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Incompressibility of Products of Pseudo-homogeneous Varieties

Nikita A. Karpenko

Abstract. We show that the conjectural criterion of p-incompressibility for products of projective homogeneous varieties in terms of the factors, previously known in a few special cases only, holds in general. Actually, the proof goes through for a wider class of varieties, including the norm varieties associated with symbols in Galois cohomology of arbitrary degree.

Let F be a field. A smooth complete irreducible F-variety X is *incompressible*, if every rational self-map $X \dashrightarrow X$ is dominant. This means that $\operatorname{cdim} X = \dim X$, where the *canonical dimension* $\operatorname{cdim} X$ is defined as the minimum of $\dim Y$ for Y running over closed irreducible subvarieties of X admitting a rational map $X \dashrightarrow Y$. The notion of canonical dimension was originally introduced in [1].

For the whole exposition, let p be a fixed prime number. The *canonical p-dimension* $\operatorname{cdim}_p X$, the p-local version of $\operatorname{cdim} X$, is defined as the minimum of $\dim Y$ for Y running over closed irreducible subvarieties of X admitting a correspondence $X \overset{p'}{\rightsquigarrow} Y$ of degree 0 and of p-prime multiplicity. For arbitrary F-varieties X and Y with irreducible X, a correspondence $X \rightsquigarrow Y$ of degree 0 is an element of the Chow group $\mathrm{CH}_d(X \times Y)$, where $d := \dim X$ (*cf.* [3, §62]); its multiplicity is its image under the push-forward homomorphism $\mathrm{CH}_d(X \times Y) \to \mathrm{CH}_d X = \mathbb{Z}$ with respect to the projection $X \times Y \to X$. We refer to [3, §75] for the basic properties of the multiplicity of a correspondence.

The notion of canonical p-dimension was originally introduced in [10]. We refer to [6] and [18, §4] for motivation, history, and a general discussion of the canonical (p-)dimension.

The variety X is *p-incompressible* if every self-correspondence $X \overset{p'}{\rightsquigarrow} X$ of degree 0 and of p-prime multiplicity is dominant, *i.e.*, if $\operatorname{cdim}_p X = \dim X$. The rational equivalence class of the closure of the graph of a rational map is a correspondence of degree 0 and of multiplicity 1; therefore we always have $\operatorname{cdim}_p X \le \operatorname{cdim} X \le \dim X$. In particular, a variety that is p-incompressible (for at least one p) is incompressible.

Studying canonical p-dimension, it is more appropriate to use the Chow group Ch with coefficients in $\mathbb{F} := \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ rather than the Chow group CH with integer coefficients. Multiplicities of correspondences, as well as degrees of classes of 0-cycles,

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then take values in $\mathbb{F}$. We also consider the Chow motives with coefficients in $\mathbb{F}$; see [3, Chapter XII].

The main result of this paper is the following criterion of p-incompressibility of the product of projective homogeneous varieties (also called twisted flag varieties) under a semi-simple affine algebraic group, contained in Theorem 7: the product $X \times Y$ of F-varieties X and Y is p-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are p-incompressible.

The story of this criterion began 9 years ago with the case of Severi–Brauer varieties proved in 2007 for the purpose of computing the essential p-dimension of finite groups; see [11] or Example 9. Several other special classes of projective homogeneous varieties have been treated afterwards; see Example 10. The proof of the general case given here is new even in the known particular cases; it is actually simpler than the available proofs of the particular cases. Moreover, it works for a wider class of varieties. Concrete examples of new and useful varieties covered by the proof are given in an appendix.

Let us now introduce the class of varieties, which we call *pseudo-homogeneous* and for which we can prove the above criterion. Recall that a field L is *p-special*, if the degree of any finite field extension of L is a power of p.

A smooth complete geometrically irreducible variety X is *pseudo-homogeneous* if it has the following two properties. For every field extension L/F containing a p-special field extension of F:

(i) the variety X_L is p-compressible provided that there exists a self-correspondence $\alpha: X_L \rightsquigarrow X_L$ of degree 0 such that the multiplicity of α differs from the multiplicity of the transpose α^t of α;

(ii) one has $\operatorname{cdim}_p X_L \geq d$, where d is the minimal integer for which there exist an element $a \in \operatorname{Ch}_d X_L$ and an element $b \in \operatorname{Ch}^d X_{L(X)}$ with $\deg(a_{L(X)} \cdot b) = 1$.

(As a variant, the assumption that L contains a p-special field can be removed from (i), but it is important for (ii) in order to make Example 6 work.)

