).

More relevant than (4.2) for the geodynamo are the Boussinesq equations with Coriolis, Lorentz and buoyancy forces included. To determine these, the induction and heat equations governing $\mathbf{B}$ and T must be added. Their averages determine the mean fields, $\overline{\mathbf{B}}$ and $\overline{T}$, which contain sources $\overline{\mathbf{M}}^b = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}}$ and $M^\theta = -\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{I}$ where $\boldsymbol{\mathcal{E}} = \overline{\mathbf{v}\times\mathbf{b}}$ is the turbulent electromotive force and $\mathbf{I} = \overline{\theta\mathbf{v}}$ is proportional to the turbulent heat flux. The ideas behind BRA apply equally well to these and, as in (4.2e),

$$\overline{\mathbf{M}}^b = -\boldsymbol{\nabla}\times(\eta^T \boldsymbol{\nabla}\times\overline{\mathbf{B}}) , \qquad M^\theta = \boldsymbol{\nabla}\boldsymbol{\cdot}(\kappa^T \boldsymbol{\nabla}\overline{T}) , \tag{4.2h,i}$$

where η^T and κ^T, the turbulent magnetic and thermal diffusivities, are also of order $\frac{1}{3}\overline{u}\lambda \sim \nu^T$. In stellar applications η^T and κ^T dominate $\overline{\eta} = \eta^T + \eta$ and $\overline{\kappa} = \kappa^T + \kappa$. For the Earth, there is no observational evidence that $\overline{\eta}$ differs substantially from η. This is not surprising if (see above) R_m is only about $7R^c_m$. Because $P_m \ll 1$, the corresponding dimensionless numbers quantifying ν and κ, the kinetic Reynolds number, $R_e \approx Vr_\mathrm{s}/\nu \approx 2 \times 10^8$ and the Peclet number, $P_e = Vr_\mathrm{s}/\kappa \approx 2 \times 10^7$, are gigantic and the SGS are important in transporting GS momentum and heat.

In stellar applications where $\overline{\rho}$ varies over many scale heights, the compressibity of the plasma must be allowed for but, if V is small compared with both $g/2\Omega_0$ and the velocity of sound c, the anelastic equations can be used and are as easy to apply as the Boussinesq equations; see Braginsky & Roberts (1995, 2007). They are now in general use in stellar dynamo simulations; e.g., Browning (2008); Browning *et al.* (2004, 2006); Brun *et al.* (2005). At the recent dynamo workshop at the Kavli Institute in Santa Barbara, an anelastic benchmark was set up so that simulators will have a ready test of their codes available. The anelastic approximation was first employed in geodynamo simulations by Glatzmaier & Roberts (1996), but the Boussinesq equations are still more commonly used.

When the Coriolis force, and of course the pressure gradient, dominate the remaining forces, the flow $\overline{\mathbf{V}}$ becomes two-dimensional with respect to the direction Oz of $\boldsymbol{\Omega}_0$, as demanded by the Proudman-Taylor theorem. [Not 'the Taylor-Proudman theorem' please! Proudman (1916) has 7 years priority over Taylor (1923)!] The early simulation of Glatzmaier (1985) used anelastic theory and in 1989 was still the most sophisticated model of solar MHD. He found that, because of the dominance of Coriolis forces, the mean angular velocity $\Omega(s, z) = \overline{V}(s, z)/s$ of the flow about Oz tended to be almost independent of z, i.e., $\Omega = \Omega(s)$, the Proudman–Taylor result. In 1989 the helioseismology bombshell burst (Libbrecht 1989): $\Omega = \Omega(\vartheta)$, where ϑ is colatitude, represents reality much better than $\Omega = \Omega(s)$. So where had the theory gone wrong?

It is now widely accepted that the BRA is too simplistic because the GS/SGS boundary is, for unavoidable numerical reasons, set at *far* too small a length scale. The SGS are significantly affected particularly by density stratification and rotation, so destroying the isotropy assumed by BRA. Lack of isotropy means that (4.2e,h,i) would better be replaced by

$$\overline{M}_i^v = \nabla_j\left(\nu_{ijkl}^T\nabla_l\overline{V}_k\right), \quad \overline{M}_i^b = -\epsilon_{ijk}\nabla_j\left(\eta_{klm}^T\nabla_m\overline{B}_l\right), \quad \overline{M}^\theta = \nabla_j\left(\kappa_{jk}^T\nabla_k\overline{T}\right), \quad (4.3\text{a,b,c})$$

