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GROUPS FORMED BY REDEFINING MULTIPLICATION

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ABSTRACT. Let G be a group with elements $1, \ldots, n$ such that the group operation agrees with ordinary multiplication whenever the ordinary product of two elements lies in G. We show that if n is odd, then G is abelian.

1. **Introduction.** In the American Mathematical Monthly [1], Rodney Forcade, Jack Lamoreaux, and Andrew Pollington conjecture that it is possible to form a multiplicative group on the set $\{1, 2, \ldots, n\}$ so that for any two integers whose product under ordinary multiplication is $k \leqq n$, their product in the group is k. Further, they believe, but have not proved, that such a group (which we shall refer to as an *FLP* group) must be abelian.

In this paper we prove the following:

THEOREM. *Every FLP group of odd order is abelian.*

2. **For sufficiently large groups, the theorem holds.** Define $H(n, y)$ to be the set of positive integers less than or equal to n all of whose prime factors are less than or equal to y. Let $\Psi(n, y)$ be the number of elements of $H(n, y)$.

LEMMA 1. *Let n be odd. If there is a number r such that $\Psi(n, r) > n/3$* and *$\Psi(n, n/r) > n/9$, then any FLP group G of order n is abelian.*

PROOF. Let q be a prime $\leqq n/r$. Then for any prime $p \leqq r$, $qp \leqq qr \leqq n$, hence $q \cdot p = p \cdot q$ where $\cdot$ denotes the group product. Thus q commutes with every prime in $H(n, r)$, and therefore with every element of $H(n, r)$. If $Z(q)$ is the set of elements which commute with q, $|Z(q)| > n/3$. Since $Z(q)$ is a subgroup of G, and G has no subgroups of index 2, $Z(q) = G$. That is, q is in the centre, $Z(G)$, of G. Hence, $H(n, r)$ is contained in $Z(G)$.

Now, $[G:Z(G)] < 9$ and is odd, hence $G/Z(G)$ is cyclic, and so G is abelian. □

Direct computation of Ψ shows that all *FLP* groups of odd order $\leqq 23000$ are abelian.

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More general results on the function $\Psi(n, y)$ can be obtained by introducing a function $\rho(u)$ (the Dickman function), which is the continuous solution of the differential difference equation:

$$\rho(u) = 1, \quad \text{for } 0 \leqq u \leqq 1$$

$$u\rho'(u) = -\rho(u - 1), \quad \text{for } u > 1.$$

It is known that ρ is strictly decreasing for $u \geqq 1$, and that $0 \leqq \rho(u) \leqq 1$. In [2] Ramaswami proves that, given $c > 0$ there is a constant $K > 0$ such that for all $u \leqq c$ and all $n \geqq 1$,

$$\Psi(n, n^{1/u}) = n(\rho(u) \pm K/\log n).$$

Notice that for $1 \leqq u \leqq 2$, $\rho(u) = 1 - \log u$, and thus $\rho(e^{2/3}) = 1 - \log(e^{2/3}) = 1/3$. Since ρ is strictly decreasing for $u \geqq 1$, for any $1 \leqq u < e^{2/3}$, and for n sufficiently large, we have $\Psi(n, n^{1/u}) > n/3$. Using the trapezoidal rule to estimate the required integral, we have shown that $\rho(e^{2/3}/(e^{2/3} - 1)) > .28038 > 1/9$, and so we can use Lemma 1 with $r = n^{1/u}$, where $u = e^{2/3} - \epsilon$, for $\epsilon > 0$ and sufficiently small, together with Ramaswami's formula, to obtain the next lemma.

LEMMA 2. *For odd n sufficiently large, any FLP group of order n is abelian.*

3. **Finding the bound.** To determine how large is "sufficiently large" in Lemma 2, we must examine the proof of Ramaswami's theorem using approximations for some number-theoretic quantities. In all that follows, p denotes a prime, and $c = a \pm b$ means that $a - b \leqq c \leqq a + b$.