Remark 1 The definition of "pseudo-homogeneous" depends on the prime p. It would be better to say "p-pseudo-homogenous", but we omit "p-" for convenience. The same applies to "split" and "A-trivial", introduced below. On the other hand, we do not abbreviate "p-incompressible".

Remark 2 A pseudo-homogeneous variety remains pseudo-homogeneous under any base field extension. On the other hand, it is not clear whether the product of two pseudo-homogeneous varieties is necessarily pseudo-homogeneous.

Remark 3 The opposite to property (i) holds for any smooth complete variety X. If X is p-compressible, then there exists a correspondence $\alpha: X \rightsquigarrow Y$ of degree 0 and multiplicity 1 to a proper closed subvariety $Y \subset X$. Considering α as a correspondence $X \rightsquigarrow X$, we have $\operatorname{mult} \alpha = 1$ and $\operatorname{mult} \alpha^t = 0$.

Remark 4 The opposite to the inequality in (ii) holds for any smooth complete variety X (*cf.* [10, Proof of Theorem 5.8, part "≤"]). Indeed, take the minimal d such that

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there exist $a \in \mathrm{Ch}_d X$ and $b \in \mathrm{Ch}^d X_{F(X)}$ with $\deg(a_{F(X)} \cdot b) = 1$. We may assume that $a = [Y]$ and $b = [Z]$ for closed subvarieties $Y \subset X$ and $Z \subset X_{F(X)}$. Since the product $[Y_{F(X)}] \cdot [Z] \in \mathrm{Ch}\, X_{F(X)}$, which is a 0-cycle class of degree 1, can be represented by a 0-cycle with support on the intersection $Y_{F(X)} \cap Z$ (see [4, §8.1]), the variety $Y_{F(X)}$ has a 0-cycle of degree 1, that is, there exists a degree 0 correspondence $X \rightsquigarrow Y$ of multiplicity 1 (see [3, p. 328] concerning the relation between correspondences and 0-cycles). Therefore $\mathrm{cdim}_p X \le \dim Y = d$.

Remark 5 Property (i) holds for any *A-trivial variety*, *cf.* [12, Definition 2.3]: the proof given in [6, Lemma 2.7] for projective homogeneous varieties goes through. More precisely, we speak about the A-triviality for the coefficient ring $\mathbb{F}$ defined as follows: a smooth complete variety X over a field F is A-trivial if for any field extension L/F with $X(L) \ne \varnothing$, the degree homomorphism $\deg\colon \mathrm{Ch}_0 X_L \to \mathbb{F}$ is an isomorphism.

Here is our basic example of pseudo-homogeneous varieties. For more examples see Appendix A.

Example 6 Any projective homogeneous variety (under an action of a semi-simple affine algebraic group) is pseudo-homogeneous. See [6, Lemma 2.7] for (i) and [7, Proposition 6.1] for (ii).

Theorem 7 *Let X and Y be pseudo-homogeneous F-varieties such that the product $X \times Y$ is also pseudo-homogeneous. Then the variety $X \times Y$ is p-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are p-incompressible. Moreover,*

$$\mathrm{cdim}_p(X \times Y) = \mathrm{cdim}_p X_{F(Y)} + \mathrm{cdim}_p Y_{F(X)}, \tag{1}$$

provided that at least one of the three varieties $X_{F(Y)}$, $Y_{F(X)}$, $X \times Y$ is p-incompressible.

Of course, for projective homogeneous X and Y, the product $X \times Y$ is also projective homogeneous so that we do not need to require the product to be pseudo-homogeneous.

Partial cases of Theorem 7 dealing with some special types of projective homogeneous varieties have recently been proved [5, 9]. For an older result in this direction see Example 9 below.

The p-incompressibility criterion, given in Theorem 7 for products of two varieties, immediately generalizes to finite products of arbitrary length.

Corollary 8 *For $n \ge 1$, let $X_1, \dots, X_n$ be F-varieties such that, for every subset $I \subset \{1, 2, \dots, n\}$, the product $\prod_{i \in I} X_i$ is pseudo-homogeneous. Then X is p-incompressible if and only if for every $i = 1, \dots, n$ the variety $(X_i)_{F(X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times X_n)}$ is p-incompressible.*

Proof Assuming that the statement holds for some $n \ge 1$, we prove it for $n+1$. Set $X := X_1 \times \cdots \times X_n$ and $Y := X_{n+1}$. If $X \times Y = X_1 \times \cdots \times X_{n+1}$ is p-incompressible, $X_{F(Y)}$ and $Y_{F(X)}$ are p-incompressible, and it follows by induction hypothesis that the variety $(X_i)_{F(X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times X_{n+1})}$ is p-incompressible for any $i = 1, \dots, n+1$.

