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Some Norm Inequalities for Operators

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Abstract. Let A_i, B_i and X_i $(i = 1, 2, \ldots, n)$ be operators on a separable Hilbert space. It is shown that if f and g are nonnegative continuous functions on $[0, \infty)$ which satisfy the relation $f(t)g(t) = t$ for all t in $[0, \infty)$, then

$$\left|\left|\left| \left|\sum_{i=1}^{n} A_i^* X_i B_i\right|^r \right|\right|\right|^2 \leq \left|\left|\left| \Big(\sum_{i=1}^{n} A_i^* f(|X_i^*|)^2 A_i\Big)^r \right|\right|\right| \left|\left|\left| \Big(\sum_{i=1}^{n} B_i^* g(|X_i|)^2 B_i\Big)^r \right|\right|\right|$$

for every $r > 0$ and for every unitarily invariant norm. This result improves some known Cauchy-Schwarz type inequalities. Norm inequalities related to the arithmetic-geometric mean inequality and the classical Heinz inequalities are also obtained.

1 Introduction

Let $B(H)$ denote the space of bounded linear operators on a separable Hilbert space H. Let $|||\,.\,|||$ denote a unitarily invariant norm defined on a norm ideal associated with it. For the sake of brevity, we will make no explicit mention of this ideal. Thus when we consider $|||T|||$ we are assuming that the operator T belongs to the norm ideal associated with $|||\,.\,|||$. For the theory of unitarily invariant norms we refer to [11], [24] or [25].

It has been shown by Bhatia and Kittaneh in [7] that if A, B are operators in $B(H)$, then

$$2|||A^*B||| \leq |||AA^* + BB^*|||, \tag{1}$$

$$|||A^*B + B^*A||| \leq |||A^*A + B^*B||| \tag{2}$$

for every unitarily invariant norm. These inequalities can be considered as noncommutative versions of the familiar arithmetic-geometric mean inequality for real numbers.

The inequality (1) has attracted the attention of several mathematicians, and different proofs of a stronger version of it have been given. See [4], [15], [19] and [22]. This stronger version asserts that if A, B and X are operators in $B(H)$, then

$$2|||A^*XB||| \leq |||AA^*X + XBB^*||| \tag{3}$$

for every unitarily invariant norm. The usual operator norm version of (3) has been proved earlier in [23].

The inequality (2) has been obtained in [7] as an application of the following Cauchy-Schwarz inequality. If A, B are operators in $B(H)$, then

$$|||A^*B|||^2 \leq |||AA^*|||\,|||BB^*||| \tag{4}$$

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for every unitarily invariant norm. See [2], [13], [14] or [26]. This inequality is a special case of the following inequality which has been proved by Horn and Mathias in [14, Example 3.1] for the finite-dimensional case. See also [2]. If A, B are operators in $B(H)$, then

$$|||\,|A^*B|^r|||^2 \le |||(AA^*)^r|||\;|||(BB^*)^r||| \tag{5}$$

for every positive real number r and for every unitarily invariant norm. Here $|T|$ stands for the positive (semidefinite) operator $(T^*T)^{1/2}$.

A recent stronger version of (5), which has been given in [5], asserts that if A, B and X are operators in $B(H)$, then

$$|||\,|A^*XB|^r|||^2 \le |||\,|AA^*X|^r|||\;|||\,|XBB^*|^r||| \tag{6}$$

for every positive real number r and for every unitarily invariant norm. For the usual operator norm and the case $r = 1$, the inequality (6) has been also obtained earlier in [23].

The inequalities (3)–(6) have been very useful in applications, especially to perturbation inequalities and geometric inequalities for operators. In addition to this, the importance of the inequalities (3)–(6) also stems from the fact that these inequalities are closely related to some classical inequalities due to Heinz. See [1], [2], [3], [5], [10], [12], [16], [20] and [23].

In this paper we present a two-fold improvement of the inequality (2). In Section 2 we invoke the arithmetic-geometric mean inequality (1) to refine the inequality (2). In Section 3 we give a stronger version of (2) by inserting a positive operator X in the proper places in (2). This is achieved by utilizing a basic Cauchy-Schwarz type inequality in [14, Theorem 2.3], from which the inequality (5) has been derived. As in the case of the inequalities (3) and (6), our stronger version of (2) seems natural enough to be widely useful. In Section 3 we establish two inequalities that are equivalent to two of the classical Heinz inequalities.

2 A Refinement of the Inequality (2)

To refine the inequality (2) we need the following folk lemma. For the reader's convenience, we provide a proof of it.

Lemma 1 *If A, B are operators in $B(H)$, then*

$$\left|\left|\left|\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right|\right|\right| = \left|\left|\left|\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}\right|\right|\right|$$

for every unitarily invariant norm.

Proof Let $U = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} I & -I \\ I & I \end{smallmatrix}\right]$, where I is the identity operator in $B(H)$. Then U is unitary and

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} = U \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix} U^*.$$

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The result now follows by the unitary invariance of $|||\,.\,|||$.

Based on the inequality (1) and Lemma 1, we have the following refinement of (2).