LEMMA 3.
(a) *If* $x > 1$, $\pi(x) < 1.2555(x/\log x)$, *where* $\pi(x)$ *is the number of primes* $\leqq x$.
(b) *There is a constant b such that*

$$\log\log x + b < \sum_{p \leqq x} \frac{1}{p} < \log\log x + b + \frac{1}{(\log x)^2}, \quad \textit{for } 1 < x \leqq 10^8,$$

and

$$\sum_{p \leqq x} \frac{1}{p} = \log\log x + b \pm \frac{1}{2(\log x)^2}, \quad \textit{for } x \geqq 286.$$

For proofs of these, see [3]. Note that, in particular, if $x > 1$ and $1 \leqq u_1 \leqq u \leqq u_1 + 1$, then

$$\log\frac{u}{u_1} - \frac{u^2}{(\log x)^2} \leqq \sum_{x^{1/u} < p \leqq x^{1/u_1}} \frac{1}{p} \leqq \log\frac{u}{u_1} + \frac{u_1^2 + u^2}{2(\log x)^2}.$$

LEMMA 4.
(a) *For* $u \leqq 1$, $\Psi(x, x^{1/u}) = [x]$.
(b) *For* $1 \leqq u \leqq 2$,

$$\Psi(x, x^{1/u}) = [x] - \sum_{x^{1/u} < p \leqq x} [x/p].$$

(c) *Let* $u_1 \geqq 2$. *Suppose there exist positive constants* X *and* K_0 *such that for all* $v \leqq u_1$,

$$\Psi(x, x^{1/v}) = x\left(\rho(v) \pm \frac{K_0}{\log x}\right),$$

for all $x \geqq X^{1/u_1}$. *Then for any* u *such that* $u_1 \leqq u \leqq u_1 + 1$, *and for all* $x \geqq X$,

$$\Psi(x, x^{1/u}) = x\rho(u) + \Psi(x, x^{1/u_1}) - x\rho(u_1)$$
$$\pm x\left\{u_1 \frac{K_0}{\log x}\left(\log\left(\frac{u}{u_1}\right) + \frac{u^2}{(\log x)^2}\right) + \frac{3u^2 + u^3 - u_1^3}{3(\log x)^2}\right\}.$$

PROOF. (a) and (b) are immediate. From [2], we get that for $u \geqq 2$

$$\Psi(x, x^{1/u}) = x\rho(u) + \Psi(x, x^{1/u_1}) - x\rho(u_1)$$
$$\pm \left\{K_0 \sum_{x^{1/u} \leqq p \leqq x^{1/u_1}} \frac{x/p}{\log(x/p)} + \left| x \int_{x^{1/u}}^{x^{1/u_1}} \frac{k(t)}{(\log t)^2} d\rho\left(\frac{\log(x/t)}{\log t}\right)\right.\right.$$
$$\left.\left. - x\left[\frac{k(t)}{(\log t)^2} \rho\left(\frac{\log (x/t)}{\log t}\right)\right]_{x^{1/u}}^{x^{1/u_1}}\right|\right\},$$

where $k(t)$ is defined by $k(t)/(\log t)^2 = \sum_{p \leqq t} 1/p - \log\log t - b$.
We have

$$K_0 \sum_{x^{1/u} \leqq p \leqq x^{1/u_1}} \frac{x/p}{\log(x/p)} \leqq u_1 K_0 \frac{x}{\log x} \sum_{x^{1/u} \leqq p \leqq x^{1/u_1}} \frac{1}{p}$$
$$= u_1 K_0 \frac{x}{\log x}\left(\log \frac{u}{u_1} \pm \frac{u^2}{(\log x)^2}\right),$$

by Lemma 3(b), while

$$\left|\int_{x^{1/u}}^{x^{1/u_1}} \frac{k(t)}{(\log t)^2} d\rho\left(\frac{\log(x/t)}{\log t}\right)\right|$$
$$\leqq \left|\int_{x^{1/u}}^{x^{1/u_1}} \frac{1}{(\log t)^2} \rho'\left(\frac{\log(x/t)}{\log t}\right) \frac{d}{dt}\left(\frac{\log(x/t)}{\log t}\right) dt\right|$$
$$\leqq \log x \left|\int_{x^{1/u}}^{x^{1/u_1}} \frac{dt}{t(\log t)^4}\right|, \text{ since } \left|\rho'\left(\frac{\log(x/t)}{\log t}\right)\right| \leqq 1 \text{ for}$$

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$$t < x^{1/u_1} = \left[\frac{\log x}{3(\log t)^3}\right]_{x^{1/u_1}}^{x^{1/u}}$$
$$= \frac{u^3 - u_1^3}{3(\log x)^2}.$$