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The other way round, if $(X_i)_{F(X_1\times\cdots\times X_{i-1}\times X_{i+1}\times X_{n+1})}$ is p-incompressible for any i, then, in particular, $Y_{F(X)}$ is p-incompressible and, by induction hypothesis, $X_{F(Y)}$ is p-incompressible. It follows that $X\times Y$ is p-incompressible. ■

Example 9 For the purpose of computing the essential dimension of finite groups, Corollary 8 for Severi–Brauer varieties $X_1,\dots,X_n$ has been obtained in [11, Theorem 2.1 with Remark 2.10] (see also [9, Remark 4.2]). A second and simpler proof was given in [9]. The third proof, given here (see the proof of Theorem 7), is particularly simple. The result has numerous further applications; see [15,16].

Example 10 For the purpose of computing the essential dimension of representations of finite groups, introduced in [14], Corollary 8 for Weil transfers of generalized Severi–Brauer varieties has been obtained in [9] under the assumption that the corresponding central simple algebras are *balanced*. Corollary 8 shows that this assumption is superfluous. Another area of applications for this result is provided in [2], where it is important that the assumption of being balanced can be dropped.

Proof of Theorem 7 Since canonical p-dimension of a variety does not change under any base-field extension of degree prime to p (see [17, Proposition 1.5] and [16, Lemma 2.2]), we may assume that F is p-special.

In order to prove Theorem 7 in whole, we only need to prove equality (1). We start proving its (more difficult) "$\geq$" part now. If the variety $X\times Y$ is p-incompressible, the "$\geq$" part is however trivial. We therefore assume that the $F(X)$-variety $Y_{F(X)}$ is p-incompressible, that is, $\operatorname{cdim}_p Y_{F(X)}=\dim Y$.

Let d be the minimal integer such that there exist

$$a\in \mathrm{Ch}_d(X\times Y)\quad\text{and}\quad b\in \mathrm{Ch}^d(X\times Y)_{F(X\times Y)}$$

with $\deg(a_{F(X\times Y)}\cdot b)=1$. Slightly abusing, but simplifying, notation, we will sometimes write a instead of $a_{F(X\times Y)}$ in the last formula or in similar situations.

Since the product $X\times Y$ is pseudo-homogeneous, we have $\operatorname{cdim}_p(X\times Y)\geq d$. Our aim is to show that $d\geq \operatorname{cdim}_p X_{F(Y)}+\dim Y$.

Let $\alpha\in \mathrm{Ch}_d(X\times Y\times X\times Y)$ be the push-forward of a under the diagonal morphism of $X\times Y$. Note that $\alpha=(a\times[X]\times[Y])\cdot\Delta$, where $\Delta\in \mathrm{Ch}_{\dim(X\times Y)}(X\times Y\times X\times Y)$ is the diagonal class.

Let β be a preimage of b under the flat pull-back

$$\mathrm{Ch}^d((X\times Y)\times(X\times Y))\twoheadrightarrow \mathrm{Ch}^d(X\times Y)_{F(X\times Y)} \tag{2}$$

along the morphism induced by the generic point of the first factor of the product $(X\times Y)\times(X\times Y)$. For surjectivity of (2), see [3, Corollary 57.11].

Let $\delta\in \mathrm{Ch}_{\dim Y}(Y_{F(X)}\times X_{F(X)}\times Y_{F(X)})$ be the class of the graph of the closed imbedding

$$\mathrm{in}\colon Y_{F(X)}\hookrightarrow X_{F(X)}\times Y_{F(X)}$$

induced by the closed rational point $\mathbf{pt}_X$ on $X_{F(X)}$ given by the generic point of X. Finally, let $\gamma\in \mathrm{Ch}^{\dim Y}(X\times Y\times Y)$ be the class of the graph of the projection

$$\mathrm{pr}_Y\colon X\times Y\to Y.$$

We consider the elements α, β, δ, and γ as correspondences and take their composition ρ (over the field $F(X)$) in the following order:

$$\rho\colon Y_{F(X)} \overset{\delta}{\rightsquigarrow} X_{F(X)} \times Y_{F(X)} \overset{\beta}{\rightsquigarrow} X_{F(X)} \times Y_{F(X)} \overset{\alpha}{\rightsquigarrow} X_{F(X)} \times Y_{F(X)} \overset{\gamma}{\rightsquigarrow} Y_{F(X)}.$$

