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Cliquishness and Quasicontinuity of Two-Variable Maps

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Abstract. We study the existence of continuity points for mappings $f: X \times Y \to Z$ whose x-sections $Y \ni y \to f(x, y) \in Z$ are fragmentable and y-sections $X \ni x \to f(x, y) \in Z$ are quasicontinuous, where X is a Baire space and Z is a metric space. For the factor Y, we consider two infinite "point-picking" games $G_1(y)$ and $G_2(y)$ defined respectively for each $y \in Y$ as follows: in the n-th inning, Player I gives a dense set $D_n \subset Y$, respectively, a dense open set $D_n \subset Y$. Then Player II picks a point $y_n \in D_n$; II wins if y is in the closure of $\{y_n : n \in \mathbb{N}\}$, otherwise I wins. It is shown that (i) f is cliquish if II has a winning strategy in $G_1(y)$ for every $y \in Y$, and (ii) f is quasicontinuous if the x-sections of f are continuous and the set of $y \in Y$ such that II has a winning strategy in $G_2(y)$ is dense in Y. Item (i) extends substantially a result of Debs and item (ii) indicates that the problem of Talagrand on separately continuous maps has a positive answer for a wide class of "small" compact spaces.

1 Introduction

Let X be a topological space and (Z, d) be a metric space. A mapping $f: X \to Z$ is said to be *cliquish* [26] if for any $\varepsilon > 0$ and any nonempty open set $U \subset X$ there is a nonempty open set $O \subset U$ such that $d(f(x), f(y)) < \varepsilon$ for all $x, y \in O$. Following [15] (see also [12]), the mapping f is said to be *fragmentable* if the restriction of f to each nonempty subspace of X is cliquish. Fragmentable mappings are said to be of the first class in Debs's paper [7]; given the common meaning of "first class functions", we will adopt here Koumoullis's terminology. Recall also that the mapping f is said to be *quasicontinuous* [14] if for every $\varepsilon > 0$, every $x \in X$, and every neighborhood V of x in X there is a nonempty open set $O \subset V$ such that $d(f(x), f(y)) < \varepsilon$ for each $y \in O$. It is well known (and easily seen) that quasicontinuous mappings are cliquish, and cliquish mappings are continuous at every point of a residual subset of X (and vice versa if X is a Baire space).

There are many studies in the literature, dating back at least to Baire [1], whose purpose is to find conditions (as weak as possible) to insure the existence of continuity points for mappings of two variables. Among them there is the following by Fudali [9]: every mapping $f: X \times Y \to Z$, where X is Baire, Y is second countable, and Z is a metric space such that for every $(x, y) \in X \times Y$ the x-section $f_x: Y \ni y \to f(x, y) \in Z$ is cliquish and the y-section $f^y: X \ni x \to f(x, y) \in Z$ is quasicontinuous, is a cliquish mapping. See also [8] for similar results and [17, 19] for closely related results involving quasicontinuous x-sections. There are easy examples showing that Fudali's result is false for metrizable Y (an example is included

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here); therefore, the following result established by Debs in [7] is of particular interest. Every mapping $f\colon X \times Y \to Z$ whose x-sections are fragmentable and y-sections are continuous is cliquish, provided that X is a "special" Baire space, $X \times Y$ is a Baire space, and Y is first countable. The interested reader is referred to [7] for the precise assumption on X.

In this note Debs's theorem is improved (see Corollary 3.5): if the y-sections of f are quasicontinuous, the x-sections are fragmentable, X is Baire, and the π-character of each point of Y is countable, then f is cliquish. This statement is a special case of one of the main results of this paper (Theorems 3.4 and 3.7), where the problem is considered under some fairly general conditions expressed in terms of two point-picking games played on the factor Y (defined in the next section). The second main result concerns the mappings f whose x-sections are continuous and y-section are quasicontinuous. It states, in particular, that such a mapping is quasicontinuous provided that the space Y has densely many points of countable π-character (Corollary 3.8).

The topic here is closely related to the following problem by Talagrand [25]: let $f\colon X \times Y \to \mathbb{R}$ be a separately continuous mapping, where X is a Baire space and Y is a compact space; is it true that f admits at least a continuity point in $X \times Y$? The reader is referred to [5] for further information about this still-open question. According to Corollary 3.8, we have a positive answer if densely many points of Y (or X) are of countable π-character. In view of the theorem by Juhász and Shelah [13], that is, $\pi_\chi(y, Y) \le t(y, Y)$ for every $y \in Y$, the answer is also positive if the compact Y admits a dense set of points of countable tightness. The definitions of the cardinal numbers $\pi_\chi(y, Y)$ and $t(y, Y)$ are recalled below.