in which tensor diffusivities appear. Unfortunately even this is not enough. Mean field electrodynamics (MFE) identifies an additional term in $\boldsymbol{\mathcal{E}}$ involving the undifferentiated components of $\overline{\mathbf{B}}$, This "alpha effect" requires a supplementary $\alpha\overline{\mathbf{B}}$ to be included in $\boldsymbol{\mathcal{E}}$ for pseudo-isotropic turbulence or, in the more general non-isotropic case, $\mathcal{E}_i = \alpha_{ij}\overline{B}_j - \eta_{ijk}^T\nabla_k\overline{B}_j$. Nearly 70 years ago, before the discovery of the $\alpha-$effect [for the history, see Rüdiger (1989)], a process analogous to the $\alpha-$effect had been proposed for momentum transport. This $\Lambda-$effect includes in the Reynolds stress tensor, $\boldsymbol{\mathcal{Q}}$, a term proportional to the components of the undifferentiated $\overline{\mathbf{V}}$. Since $\mathcal{Q}_{ij} = \mathcal{Q}_{ji}$, an isotropic $\Lambda-$effect does not exist. As $\overline{\mathbf{V}}$ is usually dominated by the zonal shear, $\boldsymbol{\Omega} = \Omega(s,z)\widehat{\boldsymbol{\phi}}$, the $\Lambda-$effect is usually represented by $Q_{ij}^\Lambda = \Lambda_{ijk}\Omega_k$, where $\Lambda_{ijk} = \Lambda_{jik}$. The inclusion of the $\Lambda-$ and $\alpha-$effects, changes (4.3a,b) to

$$\overline{M}_i^v = \nabla_j\left(\Lambda_{ikl}\Omega_l - \nu_{ijkl}^T\nabla_l\overline{V}_k\right), \qquad \overline{M}_i^b = \epsilon_{ijk}\nabla_j\left(\alpha_{kl}\overline{B}_l - \eta_{klm}^T\nabla_m\overline{B}_l\right). \qquad (4.3\text{d,e})$$

Details of this mean field theory, together with possible forms for Λ_{ijk} and ν_{ijkl}^T, may be found in Rüdiger (1989); Rüdiger & Hollerbach (2004). The $\Lambda-$effect can have a significant impact on Ω. Combined with the influence of meridional circulation, $\overline{\mathbf{V}}_P$, the observed departure of Ω from the Proudman-Taylor $\Omega(s)$ can be successfully modeled. See for example Rüdiger & Hollerbach (2004) and Rempel (2005, 2006).

Applications of the mean field MHD apparatus just described almost invariably assume that the statistical averages $\overline{\mathbf{V}}$, $\overline{\mathbf{B}}$ and $\overline{T}$ are axisymmetric; three dimensional applications are rare. The tensor diffusivities (4.3c,d,e) contain so many "free" parameters that a cynic might again wonder whether, by their judicious choice, any desired $\overline{\mathbf{V}}$, $\overline{\mathbf{B}}$ and $\overline{T}$ follow.

This brief review of dynamo theory, as applied to stars such as the Sun in particular, but more generally to any star with a convection zone, has omitted reference to the solar tachocline and to the stellar tachoclines expected at any radiative-convective interface. These zones add another layer of complexity to an already daunting theory; see, for example, Hughes *et al.* (2007). Also omitted was a discussion of small-scale dynamos for which $\overline{\mathbf{B}} \equiv \mathbf{0}$ and MFE is inapplicable. These may be relevant to the solar convection zone and to some geodynamo models.

Do some of the lessons learned from the Sun apply to dynamos in the Earth and planets? As mentioned earlier, it seems unnecessary, from the modest value of R_m in the Earth's core, to introduce into geodynamo simulations either an $\alpha-$effect or a turbulent magnetic diffusivity, of either scalar or tensor type. Because R_e and P_e are so enormous however, it is plausible that the turbulent transport of momentum and heat (and composition) are very significant. This is sometimes used to turn the computational necessity of assuming O(1) values for the Prandtl numbers, P_r $(= \nu/\kappa)$ and P_m, into a virtue, by defining them with turbulent diffusivities. Even though a significant $\Lambda-$effect appears to be unlikely because $R_o = \overline{V}_\phi/\Omega_0 r_s \ll 1$, it seems probable from the following discussion