Let $\mathbf{pt}_Y$ be the rational point on $Y_{F(Y)}$ given by the generic point of Y. As usual, we abbreviate $[\mathbf{pt}_Y]_{F(Y)(X)} \in \mathrm{Ch}_0\, Y_{F(X\times Y)}$ as $[\mathbf{pt}_Y]$. A direct computation shows that $\deg \rho_*([\mathbf{pt}_Y]) = 1$, where $\rho_*\colon \mathrm{Ch}\, Y_{F(X\times Y)} \to \mathrm{Ch}\, Y_{F(X\times Y)}$ is the homomorphism induced by ρ. Indeed,

$$[\mathbf{pt}_Y] \overset{\delta_*=\mathrm{in}_*}{\longmapsto} [\mathbf{pt}_X]\times[\mathbf{pt}_Y] \overset{\beta_*}{\longmapsto} b \overset{\alpha_*}{\longmapsto} a\cdot b \overset{\gamma_*=\mathrm{pr}_{Y*}}{\longmapsto} \mathrm{pr}_{Y*}(a\cdot b)$$

and $\deg \mathrm{pr}_{Y*}(a\cdot b) = \deg(a\cdot b) = 1$. To compute the image under β_* in the above chain, we first notice that $\beta_*([\mathbf{pt}_X]\times[\mathbf{pt}_Y]) = \beta_{F(X\times Y)} \circ [\Gamma_f]$, where Γ_f is the graph of the closed imbedding $f\colon \mathrm{Spec}\, F(X\times Y) \hookrightarrow (X\times Y)_{F(X\times Y)}$, given by the points $\mathbf{pt}_X, \mathbf{pt}_Y$, and then we apply [3, Proposition 62.4(2)] (over the field $F(X\times Y)$).

On the other hand, $\deg \rho_*([\mathbf{pt}_Y]) = \mathrm{mult}\,\rho$, *cf.* [3, Lemma 75.1]. Therefore $\mathrm{mult}\,\rho = 1$. Since the pseudo-homogeneous $F(X)$-variety $Y_{F(X)}$ is p-incompressible, it follows that $\mathrm{mult}\,\rho^t = 1$. But $\rho_*([Y]) = (\mathrm{mult}\,\rho^t)[Y]$, showing that $\rho_*([Y]) = [Y]$. We therefore have

$$[Y] \overset{\delta_*=\mathrm{in}_*}{\longmapsto} [\mathbf{pt}_X]\times[Y] \overset{\beta_*}{\longmapsto} \widetilde{b} \overset{\alpha_*}{\longmapsto} a\cdot\widetilde{b} \overset{\gamma_*=\mathrm{pr}_{Y*}}{\longmapsto} [Y]$$

for some $\widetilde{b} \in \mathrm{Ch}^{d-\dim Y}(X\times Y)_{F(X)}$. By commutativity of push-forward and flat pull-back for the cartesian square

$$\begin{array}{ccc} X\times Y & \xrightarrow{\mathrm{pr}_Y} & Y \\ \uparrow & & \uparrow \\ X_{F(Y)} & \longrightarrow & \mathrm{Spec}\, F(Y) \end{array}$$

it follows that $\deg(a'\cdot b') = 1$, where $a' \in \mathrm{Ch}_{d-\dim Y} X_{F(Y)}$ is the pull-back of the element $a \in \mathrm{Ch}_d(X\times Y)$ with respect to the morphism $X_{F(Y)} \to X\times Y$ given by the generic point of Y, and where $b' \in \mathrm{Ch}^{d-\dim Y} X_{F(X)(Y)} = \mathrm{Ch}^{d-\dim Y} X_{F(Y)(X)}$ is the pull-back of $\tilde{b} \in \mathrm{Ch}^{d-\dim Y}(X\times Y)_{F(X)}$. By Remark 4 we have $d - \dim Y \geq \mathrm{cdim}_p\, X_{F(Y)}$, that is, $d \geq \mathrm{cdim}_p\, X_{F(Y)} + \dim Y$. The "$\geq$" part of equality (1) is proved.

The proof of the "$\leq$" part, given in [9, Lemma 3.4] for projective homogeneous X and Y, also works in the current setting. We reproduce it for the reader's convenience. As in [9, Lemma 3.4], we prove the more general inequality

$$\mathrm{cdim}_p(X\times Y) \leq \mathrm{cdim}_p\, X + \mathrm{cdim}_p\, Y_{F(X)}$$

without any p-incompressibility assumption (on $X_{F(Y)}$, on $Y_{F(X)}$, or on $X\times Y$).