2 Two Games

Let Y be a topological space and $\mathcal{L}$ be a collection of nonempty subsets of Y. For $y \in Y$, we consider the following two persons infinite point-picking game $G(\mathcal{L}, y)$ on Y. Player I begins and gives $L_0 \in \mathcal{L}$, then Player II chooses a point $y_0 \in L_0$; at stage $n \ge 1$, Player I chooses $L_n \in \mathcal{L}$ and then Player II gives a point $y_n \in L_n$. A play $(L_n, y_n)_{n\in\mathbb{N}}$ is won by Player I if $y \in \overline{\{y_n : n \in \mathbb{N}\}}$; otherwise, II wins.

We will be concerned in this game with two different collections $\mathcal{L}$ of subsets of Y, namely, the collection $\mathcal{O}(Y)$ of nonempty open subsets of Y and the collection $\mathcal{A}(Y)$ of somewhere dense subsets of Y. (When the space Y is clearly identified from the context, we shall simply write $\mathcal{O}$ and $\mathcal{A}$.) Recall that a subset $F \subset Y$ is said to be somewhere dense in Y if the interior of its closure $\operatorname{Int}(\overline{F})$ in Y is nonempty. It should be mentioned that if $\mathcal{L}$ is the collection of all neighborhoods of y in Y, then $G(\mathcal{L}, y)$ is the game introduced by Gruenhage in [10]. The game $G(\mathcal{O}, y)$ is the pointwise version of the one introduced by Berner and Juhász in their paper [2] (from which the term "picking-point game" is taken): In the n-th step, Player I gives a nonempty open set $U_n \subset Y$, then II picks a point $y_n \in U_n$; I wins if $\{y_n : n \in \mathbb{N}\}$ is dense in Y.

Following the terminology of [24], the *dual game* $G^*(\mathcal{A}, y)$ of $G(\mathcal{A}, y)$ (respectively, $G^*(\mathcal{O}, y)$ of $G(\mathcal{O}, y)$) on Y is defined as follows: at stage n, Player I gives a dense open set $D_n \subset Y$ (respectively, a dense set $D_n \subset Y$) and then Player II chooses $y_n \in D_n$. Player II wins if $y \in \overline{\{y_n : n \in \mathbb{N}\}}$. Using the techniques of [23], one can

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show that these games are indeed dual, meaning that Player II has a winning strategy in the game $G^*(\mathcal{A}, y)$ (respectively, Player I has a winning strategy in $G^*(\mathcal{A}, y)$) if and only if Player I has a winning strategy in $G(\mathcal{A}, y)$ (respectively, Player II has a winning strategy in $G(\mathcal{A}, y)$). This is also true for the class $\mathcal{O}$.

The next statement and the discussion after its proof show that the difference between the games $G(\mathcal{O}, y)$ and $G(\mathcal{A}, y)$ is significant.

Proposition 2.1 *Let $y \in Y$ and $\mathcal{N}$ be a collection of subsets of Y such that*

(i) *for every neighborhood U of y in Y there is a finite collection $\mathcal{F} \subset \mathcal{N}$ such that $\operatorname{Int}(\bigcap \mathcal{F}) \neq \varnothing$ and $\bigcap \mathcal{F} \subset U$;*

(ii) *the closure of the set $A = \{z \in Y : |\{N \in \mathcal{N} : z \notin N\}| \leq \aleph_0\}$ is a neighborhood of y in Y.*

Then Player I has a winning strategy in the game $G(\mathcal{A}, y)$.