Theorem 1 *If A, B are operators in $B(H)$, then*

$$(7)\qquad 2\left|\left|\left|\begin{bmatrix} A^*B+B^*A & 0\\ 0 & 0\end{bmatrix}\right|\right|\right| \le \left|\left|\left|\begin{bmatrix} (A+B)^*(A+B) & 0\\ 0 & (A-B)^*(A-B)\end{bmatrix}\right|\right|\right|$$

for every unitarily invariant norm.

Proof Let $T = \left[\begin{smallmatrix} A & 0\\ B & 0\end{smallmatrix}\right]$, $S = \left[\begin{smallmatrix} B & 0\\ A & 0\end{smallmatrix}\right]$. Then by (1) we have $2|||T^*S||| \le |||TT^* + SS^*|||$. Thus

$$2\left|\left|\left|\begin{bmatrix} A^*B+B^*A & 0\\ 0 & 0\end{bmatrix}\right|\right|\right| \le \left|\left|\left|\begin{bmatrix} AA^*+BB^* & AB^*+BA^*\\ BA^*+AB^* & AA^*+BB^*\end{bmatrix}\right|\right|\right|.$$

Using Lemma 1, we have

$$\begin{aligned} 2\left|\left|\left|\begin{bmatrix} A^*B+B^*A & 0\\ 0 & 0\end{bmatrix}\right|\right|\right| &\le \left|\left|\left|\begin{bmatrix} AA^*+BB^*+AB^*+BA^* & 0\\ 0 & AA^*+BB^*-AB^*-BA^*\end{bmatrix}\right|\right|\right|\\ &= \left|\left|\left|\begin{bmatrix} (A+B)(A+B)^* & 0\\ 0 & (A-B)(A-B)^*\end{bmatrix}\right|\right|\right|\\ &= \left|\left|\left|\begin{bmatrix} (A+B)^*(A+B) & 0\\ 0 & (A-B)^*(A-B)\end{bmatrix}\right|\right|\right|, \end{aligned}$$

since $|||RR^*||| = |||R^*R|||$ for every operator R. The proof of (7) is now complete.

To see how the inequality (7) is an improvement of (2), we need to recall that for positive operators C, D in $B(H)$ we have

$$(8)\qquad \left|\left|\left|\begin{bmatrix} C & 0\\ 0 & D\end{bmatrix}\right|\right|\right| \le \left|\left|\left|\begin{bmatrix} C+D & 0\\ 0 & 0\end{bmatrix}\right|\right|\right|$$

for every unitarily invariant norm. See [6, Theorem 1] or [8, Lemma 4]. Since$(A+B)^*(A+B) + (A-B)^*(A-B) = 2(A^*A + B^*B)$, it follows from (7) and (8) that

$$\begin{aligned} 2\left|\left|\left|\begin{bmatrix} A^*B+B^*A & 0\\ 0 & 0\end{bmatrix}\right|\right|\right| &\le \left|\left|\left|\begin{bmatrix} (A+B)^*(A+B) & 0\\ 0 & (A-B)^*(A-B)\end{bmatrix}\right|\right|\right|\\ &\le 2\left|\left|\left|\begin{bmatrix} A^*A+B^*B & 0\\ 0 & 0\end{bmatrix}\right|\right|\right|, \end{aligned}$$

which clearly refines (2).

Specializing (7) to the usual operator norm and the Schatten p-norms, we obtain

$$(9)\qquad 2\|A^*B+B^*A\| \le \max\left(\|A+B\|^2, \|A-B\|^2\right),$$

$$(10)\qquad 2^p\|A^*B+B^*A\|_p^p \le \|A+B\|_{2p}^{2p} + \|A-B\|_{2p}^{2p}$$

for all p with $1 \le p < \infty$.

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3 A Stronger Version of the Inequality (2)

To obtain a stronger version of (2) along the lines of (3) and (6), we need the following basic inequality, which has been established in [14, Theorem 23] for the finite-dimensional case. However, a slight modification of the argument in [14] extends the result to the infinite-dimensional case.

Lemma 2 *Let A, B and C be operators in $B(H)$ with A, B positive. If the operator matrix* $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ *is positive, then*

$$|||\,|C|^r\,|||^2 \le |||A^r|||\;|||B^r||| \tag{11}$$

for every $r > 0$ and for every unitarily invariant norm.

As mentioned in Section 1, the inequality (5) follows as a consequence of (11). It should be mentioned here that the proof of (6) given in [5] depends on (5), the polar decomposition of an operator, and some basic majorization relations between eigenvalues and singular values of compact operators. Here we give a natural generalization of (5) by applying (11) to certain positive operators.

Lemma 3 *Let A, B and X be operators in $B(H)$. If f and g are nonnegative continuous functions on $[0,\infty)$ which satisfy the relation $f(t)g(t) = t$ for all t in $[0,\infty)$, then*

$$|||\,|A^*XB|^r\,|||^2 \le |||\big(A^* f(|X^*|)^2 A\big)^r|||\;|||\big(B^* g(|X|)^2 B\big)^r||| \tag{12}$$

for every $r > 0$ and for every unitarily invariant norm.