We set $x := \mathrm{cdim}_p\, X$ and $y := \mathrm{cdim}_p\, Y_{F(X)}$. Since the variety X is pseudo-homogeneous, we can find $a_X \in \mathrm{Ch}_x\, X$ and $b_X \in \mathrm{Ch}^x\, X_{F(X)}$ with $\deg(a_X\cdot b_X) = 1$. Similarly, since the variety $Y_{F(X)}$ is pseudo-homogeneous, we can find $a_Y \in \mathrm{Ch}_y\, Y_{F(X)}$ and $b_Y \in \mathrm{Ch}^y\, Y_{F(X)(Y)}$ with $\deg(a_Y\cdot b_Y) = 1$. Let $\alpha_Y \in \mathrm{Ch}_{\dim X+y}(X\times Y)$ be a preimage of a_Y under the pull-back along the morphism $Y_{F(X)} \to X \times Y$ induced by the generic point of X. We set

$$a := (a_X \times [Y]) \cdot \alpha_Y \in \mathrm{Ch}_{x+y}(X \times Y) \quad \text{and} \quad b := b_X \times b_Y \in \mathrm{Ch}^{x+y}(X \times Y)_{F(X\times Y)}.$$

We have the relation $\deg(a \cdot b) = \deg(a_X \cdot b_X) \cdot \deg(a_Y \cdot b_Y) = 1$, showing by Remark 4 that $\mathrm{cdim}_p(X \times Y) \le x + y$. ∎

A Generically Upper-split Varieties

By a *Tate motive* we mean any shift $\mathbb{F}\{i\}$ ($i \in \mathbb{Z}$) of the motive of the point $\mathbb{F} := M(\operatorname{Spec} F)$. (We follow the tradition of denoting the motive of the point by the same letter as the coefficient ring.) A motive is *split* if it is isomorphic to a finite direct sum of Tate motives. We say that a smooth complete variety is split provided that its motive is split. A smooth complete geometrically irreducible variety X is *generically split* if the motive of the $F(X)$-variety $X_{F(X)}$ is split.

We generalize the notion of a generically split variety as follows.

Definition A.1 A smooth complete geometrically irreducible variety X is *generically upper-split* if there exists a direct summand U of the total motive $M(X)$ of X such that $\mathrm{Ch}^0\, U \neq 0$ and $U_{F(X)}$ is split.

Clearly, any base field change of a generically upper-split variety is again a generically upper-split variety. Also the product of two generically upper-split varieties is generically upper-split again.

Note that U in the above definition is a *generically split motive* in the sense of [19, Definition 1.1] and therefore satisfies the *nilpotence principle* by [19, Proposition 3.1]: any endomorphism of U vanishing over a field extension of F is nilpotent.

Example A.2 Let F be a field of characteristic 0 and let X be a *norm variety* (see [12, §4]) or, more generally, any p-generic splitting variety of a non-zero symbol in the Galois cohomology group $H^{n+1}(F, \mu_p^{\otimes n})$ for some $n \ge 1$. Then X is generically upper-split. Indeed, X is smooth, complete, and geometrically irreducible by the very definition of a p-generic splitting variety. Moreover, by [12, Theorem 4.1], there is a direct summand R (called the Rost motive of the symbol) in the motive of X such that

$$R_{F(X)} \simeq \bigoplus_{i=0}^{p-1} \mathbb{F}\{i \cdot (p^n - 1)/(p-1)\}.$$

In particular, $R_{F(X)}$ is split and $\mathrm{Ch}^0\, R = \mathrm{Ch}^0\, R_{F(X)} = \mathbb{F}$ is non-zero. Note that the structure of the total motive of X is a complete mystery and is understood only in very special situations (namely, when $p = 2$ and X is a projective quadric; when $n = 1$ and X is a Severi–Brauer variety, and finally, when $n = 2$ and X is a smooth equivariant compactification of the special linear group of a central division algebra of prime degree [13]).

We define the *upper motive* $U(X)$ of a generically upper-split variety X as the motive $U(X)$ from the following.

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Lemma A.3 *For any generically upper-split variety X, there exists an* indecomposable *direct summand $U(X)$ of $M(X)$ such that $U(X)_{F(X)}$ is split and* $\mathrm{Ch}^0\, U(X) \neq 0$. *Moreover, the isomorphism class of $U(X)$ is determined by X.*

Proof To prove existence of $U(X)$, let us note that any motive U as in Definition A.1 satisfies the Krull–Schmidt principle and, in particular, decomposes into a finite direct sum of indecomposable motives, *cf.* [8, §2a]. Since $\mathrm{Ch}^0\, U \neq 0$, at least one of the summands (in fact, precisely one, because the $\mathbb{F}$-vector space $\mathrm{Ch}^0\, U = \mathrm{Ch}^0\, X$ is 1-dimensional) also has non-zero Ch^0, and therefore can be taken for $U(X)$.