Proof Let us fix a bijective map $\mathbb{N} \ni n \to (\phi(n), \psi(n)) \in \mathbb{N} \times \mathbb{N}$ such that $n > \phi(n)$ for every $n \geq 1$. Put $\tau_y(\varnothing) = A$. For $y_0 \in A$, *i.e.*, the answer of Player II, let $\mathcal{S}_0 = \{S^0_n : n \in \mathbb{N}\}$ be an enumeration of the collection of all sets of the form $\bigcap \mathcal{F}$, where $\mathcal{F}$ is a finite subcollection of $\mathcal{N}_0 = \{N \in \mathcal{N} : y_0 \notin N\}$ (we adopt the convention $\bigcap \varnothing = Y$). Define $\tau_y(y_0) = S^0_0 \cap A$ if $S^0_0 \cap A \in \mathcal{A}$ and $\tau_y(y_0) = A$ otherwise. At stage $n \geq 1$, to define $\tau_y(y_0, \dots, y_n)$ put $\mathcal{N}_n = \{N \in \mathcal{N} : y_n \notin N\}$ and denote by $\mathcal{S}_n = \{S^n_k : k \in \mathbb{N}\}$ the collection of all intersections $\bigcap \mathcal{F}$, where $\mathcal{F}$ is a finite subcollection of $\bigcup_{i \leq n} \mathcal{N}_i$. Then put $\tau_y(y_0, \dots, y_n) = A \cap S^{\phi(n)}_{\psi(n)}$ if $A \cap S^{\phi(n)}_{\psi(n)} \in \mathcal{A}$ and $\tau_y(y_0, \dots, y_n) = A$ otherwise. The definition of τ_y is complete.

To show that τ_y is a winning strategy, let $(y_n)_{n \in \mathbb{N}} \subset Y$ be a play which is compatible with τ_y and let $U \subset \overline{A}$ be a neighborhood of y in Y. There is a finite set $\mathcal{F} \subset \mathcal{N}$ such that $\bigcap \mathcal{F} \subset U$ and $\operatorname{Int}(\bigcap \mathcal{F}) \neq \varnothing$. Let $\mathcal{F}_1 = \mathcal{F} \cap (\bigcup_{n \in \mathbb{N}} \mathcal{N}_n)$. We assume that $\mathcal{F}_1 \neq \varnothing$ (otherwise, $\{y_n : n \in \mathbb{N}\} \subset U$). Put $S = \bigcap \mathcal{F}_1$ and choose $n \in \mathbb{N}$ such that $S = S^{\phi(n)}_{\psi(n)}$; since $\varnothing \neq \operatorname{Int}(\bigcap \mathcal{F}) \subset \overline{A}$, the set $S \cap A$ belongs to $\mathcal{A}$. It follows that $y_{n+1} \in S$, hence $y_{n+1} \in U$ since $y_{n+1} \in N$ for every $N \in \mathcal{F} \setminus \mathcal{F}_1$. ∎

Recall that a network at y in Y is a collection $\mathcal{N}$ of subsets of Y such that every neighborhood of y in Y contains some nonempty member of $\mathcal{N}$. A π-base at y in Y is a network at y, all members of which are open. The space Y is said to have a countable π-character at $y \in Y$, in symbol $\pi_\chi(y, Y) \leq \aleph_0$, if y has a countable π-base in Y.

Proposition 2.1 applies in the case $\pi_\chi(y, Y) \leq \aleph_0$ as well as in many other cases. To illustrate this, let us consider for a cardinal number κ the Cantor cube 2^κ of weight κ. It is well known that $\pi_\chi(y, 2^\kappa) = \kappa$ for every $y \in 2^\kappa$ (see [11]); since 2^κ is a regular space, it follows that if κ is uncountable, then $\mathcal{A}(2^\kappa)$ does not include any countable π-network at any point of 2^κ. Also, if κ is uncountable, then Player II has a winning strategy in the game $G(\mathcal{O}, y)$ for every $y \in 2^\kappa$: It suffices to confront Player II in the dual game $G^*(\mathcal{O}, y)$ to the dense subset of 2^κ given by

$$\Sigma(\overline{y}) = \{z \in 2^\kappa : |\{i \in \kappa : z(i) \neq \overline{y}(i)\}| \leq \aleph_0\},$$

where $\overline{y} = 1 - y$. However, Player I always has a winning strategy in the games

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$G(\mathcal{A}, y)$, $y \in 2^\kappa$. Indeed, for $y \in 2^\kappa$, the collection

$$\mathcal{N} = \{\{z \in 2^\kappa : z(i) = y(i)\} : i \in \kappa\}$$

satisfies the assumption of Proposition 2.1. Observe also that Player I has a winning strategy in the games $G(\mathcal{O}, y)$ played on the dense subspace $\Sigma(0)$ of 2^κ, for every $y \in \Sigma(0)$ [10].

Clearly, if Player I has a winning strategy in the game $G(\mathcal{L}, y)$ on Y, then the collection $\mathcal{L}$ is a π-network at y in Y. (Of course, this holds even if Player II does not have a winning strategy in this game.) The next lemma, needed below, gives us a bit more. Let $Y^{<\omega}$ stand for the set of finite sequences in Y.