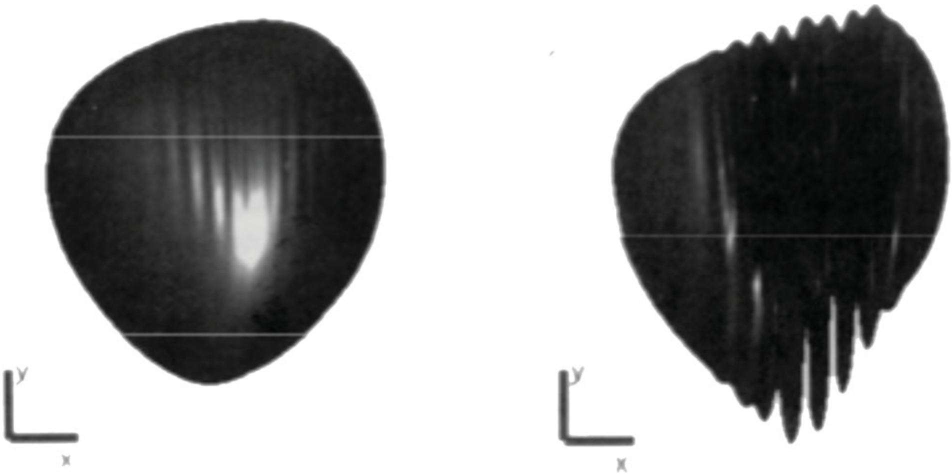

Figure 1. *View from the $z-$axis of the breakup of an initially spherical blob of buoyant fluid.* Left: at time t =1.25, plate-like structures begin to form; Right: at time t = 1.75, the blob has nearly disintegrated into individual plates. The increasing elongation of the blob in the $z-$direction is not apparent in this projection. The $z-$axis is parallel to $\mathbf{\Omega}_0$ and the $y-$axis is parallel to $\overline{\mathbf{B}}$. The dimensionless unit of time is a/v, where a = initial radius of blob, $v = g\theta_0/2\Omega_0$, θ_0 being the temperature excess of the blob. The Ekman number is 6.87×10^{-6} (St Pierre 1996). Reproduced with the permission of Taylor and Francis, publishers of *Geophysical and Astrophysical Fluid Dynamics*; http://www.informaworld.com

that isotropic diffusivities ν^T and κ^T cannot adequately describe turbulent transport of GS momentum and heat.

The nature of turbulent convection in the Earth's core has been considered in more detail by Braginsky & Roberts (1995, 2003) and Loper (2007). To explore the SGS and their anisotropies for $P_m \ll 1$, Braginsky & Meytlis (1990) devised a simple model in which $\overline{\mathbf{B}}$ is in the $y-$direction, with $\mathbf{\Omega}_0$, $\mathbf{g}$ and $\nabla\overline{T}$ in the $z-$direction, and $\beta \equiv \partial_z\overline{T} > 0$. For large E_ℓ, the instabilities of this state are highly anisotropic, being platelike, the thickness of the plates being smaller in the $x-$ or $\mathbf{\Omega}_0 \times \overline{\mathbf{B}}-$direction than in the perpendicular directions by $\mathrm{O}(E_\ell^{-1})$. By an argument too lengthy to give here, Braginsky and Meytlis concluded that

$$\kappa_{xx} \sim \eta\left(\frac{g\widetilde{\alpha}\beta}{4\Omega_0^2}\right)E_\ell^2\,, \qquad \kappa_{yy} \sim \kappa_{zz} \sim \eta\left(\frac{g\widetilde{\alpha}\beta}{4\Omega_0^2}\right)E_\ell^4\,. \tag{4.4a,b}$$

These results depend strongly on $\overline{B}$. Even though κ_{xx} is much less than κ_{yy} and κ_{zz}, it generally greatly exceeds the molecular κ. The argument leading to (4.4) also gives equipartition on the smallest SGS. Matsushima *et al.* (1999) and Matsushima (2001, 2004, 2005) investigated these questions by computer models. Shimizu, reported by Loper (2007), has sugested a potentially useful scaling. That the system is highly dispersive is apparent in the simulation of St Pierre (1996) of the rise and disintegration of a buoyant blob of fluid (see Figure). Whether similar ideas have a broader, astrophysical relevance remains to be seen.