To prove uniqueness, given a second $U(X)'$ with the same properties as $U(X)$, we proceed as in [8, end of §2b] to prove that the composition $U(X) \to M(X) \to U(X)'$ of the imbedding of a direct summand followed by the projection onto a direct summand is an isomorphism. Indeed, similarly as in [8, Corollary 2.2], some power of the composition $U(X) \to U(X)' \to U(X)$ is a projector. Moreover, it is non-zero because its multiplicity is $1 \neq 0$. By indecomposability of $U(X)$, it follows that the projector is the identity so that the composition is an isomorphism. The composition $U(X)' \to U(X) \to U(X)'$ is an isomorphism by the same reason. The statement follows. ∎

The *dimension* $\dim U(X)$ of the upper motive $U(X)$ is defined as the maximal d such that the Tate motive $\mathbb{F}\{d\}$ is a direct summand of $U(X)_{F(X)}$. Given a motive M, we write M^* for its dual. Recall that the total motive $M(X)$ of an arbitrary smooth complete equi-dimensional variety X satisfies the duality formula $M(X) \simeq M(X)^*\{\dim X\}$. The same formula holds for the upper motive of a projective homogeneous variety over a p-special field [6, Proposition 5.2]. It turns out that it also holds in the case of a generically upper-split variety.

Theorem A.4 *The upper motive $U(X)$ of an arbitrary generically upper-split variety X satisfies the duality formula $U(X) \simeq U(X)^*\{\dim U(X)\}$.*

Proof The proof is similar to [8, proof of Theorem 3.5], but differs in the two following aspects (as well as in notation). First, since $U(X)$ splits already over $F(X)$, the proof here is not inductive. And second, the total motive of the variety in [8, proof of Theorem 3.5] is *geometrically split* (meaning that it splits over some extension of the base field), whereas here we only know that the summand $U(X)$ of the total motive $M(X)$ has this property. However, since all computations are done "inside" of $U(X)$, this second difference does not create any complications.

Proceeding as in [8, proof of Theorem 3.5], we construct some morphisms

$$\alpha\colon U(X) \to U(X)^*\{\dim U(X)\} \quad \text{and} \quad \beta\colon U(X)^*\{\dim U(X)\} \to U(X)$$

with the composition $\beta \circ \alpha$ given by a correspondence of multiplicity 1. A power of the composition is a non-zero projector and therefore, by indecomposability of $U(X)$, an isomorphism. Thus $U(X)$ is identified with a direct summand of $U(X)^*\{\dim U(X)\}$. Since $U(X)^*\{\dim U(X)\}$ is indecomposable, $U(X)$ is actually identified with the whole of it.

In order to construct α and β, we set $d := \dim U(X)$ and pick up a direct summand $\mathbb{F}\{d\}$ in $U(X)_{F(X)}$. The projection $U(X)_{F(X)} \to \mathbb{F}\{d\}$ is given by some

$$a \in \operatorname{Hom}\big(U(X)_{F(X)}, \mathbb{F}\{d\}\big) = \operatorname{Ch}^d U(X)_{F(X)} \subset \operatorname{Ch}^d X_{F(X)}$$

and the imbedding $\mathbb{F}\{d\} \to U(X)_{F(X)}$ is given by some

$$b \in \operatorname{Hom}\big(\mathbb{F}\{d\}, U(X)_{F(X)}\big) = \operatorname{Ch}_d U(X)_{F(X)} \subset \operatorname{Ch}_d X_{F(X)}.$$

Note that $\deg(a \cdot b) = 1$.

Let α' be any preimage of a under the pull-back epimorphism $\operatorname{Ch}^d(X \times X) \twoheadrightarrow \operatorname{Ch}^d X_{F(X)}$ given by the generic point of the first factor. We take for α the composition $\pi^t \circ \alpha' \circ \pi$, where π is the projector of $U(X)$.

The construction of β is more tricky. Let **pt** be the rational point on $X_{F(X)}$, given by the generic point of X, and let β' be a preimage of $b \times [\mathbf{pt}]$ under the pull-back

$$\operatorname{Ch}_{\dim X + d}(X \times X \times X) \twoheadrightarrow \operatorname{Ch}_d(X \times X)_{F(X)},$$

given by the generic point of the second factor of $X \times X \times X$. We consider β' as a correspondence $X \rightsquigarrow X \times X$ and take the composition $\beta' \circ \pi^t$. Let β'' be the pullback of $\beta' \circ \pi^t$ along the closed imbedding $X \times X \hookrightarrow X \times X \times X$, $(x, y) \mapsto (x, x, y)$, given by the diagonal of the first factor in $X \times X$. We set $\beta := \pi \circ \beta'' \circ \pi^t$.