Lemma 2.2 *Suppose that Player I has a winning strategy τ in the game $G(\mathcal{L}, y)$. Then, for every neighborhood V of y in Y, Player I has a winning strategy σ in the game $G(\mathcal{L}, y)$ such that $\sigma \subset V$, that is, $\sigma(s) \subset V$ for every $s \in Y^{<\omega}$.*

Proof Fix some $L_0 \in \mathcal{L}$ such that $L_0 \subset V$. For every finite sequence $s = (y_0, \dots, y_n) \in Y^{<\omega}$ such that $y \in \overline{\{y_0, \dots, y_n\}}$ (no separation axiom is assumed), put $\sigma(s) = L_0$. For the remaining sequences in $Y^{<\omega}$, including the empty sequence (that is, the first move of Player I), we proceed as follows. Let $s \in Y^k$ be such a sequence ($k \in \mathbb{N}$). If $\tau(s) \subset V$, put $\sigma(s) = \tau(s)$ and $t_s = \varnothing$. If not, write $s = (y_0, \dots, y_k)$ (if $s \neq \varnothing$) and choose a finite sequence $t_s = (x^s_0, \dots, x^s_{n_s}) \in Y^{n_s}$ such that the sequence (s, t_s) is compatible with τ, $\{x^s_0, \dots, x^s_{n_s}\} \cap V = \varnothing$ and $\tau(s, t_s) \subset V \setminus \overline{\{y_0, \dots, y_k\}}$ (or $\tau(s, t_s) \subset V$ if $s = \varnothing$); such a sequence exists since τ is a winning strategy. Then define $\sigma(s) = \tau(s, t_s)$. The definition of σ is complete.

Let $(y_n)_{n\in\mathbb{N}} \subset Y$ be a sequence which is compatible with σ and let $W \subset V$ be a neighborhood of y in Y. We may suppose that $y \notin \overline{\bigcup_{n\in\mathbb{N}} \{y_0, \dots, y_n\}}$. Put $s_n = (y_0, \dots, y_n)$ ($n \in \mathbb{N}$); then, the sequence $(z_n)_{n\in\mathbb{N}}$ starting with $t_\varnothing$ and obtained from $(y_n)_{n\in\mathbb{N}}$ by inserting each t_{s_n} just after y_n, is compatible with τ. Hence there is $p \in \mathbb{N}$ such that $z_p \in W$; since no term of the sequences $t_\varnothing$ and t_{s_n} ($n \in \mathbb{N}$) belongs to V, $z_p \in \{y_n : n \in \mathbb{N}\}$. ■

3 Main Results

The main results rest on the following proposition. In its proof we shall make use of the description of first category sets in terms of the Banach–Mazur game. For a space X and $R \subset X$, a play in the game $BM(R)$ (on X) is a sequence $(V_n, U_n)_{n\in\mathbb{N}}$ of pairs of nonempty open subsets of X produced alternately by two players β and α as follows: β is the first to move and gives V_0; then Player α gives $U_0 \subset V_0$. At stage $n \geq 1$, the open set $V_n \subset U_n$ being chosen by β, Player α gives $U_n \subset V_n$. Player α wins the play if $\bigcap_{n\in\mathbb{N}} U_n \subset R$. It is well known that X is $BM(R)$-α-favorable (*i.e.,* α has a winning strategy in the game $BM(R)$) if and only if R is a residual subset of X. The reader is referred to [18].

Let us say that the space Y is *fragmented* by $\Delta \subset Y \times Y$ if every nonempty subspace of Y admits a nonempty (relatively) open subset U such that $U \times U \subset \Delta$. In the next statement, X, Y are topological spaces, $(\Delta_x)_{x\in X}$ is an X-indexed collection of subsets of $Y \times Y$ and $\mathcal{L}$ is collection of nonempty subsets of Y such that for every $y \in Y$, Player I has a winning strategy in the game $G(\mathcal{L}, y)$.