5. Asymptopia

Asymptotic methods were applied early in the history of dynamo theory at a time when powerful electronic computers were nonexistent and when, partly because of Cowling's

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theorem, the very existence of homogeneous fluid dynamos was in doubt. One of the first proofs that homogeneous fluid dynamos exist relied on asymptotic methods, and the first kinematic spherical geodynamo models were based on them (Braginsky 1964b).

Asymptotic (or singular perturbation) methods apply in the limiting case when one or more parameters tend to zero (or their reciprocals tend to zero), and when setting them zero would lower the differential order of the governing equations. For example, when $\nu \to 0$, viscous boundary layers are present. whose thicknesses tend to zero with ν. There are no boundary layers when $\nu = 0$, but the differential order of the governing *ideal equations* is less, so that boundary conditions must be dropped, but which? This is sometimes easily decided, but not always. The resolution of such matters has provided a happy hunting ground for a generation of applied mathematicians, a kind of utopia that led me to coin the word heading this Section. Asymptopia delivers ideal equations and boundary conditions, independent of the diffusivities or dependent only on their ratios. For example, for the kinematic Braginsky (1964b) model, it posed equations independent of the parameter ($R_m^{-1/2}$) whose smallness was used to derive them. These could then be solved by the primitive computers available at that time.

The smallness of E for naturally-occurring MHD dynamos immediately suggests asymptopia, but there are severe obstacles. Small ν means turbulence and the problems of the SGS described in §4, problems exacerbated by small κ and (except for the geodynamo) small η. Although analytic progress is impossible, one may, as an article of faith, believe that, when ν, κ and η are small, the concepts of asymptopia are valid, and that the behavior of the large GS can be characterized by parameters independent of the diffusivities (or dependent only on their ratios), even though all diffusivities are essential for the SGS and boundary layers. Such characterizations constitute *scaling theory*.

Consider, as an example of scaling theory (Jones 2007), non-rotating, non-magnetic convection in a plane layer of depth D. In the limit of infinite Rayleigh number, $R_a = g\widetilde{\alpha}D^3\Delta\overline{T}/\nu\kappa$, the boundary layers on the walls are infinitely thin, and elsewhere the typical convective velocity V and departure in the temperature from $\overline{T}$ are

$$V \sim (gD)^{1/3}\left(\frac{\widetilde{\alpha}F_c}{\rho C_p}\right)^{1/3}, \quad \theta \sim \frac{1}{\widetilde{\alpha}(gD)^{1/3}}\left(\frac{\widetilde{\alpha}F_c}{\rho C_p}\right)^{1/3}, \tag{5.1}$$

where F_c is the convective heat flux and C_p is specific heat. These results are independent of ν, κ and $\nu/\kappa = P_r$.

Scaling theory has achieved prominence recently through the influential papers of Christensen & Aubert (2006) and Christensen *et al.*, (2009). For the thermally–driven dynamo, five of their dimensionless parameters are a Rossby number R_o and

$$Ra_Q^* = \frac{r_\mathrm{s}}{r_\mathrm{i}}.\frac{g\widetilde{\alpha}F_c}{\rho C_p\Omega_0^3D^2}, \qquad Lo = \frac{V_A}{\Omega_0 D}, \tag{5.2a,b}$$

$$Nu^* = \frac{r_\mathrm{s}}{r_\mathrm{i}}.\frac{F_c}{\rho C_p(\Delta T)\Omega_0 D}, \quad f_\mathrm{ohm} = \frac{Q^\eta}{Q^\eta + Q^\nu}, \tag{5.2c,d}$$

where $D = r_\mathrm{s} - r_\mathrm{i}$. One of these depends on P_m but the rest are independent of the diffusivities and their ratios. In (5.2c,b), the modified Rayleigh number Ra_Q^* quantifies buoyancy, the Nusselt number Nu^* heat flux, and the Lorentz number Lo field strength.

Christensen & Aubert (2006) derive their scaling laws by analyzing 66 dynamo integrations, all in geo-geometry $r_\mathrm{i}/r_\mathrm{s} = 0.35$, the SIC being electrically insulating in all but 5 cases. Convection was driven between fixed, noslip boundaries by an assigned temperature difference $\Delta\overline{T}$ between them, and with $5 < R_a/R_a^c < 50$. In all except one case

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($E \equiv \nu/\Omega_0 D^2 = 10^{-6}$), the Ekman number E was between 3×10^{-6} and 3×10^{-4}. The range $0.06 \leqslant P_m \leqslant 10$ was investigated.