It remains to check that $\operatorname{mult}(\beta \circ \alpha) = 1$. Recall that this multiplicity coincides with the degree of the 0-cycle class $(\beta \circ \alpha)_*([\mathbf{pt}])$ and $(\beta \circ \alpha)_* = \pi_* \circ \beta''_* \circ \pi^t_* \circ \alpha'_* \circ \pi_*$. We have $\pi_*([\mathbf{pt}])$ is a 0-cycle class of degree 1, the degree of the product $b \cdot \alpha'_*(\pi_*([\mathbf{pt}]))$ coincides with the degree of the product $b \cdot \alpha'_*([\mathbf{pt}])$ which is 1 because $\alpha'([\mathbf{pt}]) = a$. As a consequence, $\pi^t_*(\alpha'_*(\pi_*([\mathbf{pt}]))) = a$. Finally, the degree of the 0-cycle class $\beta''_*(a)$ is 1. ∎

The following statement provides the generically upper-split analogue of [6, Theorem 5.1].

Corollary A.5 *Any generically upper-split variety X is pseudo-homogeneous and satisfies* $\operatorname{cdim}_p X = \dim U(X)$.

Proof We start by proving the equality. Let Y be a closed subvariety of X with $\dim Y = \operatorname{cdim}_p X$ and with a correspondence $\alpha: X \rightsquigarrow Y$ of degree 0 and multiplicity 1. We consider α as a correspondence $X \rightsquigarrow X$. If $\pi: X \rightsquigarrow X$ is a projector determining $U(X)$, an appropriate power of the composition $\pi \circ \alpha \circ \pi$ gives a non-zero motive isomorphic to a direct summand of $U(X)$ and therefore to $U(X)$ itself. It follows that $\operatorname{Ch}^i U(X)_L = 0$ for any $i > \dim Y$ and any field extension L/F. In particular, taking $L = F(X)$, we get $\dim U(X) \le \dim Y = \operatorname{cdim}_p X$.

The opposite inequality is proved, based on Theorem A.4, exactly as in [6, proof of Theorem 5.1]. In the course of the proof, elements $a \in \operatorname{Ch}_d X$ and $b \in \operatorname{Ch}^d X_{F(X)}$ are constructed with $d = \operatorname{cdim}_p X$ and $\deg(a \cdot b) = 1$. This shows that X satisfies property (ii) of the definition of a pseudo-homogeneous variety.

To show that property (i) is also satisfied, assume that there is a self-correspondence $\alpha: X \rightsquigarrow X$ with $\operatorname{mult} \alpha = 1$ and $\operatorname{mult} \alpha^t = 0$. Proceeding with α as in the beginning of the current proof, we represent $U(X)$ by a projector in the form of a power of $\pi \circ \alpha \circ \pi$. The multiplicity of the transpose of this projector is 0 showing that $\dim U(X) < \dim X$. Since we already proved the formula $\dim U(X) = \operatorname{cdim}_p X$, it follows that X is p-compressible. ■

Corollary A.6 *The conclusion of Corollary 8 holds for arbitrary generically upper-split varieties $X_1, \dots, X_n$.*

In particular, the conclusion of Corollary 8 holds for norm varieties $X_1, \dots, X_n$; see Example A.2. (Note that the corresponding symbols in the Galois cohomology need not be of the same degree.) This makes it possible to determine the canonical p-dimension of an arbitrary finite direct product $X = X_1 \times \cdots \times X_n$ of norm varieties. Indeed, if for any i, X_i considered over the function field of the product of the remaining varieties is p-incompressible, then we have $\operatorname{cdim}_p X = \dim X$. Otherwise, if X_i considered over the function field of the product of the remaining varieties is p-compressible for some i, then we remove this X_i which does not affect the canonical p-dimension of the product. Proceeding this way, we eventually end up with a p-incompressible product, whose dimension is the canonical p-dimension of the original product. This can be viewed as a generalization to symbols in Galois cohomology of arbitrary degree of the result of Example 9 related to the Brauer group.