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Proposition 3.1 *Let R be a second category subset of X such that Y is fragmented by Δ_x for each $x \in R$. Then for every nonempty open set $V \subset Y$, there exist a nonempty open set $U \subset X$, $y \in V$, and $L \in \mathcal{L}$, with $L \subset V$, such that for every $b \in L$ the set $\{x \in U : (b, y) \in \Delta_x\}$ is dense in U.*

Proof Assume, on the contrary, that the claim is false for some nonempty open set $V \subset Y$, and let us prove that $X \setminus R$ is a residual subset of X, *i.e.*, R is of the first category in X. For each $y \in V$ and $L \in \mathcal{L}$, with $L \subset V$, let $D(y, L)$ be the set of $x \in X$ for which there is $b \in L$ such that $(b, y) \notin \Delta_a$ for every a in some neighborhood of x in X; by our assumption, the open set $D(y, L)$ is dense in X.

For each (x, y, L) (where $(x, y) \in X \times V$, $L \in \mathcal{L}$, and $L \subset V$) such that $x \in D(y, L)$, choose a point $c_L(x, y) \in L$ and an open neighborhood $O(L, x, y)$ of x in X such that $(c_L(x, y), y) \notin \Delta_a$ for every $a \in O(L, x, y)$. Let us fix for each $y \in V$ a winning strategy $\tau_y \subset V$ for Player I in the game $G(\mathcal{L}, y)$ (Lemma 2.2). We shall define a winning strategy σ for Player α in the Banach–Mazur game $BM(X \setminus R)$ on X as follows. Let V_0 be the first move of Player β in the game $BM(X \setminus R)$ and put $\sigma(V_0) = V_0 \cap D_0(y_0, L_0^{y_0})$, where y_0 is an arbitrary (but fixed) point of V and $L_0^{y_0} = \tau_{y_0}(\varnothing)$. Define $F_0 = \{y_0\}$.

At stage 1, if V_1 is the response of β to $\sigma(V_0)$, first choose $x_1 \in V_1$, put $y_1 = c_{L_0^{y_0}}(x_1, y_0)$ and $F_1 = \{y_1\}$. Then define $\sigma(V_0, V_1)$ to be the nonempty open subset of X given by

$$V_1 \cap O(L_0^{y_0}, x_1, y_0) \cap D(y_0, L_1^{y_0}) \cap D(y_1, L_0^{y_1}),$$

where $L_1^{y_0} = \tau_{y_0}(y_1)$ and $L_0^{y_1} = \tau_{y_1}(\varnothing)$.

At stage 2, if V_2 is the response of β to $\sigma(V_0, V_1)$, first choose $x_2 \in V_2$, put $y_2 = c_{L_1^{y_0}}(x_2, y_0)$, $y_3 = c_{L_0^{y_1}}(x_2, y_1)$, and $F_2 = \{y_2, y_3\}$; then define $\sigma(V_0, V_1, V_2)$ to be the nonempty open set given by

$$V_2 \cap O(L_1^{y_0}, x_2, y_0) \cap O(L_0^{y_1}, x_2, y_1) \cap D(y_0, L_2^{y_0}) \cap D(y_1, L_1^{y_1}) \cap \Big[\bigcap_{y \in F_2} D(y, L_0^{y}) \Big],$$

where $L_2^{y_0} = \tau_{y_0}(y_1, y_2)$, $L_1^{y_1} = \tau_{y_1}(y_3)$, and $L_0^{y} = \tau_y(\varnothing)$ for $y \in F_2$.

Continuing inductively, the notations will become more and more complicated but the process allows us to define a strategy σ for Player α in the game $BM(X \setminus R)$ with the following property: To each play $s = (V_n)_{n\in\mathbb{N}}$ for β against σ corresponds a set $F_s = \bigcup_{n\in\mathbb{N}} F_n \subset Y$ such that for each $y \in F_s$ there is a play $(y_n)_{n\in\mathbb{N}} \subset F_s$ of Player II in the game $G(\mathcal{L}, y)$ against the strategy τ_y such that $(y_n, y) \notin \Delta_x$ for every $x \in \bigcap_{n\in\mathbb{N}} V_n$ and $n \in \mathbb{N}$.

To conclude, let $s = (V_n)_{n\in\mathbb{N}}$ be a play for Player β against σ and let us show that $\bigcap_{n\in\mathbb{N}} V_n \subset X \setminus R$. Let $x \in \bigcap_{n\in\mathbb{N}} V_n$ and suppose that $x \in R$. There is an open set $O \subset Y$ such that $O \cap F_s \neq \varnothing$ and $(O \times O) \cap (F_s \times F_s) \subset \Delta_x$. Let $y \in O \cap F_s$; since $y \in \overline{\{y_n : n \in \mathbb{N}\}}$, there is $n \in \mathbb{N}$ such that $(y_n, y) \in \Delta_x$, which is a contradiction. ■

Throughout the rest of the paper, $f\colon X \times Y \to Z$ is a mapping, where (Z, d) is a metric space. Let $\varepsilon > 0$. We shall apply Proposition 3.1 to the collection of subsets of $Y \times Y$ of the form $\Delta_x = \{(y, z) \in Y \times Y : d(f(x, y), f(x, z)) < \varepsilon\}$, $x \in X$. Clearly, the "ε-fragmentability" of the mapping f_x for $x \in X$ as defined in the introduction means that Y is fragmented by Δ_x.