Finally, by Lemma 3(b),

$$\left|\left[\frac{k(t)}{(\log t)^2}\rho\left(\frac{\log(x/t)}{\log t}\right)\right]_{x^{1/u}}^{x^{1/u_1}}\right| \leqq \frac{u^2}{(\log x)^2},$$

since if $x^{1/u_1} \leqq 10^8$ then $0 < k(x^{1/u}) < 1$ and $0 < k(x^{1/u_1}) < 1$, while if $x^{1/u_1} > 10^8$ then $x^{1/u} > 10^4 > 286$, so that $|k(x^{1/u})| < 1/2$ and $|k(x^{1/u_1})| < 1/2$. □

Now, for $1 \leqq u \leqq 2$,

$$\Psi(x, x^{1/u}) = [x] - \sum_{x^{1/u}<p\leqq x} [x/p]$$
$$= [x] - x \sum_{x^{1/u}<p\leqq x} \frac{1}{p} + \sum_{x^{1/u}<p\leqq x} \left(\frac{x}{p} - [x/p]\right)$$

So, by Lemma 3(b), for $1 \leqq u \leqq 2$,

$$\Psi(x, x^{1/u}) > x\left(\rho(u) - \frac{u^2 + 1}{2(\log x)^2} - \frac{1}{x}\right)$$
$$\geqq x\left(\rho(u) - \frac{2.5}{(\log x)^2} - \frac{1}{x}\right),$$

while

$$\Psi(x, x^{1/u}) < x\left(\rho(u) + \frac{4}{(\log x)^2}\right) + \pi(x).$$

Thus,

(1) $$\Psi(x, x^{1/u}) = x\left(\rho(u) \pm \frac{2.0555}{\log x}\right), \text{ for } x \geqq e^5,\ 1 \leqq u \leqq 2,$$

by Lemma 3(a) and

(2) $$\Psi(x, x^{1/u}) > x(\rho(u) - .02505), \text{ for } x \geqq e^{10},\ 1 \leqq u \leqq 2.$$

Then in Lemma 4(c), if we take $X = e^{10}$, $K_0 = 2.0555$, and $u_1 = 2$ we have for $2 \leqq u \leqq 3$ and $x \geqq e^{10}$,

$$\Psi(x, x^{1/u}) = x\rho(u) + \Psi(x, \sqrt{x}) - x\rho(2)$$
$$\pm x\left\{\frac{4.111}{\log x}\left(\log\left(\frac{u}{u_1}\right) + \frac{u^2}{(\log x)^2}\right) + \frac{3u^2 + u^3 - u_1^3}{3(\log x)^2}\right\}$$

$$> x\left\{\rho(u) - .02505 - \frac{4.111}{\log x}\left(\log\left(\frac{u}{2}\right) + \frac{u^2}{(\log x)^2}\right) - \frac{3u^2 + u^3 - 8}{3(\log x)^2}\right\}, \text{ by (1).}$$

Therefore, if $2 \leqq u \leqq 2.12$

(3) $$\Psi(x, x^{1/u}) \geqq x(\rho(u) - .1176), \text{ for } x \geqq e^{10}.$$

Thus, for $n \geqq e^{10}$, in order to get $\Psi(n, n^{1/u}) > n/3$, it suffices that $\rho(u) \geqq 1/3 + .02505$, by (2). Since $\rho(u) = 1 - \log u$ for $1 \leqq u \leqq 2$, we get $\rho(u) \geqq 1/3 + .02505$ for $1 \leqq u \leqq \exp(1 - 1/3 - .02505)$, in particular for $u = 1.897$. Now we will use (3) to show that for $n \geqq e^{10}$, $\Psi(n, n^{(1.897-1)/1.897}) > n/9$ by showing that $\rho(1.897/.897) - .1176 > 1/9$. Here, $2 < 1.897/.897 < 2.115$, so $\rho(1.897/.897) > \rho(2.115)$, and by approximating the integral we get that $\rho(2.115) - .1176 > .2540 - .1176 = .1364 > 1/9$. Hence, for $n > e^{10}$, we get that for $u_0 = 1.897$, $\Psi(n, n^{1/u_0}) > n/3$, while $\Psi(n, n^{(u_0-1)/u_0}) > n/9$. We have verified the theorem directly for $n \leqq 23{,}000$ using Lemma 1, so the proof of the theorem is now complete.

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