Remark A.7 An anonymous referee suggested considering the following property $(*)$ of a smooth complete geometrically irreducible F-variety X. There is an indecomposable geometrically split direct summand U of the total motive $M(X)$, such that $\operatorname{Ch}^0 U \neq 0$ and $U_{F(X)}$ contain $\mathbb{F}\{\dim U\}$ as a direct summand. We say that X satisfies $(**)$ if X satisfies $(*)$ over any field extension of F. Clearly, any projective homogeneous variety as well as any generically upper-split variety satisfies $(**)$.

Given X with $(**)$, it is not clear if U satisfies nilpotence principle, but we can avoid this problem by working with the *reduced* Chow motives, *i.e.*, the Chow motives constructed out of the reduced Chow groups in place of the usual Chow groups, *cf.* [6, §3]. Then the isomorphism class of U is determined by X (*cf.* Lemma A.3), U satisfies the duality formula of Theorem A.4 as well as the equality of Corollary A.5, and, finally, X is pseudo-homogeneous.

So far, we do not see new interesting examples of varieties, covered by this approach; this is why we do not pursue it in more details here.

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References

[1] G. Berhuy and Z. Reichstein, *On the notion of canonical dimension for algebraic groups.* Adv. Math. **198**(2005), no. 1, 128–171. http://dx.doi.org/10.1016/j.aim.2004.12.004
[2] I. Biswas, A. Dhillon, and N. Hoffmann, *On the essential dimension of coherent sheaves.* arxiv:1306.6432
[3] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms.* American Mathematical Society Colloquium Publications 56, American Mathematical Society, Providence, RI, 2008.

[4] W. Fulton, *Intersection theory.* Second ed. Ergebnisse der Mathematik und ihrer Grenzgebiete 2, Springer-Verlag, Berlin, 1998.
[5] N. A. Karpenko, *Incompressibility of products by Grassmannians of isotropic subspaces.* http://www.ualberta.ca/~karpenko/publ/pbg-r.pdf
[6] ———, *Canonical dimension*. In: Proceedings of the International Congress of Mathematicians. II, New Delhi, Hindustan Book Agency, 2010. pp. 146–161.
[7] ———, *Sufficiently generic orthogonal Grassmannians*. J. Algebra **372**(2012), 365–375. http://dx.doi.org/10.1016/j.jalgebra.2012.09.021
[8] ———, *Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties*. J. Reine Angew. Math. **677**(2013), 179–198.
[9] ———, *Incompressibility of products of Weil transfers of generalized Severi-Brauer varieties.* Math. Z. (2014), 1–11. http://dx.doi.org/10.1017/S1474748011000090
[10] N. A. Karpenko and A. S. Merkurjev, *Canonical p-dimension of algebraic groups*. Adv. Math. **205**(2006), no. 2, 410–433. http://dx.doi.org/10.1016/j.aim.2005.07.013
[11] ———, *Essential dimension of finite p-groups*. Invent. Math. **172**(2008), no. 3, 491–508. http://dx.doi.org/10.1007/s00222-007-0106-6
[12] ———, *On standard norm varieties*. Ann. Sci. Éc. Norm. Supér. (4) **46**(2013), no. 1, 175–214.
[13] ———, *Motivic decomposition of compactifications of certain group varieties*. J. Reine Angew. Math., to appear. http://dx.doi.org/10.1515/crelle-2016-0015
[14] N. A. Karpenko and Z. Reichstein, *A numerical invariant for linear representations of finite groups.* Comment. Math. Helv. **90**(2015), no. 3, 667–701. With an appendix by Julia Pevtsova and Z. Reichstein. http://dx.doi.org/10.4171/CMH/367
[15] R. Lötscher, M. MacDonald, A. Meyer, and Z. Reichstein, *Essential dimension of algebraic tori*. J. Reine Angew. Math. **677**(2013), 1–13. http://dx.doi.org/10.1515/crelle.2012.010
[16] ———, *Essential p-dimension of algebraic groups whose connected component is a torus*. Algebra Number Theory **7**(2013), no. 8, 1817–1840. http://dx.doi.org/10.2140/ant.2013.7.1817
[17] A. S. Merkurjev, *Essential dimension*. In: Quadratic Forms: Algebra, Arithmetic, and Geometry. Contemp. Math. 493, American Mathematical Society, Providence, RI, 2009, pp. 299–326.
[18] ———, *Essential dimension: a survey*. Transform. Groups **18**(2013), no. 2, 415–481. http://dx.doi.org/10.1007/s00031-013-9216-y
[19] A. Vishik and K. Zainoulline, *Motivic splitting lemma*. Doc. Math. **13**(2008), 81–96.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, AB
e-mail: karpenko@ualberta.ca