Remark 3.2 Let $y \in Y$ be such that f_x is continuous at y for every $x \in R$ (notations of Proposition 3.1).

(i) If Player I has a winning strategy τ_y in the game $G(\mathcal{L}, y)$ on Y, then, involving only the strategy τ_y, the same method in the above proof allows us to establish the following assertion:

$$(*) \quad \text{For every } \varepsilon > 0 \text{ and any neighborhood } V \text{ of } y \in Y\text{, there is a nonempty open set } U \subset X \text{ and } L \in \mathcal{L} \text{ with } L \subset V \text{ such that for each } b \in L\text{, the set } \{x \in U : d(f(x,b), f(x,y)) < \varepsilon\} \text{ is dense in } U.$$

(ii) If y has a countable network $\mathcal{N} \subset \mathcal{L}$, then the above property $(*)$ can be proved easily as follows: Proceeding by contradiction as in the proof of Proposition 3.1, since the open subset $D(y, L)$ of X is dense in X for every $L \in \mathcal{L}$ with $L \subset V$, there is $x \in R$ such that $x \in D(y, L)$ for every $L \in \mathcal{N}$ with $L \subset V$. This gives a countable set $\{y_n : n \in \mathbb{N}\} \subset Y$ such that $y \in \overline{\{y_n : n \in \mathbb{N}\}}$ and $d(f(x, y_n), f(x, y)) \geq \varepsilon$ for every $n \in \mathbb{N}$, which is absurd since f_x is continuous at y.

The following interesting concept is formulated in [16] (quite similar concepts were studied by K. Bögel [3,4]): The mapping $f \colon X \times Y \to Z$ is said to be horizontally continuous at $(a, b) \in X \times Y$ if for every neighborhood W of $f(a, b)$ in Z and every neighborhood $U \times V$ of (a, b) in $X \times Y$, there are a nonempty open set $O \subset U$ and $y \in V$ such that $f(O \times \{y\}) \subset W$.

The mapping $f \colon X \times Y \to Z$ is said to be *lower quasicontinuous* ([6]) with respect to the variable x at the point $(a, b) \in X \times Y$ if for every neighborhood W of $f(a, b)$ in Z and every neighborhood $U \times V$ of (a, b) in $X \times Y$, there is a nonempty open set $O \subset U$ such that for each $x \in O$ there is $y \in V$ such that $f(x, y) \in W$. Lower quasicontinuity with respect to the variable y is defined similarly. Note that the quasicontinuity of f^b at $a \in X$ implies that f is horizontally quasicontinuous at (a, b), which in turn implies that f is lower quasicontinuous with respect to the variable x at (a, b).

We continue to assume (in Proposition 3.3 and Theorem 3.4 below) that for every $y \in Y$, Player I has a winning strategy in the game $G(\mathcal{L}, y)$.

Proposition 3.3 *Suppose that f_x is fragmentable for each x in a second category set R in X, f^y is cliquish for every $y \in Y$, and f is horizontally quasicontinuous. Let $V \subset Y$ be a nonempty open set. Then there exist $b \in Y$ and a nonempty open set $O \times W \subset X \times V$ such that $d(f(x, y), f(x', b)) \leq \varepsilon$ for every $x, x' \in O$ and $y \in W$, in each of the following:*

(i) $\mathcal{L} = \mathcal{O}$.

(ii) $\mathcal{L} = \mathcal{A}$, *$f^y$ is quasicontinuous for every $y \in Y$ and f is lower quasicontinuous with respect to the variable y.*

Proof By Proposition 3.1, there are $b \in V$, a nonempty open set $U \subset X$, and $L \in \mathcal{L}$ with $L \subset V$, such that for every $y \in L$ the set

$$D_y = \{x \in U : d(f(x, y), f(x, b)) \leq \varepsilon/2\}$$

is dense in U. Since f^b is cliquish in both cases, there is a nonempty open set $O \subset U$ such that $\operatorname{diam}(f^b(O)) \leq \varepsilon/2$.