The Christensen-Aubert study has suggested several interesting power law dependencies that pose theoretical challenges, e.g., that the dynamo fails if $P_m < 450E^{3/4}$. Unless the optimal exponents are modified, there is less hope of deducing others of their empirical laws, e.g., $R_o \propto {Ra_Q^*}^{0.43} P_m^{-0.13}$ and $Nu^* = 0.076{Ra_Q^*}^{0.53}$. The latter suggests that the exponent should be $\frac{1}{2}$, leading to the perhaps surprising conclusion that the convective heat flux is independent of κ. Their best fit for the field strength was $Lo = 0.76{Ra_Q^*}^{0.32} P_m^{0.11} f_{\rm ohm}^{1/2}$; their next best was $Lo = 0.92{Ra_Q^*}^{0.34} f_{\rm ohm}^{1/2}$. Changing 0.34 to $\frac{1}{3}$ in the latter, and setting $f_{\rm ohm} = 1$ for the Earth (see §3), leads to their proposed alternative to (2.1a):

$$V_A = 0.9(gD)^{1/3}(\widetilde{\alpha} F_c/\rho C_p)^{1/3}\,, \qquad \text{or} \qquad V_A = 0.9(gDF_b/\rho)^{1/3}\,, \tag{5.3}$$

where $F_b = \widetilde{\alpha} F_c / C_p$ is the buoyancy flux, which from their Nu^* is $6-8\times 10^{-9}$kg m^{-2}s^{-1}. This makes B only about 1 mT and independent of both Ω_0 and η. It is determined mainly by the heat flux F_c (or the buoyancy flux F_b when both sources of buoyancy operate); (5.3) may be thought of as a power balance rather than a force balance such as (2.1a).

Sixty six dynamo models is a lot! And they appear to cover adequately the range of parameters that is computationally accessible. They do not, and cannot, cover the enormous parameter range over which naturally-occurring dynamos roam. Of course, the derivation of scaling laws is partly motivated by a wish to apply them outside the computationally accessible domain, so the question is one of degree: how far outside that domain can they be trusted? The Earth's P_m of 10^{-6} lies far beneath the $0.06 \leqslant P_m \leqslant 10$ range of the models, and $10^{-6} \leqslant E \leqslant 3 \times 10^{-4}$ does not include the $E \sim 10^{-15}$ of the Earth (or 10^{-9} if ν^T defines E instead of ν). There is some danger that other similarity laws may apply beyond the computationally accessible domain. An example where this may be happening is given below. Extrapolation to stars and other cosmic contexts is even more extreme and therefore even more problematic (quite apart from the unexplored effects of compressibility on the scaling laws).

The 66 models assign the core surface temperature $\overline{T}(r_{\rm s})$ but, because of the role of the mantle in transmitting heat, it is more realistic to specify the heat flux $q(r_{\rm s})$ on the core boundary [e.g., Braginsky & Roberts (2007)]. When $E \ll 1$, this may make a significant difference. In an effort to move towards geophysically more realistic parameter values, Kageyama *et al.* (2008) used the Earth Simulator to integrate a model for $E = 2.3 \times 10^{-7}$, but they assigned a uniform $\overline{T}(r_{\rm s})$. Disappointingly, the resulting field had a small scale, i.e., was less Earthlike than previous models, even though their E was more Earthlike. Sakuraba and Roberts (to appear) present results from a small E simulation ($E = 5 \times 10^{-7}$, $P_m = 0.2$) but in which $q(r_{\rm s})$ is assumed uniform rather than $\overline{T}(r_{\rm s})$. The resulting field is dipole dominated and is generally more Earthlike. We speculate that, although the characters of the constant$-q(r_{\rm s})$ and constant$-\overline{T}(r_{\rm s})$ models are similar for the values of E common in today's simulations, they will increasingly differ as E is reduced further, and possibly their scaling laws will differ too? Exciting times lie ahead!

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Discussion

Koutchmy: Planet Mars is rotating as fast as the Earth. Why is its magnetic field much weaker?

Roberts: The magnetic field of Mars arises from remnant magnetization of minerals in its crust. This magnetization was probably acquired earlier in the planet's history when it operated a dynamo in its electrically conducting core. That core may have solidified; or conceivably the mantle of Mars did not allow enough heat to emerge from the core to set up sufficiently vigorous convection to permit a Martian dynamo to operate.

Paul Roberts

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Anders Johansen

Yi-Jiun Su's problems are much smaller

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