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Value Distribution of the Riemann Zeta Function

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Abstract. In this note, we give a new short proof of the fact, recently discovered by Ye, that all (finite) values are equidistributed by the Riemann zeta function.

In recent years, the general value distribution of the Riemann zeta function has been investigated from the point of view of Nevanlinna theory by Ye Zhuan [6], and subsequently by Liao Liangwen and Yang Chung-Chun [2]. It was shown by Ye that the distribution of a-values is the same for all finite values a, in the sense that no finite value is Nevanlinna deficient. Thus, if some aspect of value distribution can be shown to hold for generic a-values, or even for one particular a-value, then there is reasonable hope that it holds also for the zeros of the Riemann zeta function. In this note, we give a new short proof that no finite value is deficient for the Riemann zeta function. Our proof employs properties of the distribution of zeros as well as their multiplicities, for the Riemann zeta function. Hence, we use distribution of zeros to gain knowledge on general value distribution with the hope that the debt can eventually be repaid.

We shall employ standard notation regarding Nevanlinna theory and the Riemann zeta function.

One can compute directly the deficiency of the value 0, but this is not necessary since the following theorem asserts that the deficiencies of all finite values vanish.

Theorem 1 *The Riemann zeta function has no finite deficient values.*

First of all, the deficiency of the value ∞ for the Riemann zeta function is one.

If f is a meromorphic function in the plane, following Nevanlinna, we set

$$N_1(r, f) = N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f').$$

We shall write $N_1(r)$ instead of $N_1(r, f)$ when the function f is clear from context.

From [4], we have that the inequality

$$\sum_1^q \delta(f, a_\nu) + \varliminf_{r\to\infty} \frac{N_1(r, f)}{T(r, f)} \leq 2$$

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is valid for any function f meromorphic in the plane.

Since we already know that $\delta(\zeta,\infty) = 1$, in order to prove the theorem it is sufficient to prove that

$$\underline{\lim}_{r\to\infty} \frac{N_1(r,\zeta)}{T(r,\zeta)} \geq 1. \tag{1}$$

The only pole of ζ' is a pole of order two in $z = 1$. Hence, $2N(r,\zeta) - N(r,\zeta') = 0$ and consequently,

$$N_1(r,\zeta) = N\Big(r, \frac{1}{\zeta'}\Big).$$

Now, we have to study $N(r, \frac{1}{\zeta'})$, which is equivalent to studying the zeros of $\zeta'(z)$. Berndt [1][1] has shown that if $k \geq 1$, then if $T \to \infty$,

$$\mathcal{N}_k(T) = \frac{1}{2\pi} T \log T - \Big(\frac{1 + \log 4\pi}{2\pi}\Big) T + O(\log T),$$

where $\mathcal{N}_k(T)$ counts the number of zeros of $\zeta^{(k)}(s)$ such that $0 < t < T$, where $t = \mathrm{Im}(s)$.

The function $\mathcal{N}_k(T)$ counts only the zeros whose imaginary part lies in $0 < t < T$. But $\zeta'(\bar{z}) = \overline{\zeta'(z)}$, so if $\zeta'(z) = 0$, then $\zeta'(\bar{z}) = 0$. It follows that $2\mathcal{N}_k(T)$ counts the number of zeros of $\zeta^{(k)}(s)$, with imaginary part t, such that $t \in (-T, T) \setminus \{0\}$.

We want to prove that $n(r, 1/\zeta') \geq \frac{r}{\pi} \log r + O(\log r)$. A theorem of Verma [5] asserts that for each integer $k \geq 0$, there exists an r_k such that $\zeta^{(k)}(s)$ has no non-real zero in the region $Re(s) \leq 0$ except at most a finite number in the semi-circle $|s| \leq r_k$. Thus, for k=1, there exists $\sigma_1 = r_1$ such that there is no non-real zero in the left half-plane $Re(s) < -\sigma_1$. Moreover, since $\zeta'(s)$ is a Dirichlet series, there exists α_1 such that for all s in $Re(s) > \alpha_1$, $\zeta'(s) \neq 0$. This also follows from the lower estimate for $|\zeta'(s)|$ [5, p. 218]. Let us denote $\gamma_1 = \max(\sigma_1, \alpha_1)$. Hence, all the non-real zeros of ζ' are such that $|Re(s)| < \gamma_1$.

If we inscribe a rectangle in a circle of radius r, with $r > \gamma_1$, we find that

$$n\Big(r, \frac{1}{\zeta'}\Big) \geq \frac{(r^2 - \gamma_1^2)^{1/2}}{\pi} \log(r^2 - \gamma_1^2)^{1/2} + 0(r).$$

It follows that

$$N_1(r,\zeta) = N\Big(r, \frac{1}{\zeta'}\Big) \geq \frac{r}{\pi} \log r - \frac{\gamma_1}{\pi} \log \gamma_1.$$

Since

$$T(r,\zeta) = \frac{r \log r}{\pi} + O(r),$$

we have (1) and so we can conclude that the only deficient value of the Riemann zeta function is the value infinity.

[1] The referee has pointed out that Liao and Ye used Berndt's result in their recent publication [3]

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