To prove (i), we take $W = L$. Suppose that $d(f(x_0, y_0), f(x_1, b)) > \varepsilon$ for some $x_0, x_1 \in O$ and $y_0 \in L$. Since f is horizontally quasicontinuous at (x_0, y_0) and L is open, there is a nonempty open set $O_1 \subset O$ and $y_1 \in L$ such that $d(f(a, y_1), f(x_1, b)) > \varepsilon$ for every $a \in O_1$. Let $a_1 \in O_1 \cap D_{y_1}$; it follows from $d(f(a_1, y_1), f(a_1, b)) \leq \varepsilon/2$ that $d(f(a_1, b), f(x_1, b)) > \varepsilon/2$, which is a contradiction.

To prove (ii), we take $W = \operatorname{Int}(\overline{L} \cap V)$; since $L \in \mathcal{A}$ and $L \subset V$, W is nonempty. Suppose that $d(f(x_0, y_0), f(x_1, y)) > \varepsilon$ for some $x_0, x_1 \in O$ and $y_0 \in W$. There is a nonempty open set $W_1 \subset W$ such that for each $y \in W_1$ there is $a \in O$ such that $d(f(a, y), f(x_1, b)) > \varepsilon$; taking $y_1 \in W_1 \cap L$, we obtain $d(f(a_1, y_1), f(x_1, b)) > \varepsilon$ for some $a_1 \in O$. Since f^{y_1} is quasicontinuous, there is a nonempty open set $O_1 \subset O$ such that $d(f(a, y_1), f(x_1, b)) > \varepsilon$ for every $a \in O_1$; as in the proof of (i), taking $a \in O_1 \bigcap D_{y_1}$ gives the contradiction $d(f(a, b), f(x_1, b)) > \varepsilon/2$. ■

Now we state the first main result of this note.

Theorem 3.4 *Suppose that f is horizontally quasicontinuous, f^y is cliquish for every $y \in Y$, and f_x is fragmentable for each x in the Baire space X. Then f is cliquish in each of the following:*

(i) $\mathcal{L} = \mathcal{O}$;

(ii) $\mathcal{L} = \mathcal{A}$, *$f^y$ is quasicontinuous for every $y \in Y$, and f is lower quasicontinuous with respect to the variable y.*

Proof Let $U \times V \subset X \times Y$ be a nonempty open set and $\varepsilon > 0$. By Proposition 3.3 (for $X = R = U$), there are $b \in Y$ and a nonempty open set $O \times W \subset U \times V$ such that $d(f(x, y), f(x', b)) \leq \varepsilon$ for every $x, x' \in O$ and $y \in W$. For every $x, x' \in O$ and $y, y' \in W$, we have

$$d(f(x, y), f(x', y')) \leq d(f(x, y), (x', b)) + d(f(x', b), f(x', y')) \leq 2\varepsilon. \qquad ■$$

In view of Proposition 2.1 and Theorem 3.4(i), we obtain the following improvement of Debs's result mentioned in the introduction.

Corollary 3.5 *Suppose that for each $y \in Y$, $\pi_\chi(y, Y) \leq \aleph_0$ and f^y is quasicontinuous, and X is a Baire space and f_x is fragmentable for each $x \in X$. Then f is cliquish.*

The concept of cliquish mapping extends in a natural way to mappings taking their values in uniform spaces. Therefore, Theorem 3.4 and Corollary 3.5 hold more generally for every uniform space Z. Let us also note that the assumption on the y-sections of f in Theorem 3.4 allows us to assume in this statement that the x-sections are fragmentable for x belonging to a dense Baire subspace of X.

If the x-sections of the mapping f are continuous, then the cliquishness of f in Theorem 3.4 can be significantly improved, as we propose to show in what follows. We need a variant of Proposition 3.3.

Proposition 3.6 *Suppose that f_x is continuous for each $x \in X$, f^y is quasicontinuous for every $y \in Y$ and X is a Baire space. Let $b \in Y$ be such that Player I has a winning strategy in the game $G(\mathcal{A}, b)$ on Y and let V be a neighborhood of b in Y. Then, for every nonempty open set $U \subset X$, there is a nonempty open set $O \times W \subset U \times V$ such that $d(f(x,y), f(x,b)) \leq \varepsilon$ for every $(x,y) \in O \times W$.*

Proof By Remark 3.2(i), there are $L \in \mathcal{A}$, with $L \subset V$, and a nonempty open set $O \subset U$ such that for every $y \in L$ the set $\{x \in U : d(f(x,y), f(x,b)) \leq \varepsilon/2\}$ is dense in U. It remains to follow the proof of Proposition 3.3(ii) (more simply, because here the f_x's are continuous). ■

The following is a variant of Theorem 3.4(ii) (the assumption on the x-sections of f is strengthened, but there are fewer constraints on Y and the conclusion is stronger).

Theorem 3.7 *Suppose that for each point y in a dense subset of Y, Player I has a winning strategy in the game $G(\mathcal{A}, y)$. If X is a Baire space, f_x is continuous for every $x \in X$ and f^y is quasicontinuous for every $y \in Y$, then f is quasicontinuous.*

Proof Let $(a,b) \in X \times Y$, $U \times V$ be a neighborhood of $(a,b) \in X \times Y$, and $\varepsilon > 0$. We may suppose that $d(f(a,y), f(a,b)) < \varepsilon$ for every $y \in V$. Let $c \in V$ be such that Player I has a winning strategy in the game $G(\mathcal{A}, c)$. Since f^c is quasicontinuous, there is a nonempty open set $O_1 \subset U$ such that $d(f(x,c), f(a,c)) < \varepsilon$ for every $x \in O_1$. Let $O_2 \times W \subset O_1 \times V$ be a nonempty open set such that $d(f(x,y), f(x,c)) < \varepsilon$ for every $(x,y) \in O_2 \times W$ (Proposition 3.6). For every $(x,y) \in O_2 \times W$, we have

$$\begin{aligned} d(f(x,y), f(a,b)) &\leq d(f(x,y), f(x,c)) + d(f(x,c), f(a,c)) + d(f(a,c), f(a,b)) \\ &\leq 3\varepsilon. \end{aligned}$$ ■

The following is a consequence of Theorem 3.7; it can be also obtained (directly and more simply) by using Remark 3.2(ii).

Corollary 3.8 *Suppose that for every y in a dense subset of Y, the collection $\mathcal{A}$ includes a countable network at y in Y. If X is a Baire space, f^y is quasicontinuous for each $y \in Y$ and f_x is continuous for each $x \in X$, then f is quasicontinuous.*

Corollary 3.8 shows that the question of Talagrand [25] mentioned in the introduction has a positive answer if one of the two spaces X and Y has a dense subset of points of countable π-character (since Y is compact, the product $X \times Y$ is Baire, hence the quasicontinuous mapping $f \colon X \times Y \to Z$ has at least a continuity point). Related to this, let us recall from the theorem of Šapirovskiĭ [22] that all compact spaces Y that cannot be continuously mapped onto the Tychonoff cube $[0,1]^{\omega_1}$ satisfy the conditions of Corollary 3.8. This is also the case of all hereditarily normal compact spaces, by another result of Šapirovskiĭ [20, 21]. Taking into account the theorem of Juhász and Shelah [13] that $\pi_\chi(y, Y) \leq t(y, Y)$ for every y in the compact space Y, the answer to Talagrand's question is also positive if Y has a dense set of points of countable tightness. The tightness $t(y, Y)$ of y in Y is the smallest cardinal κ such that whenever $y \in \overline{A}$, $A \subset Y$, there is a set $B \subset A$ with $|B| \leq \kappa$ such that $y \in \overline{B}$.

Example 3.9 In conclusion, we return to the question raised in the introduction whether it is possible to assume in the main results that the x-sections of the mappings $f : X \times Y \to Z$ are only cliquish, as is the case if the factor Y is (locally) second countable [9]. Unfortunately, this is not possible even for metrizable Y. To show this, let Y be a metrizable space such that the interior of every separable subspace of Y is empty and take X to be the countably compact (hence Baire) subspace $X = \Sigma(0)$ of the Cantor space 2^Y. Then the evaluation mapping $X \times Y \ni (x, y) \to x(y) \in \{0, 1\}$ is cliquish in the variable y, continuous in the variable x, but not cliquish if Y is dense in itself. Furthermore, inverting the roles of X and Y, and taking Y to be a Baire space, *e.g.,* completely metrizable, we obtain an example showing that the assumption concerning the y-sections of the mapping f in Theorems 3.4 and 3.7 cannot be replaced by the lower quasicontinuity of f with respect to the variable x.

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