Proof Let $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $Y = \begin{bmatrix} f(|X^*|)^2 & X \\ X^* & g(|X|)^2 \end{bmatrix}$. Then Y is positive (see [18, Theorem 1]) and so

$$\begin{bmatrix} A^* f(|X^*|)^2 A & A^*XB \\ B^*X^*A & B^*g(|X|)^2 B \end{bmatrix} = T^*YT$$

is positive. The result now follows by invoking Lemma 2.

Observe that important special cases follow from (12) by letting $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, where α is a real number with $0 \le \alpha \le 1$. In particular, the case $f(t) = g(t) = t^{1/2}$ ensures that

$$|||\,|A^*XB|^r\,|||^2 \le |||(A^*|X^*|A)^r|||\;|||(B^*|X|B)^r|||. \tag{13}$$

Using the fact that $|||(T^*T)^r||| = |||(TT^*)^r|||$ for every operator T and for every $r > 0$, we remark that our inequality (13) is equivalent to the inequality (13) in [5]. Moreover, as one can infer from the proof of (6) given in [5], the right hand side of (13) is dominated by that of (6).

Now we are in a position to establish a general Cauchy-Schwarz inequality, from which we obtain our promised stronger version of (2).

Theorem 2 *Let A_i, B_i and X_i $(i = 1, 2, \dots, n)$ be operators in $B(H)$. If f and g are as in Lemma 3, then*

$$|||\,\Big|\sum_{i=1}^n A_i^* X_i B_i\Big|^r\,|||^2 \le |||\Big(\sum_{i=1}^n A_i^* f(|X_i^*|)^2 A_i\Big)^r|||\;|||\Big(\sum_{i=1}^n B_i^* g(|X_i|)^2 B_i\Big)^r||| \tag{14}$$

for every $r > 0$ and for every unitarily invariant norm.

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Proof Let

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ B_n & 0 & \cdots & 0 \end{bmatrix}, \text{ and}$$

$$X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix}.$$

Then

$$A^*XB = \begin{bmatrix} \sum_{i=1}^n A_i^* X_i B_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$A^* f(|X^*|)^2 A = \begin{bmatrix} \sum_{i=1}^n A_i^* f(|X_i^*|)^2 A_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \text{and}$$

$$B^* g(|X|)^2 B = \begin{bmatrix} \sum_{i=1}^n B_i^* g(|X_i|)^2 B_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Now the inequality (14) follows from (12).

As an application of Theorem 2 we get the following corollary.

Corollary 1 *If A, B and X are operators in $B(H)$, then*

$$\text{(15)} \qquad |||\,|A^*XB + B^*XA|^r\,|||^2 \le |||(A^*|X^*|A + B^*|X^*|B)^r|||\;|||(A^*|X|A + B^*|X|B)^r|||$$

for every $r > 0$ and for every unitarily invariant norm.

Proof This follows from Theorem 2 by letting $n = 2$, $f(t) = g(t) = t^{1/2}$, $A_1 = A$, $A_2 = B$, $B_1 = B$, $B_2 = A$, and $X_1 = X_2 = X$.

Our stronger version of (2) can be stated as follows.

Corollary 2 *If A, B and X are operators in $B(H)$ with X normal, then*

$$\text{(16)} \qquad |||\,|A^*XB + B^*XA|^r\,||| \le |||(A^*|X|A + B^*|X|B)^r|||$$

for every $r > 0$ and for every unitarily invariant norm. In particular, if X is positive, then

$$\text{(17)} \qquad |||A^*XB + B^*XA||| \le |||A^*XA + B^*XB|||$$

for every unitarily invariant norm.

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Proof The inequality (16) follows from (15) since $|X| = |X^*|$. The inequality (17) is a special case of (16).

We conclude this section with the following remarks.

Remark 1 The positivity assumption on X is essential for (17) to hold. Without this assumption, (17) is false even for self-adjoint operators X. To see this, consider the example $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, and $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\|A^*XB + B^*XA\| = 1 + \sqrt{2}$ and $\|A^*XA + B^*XB\| = 2$.

Remark 2 Using the fact that $|||T^*||| = |||T|||$ for every operator T, one can conclude from (3) that if A, B and X are operators in $B(H)$ with X self-adjoint, then

$$|||A^*XB + B^*XA||| \leq |||AA^*X + XBB^*||| \tag{18}$$

for every unitarily invariant norm. For the general case where X is not necessarily self-adjoint, replace A, B and X in (18) by $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$, and $\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}$, respectively, to get

$$\left|\left|\left| \begin{bmatrix} 0 & A^*XB + B^*XA \\ A^*X^*B + B^*X^*A & 0 \end{bmatrix} \right|\right|\right| \leq \left|\left|\left| \begin{bmatrix} 0 & AA^*X + XBB^* \\ AA^*X^* + X^*BB^* & 0 \end{bmatrix} \right|\right|\right|, \tag{19}$$

which is a generalization of (18).

When specialized to the usual operator norm and the Schatten p-norms, the inequality (19) yields

$$\|A^*XB + B^*XA\| \leq \max\big(\|AA^*X + XBB^*\|, \|AA^*X^* + X^*BB^*\|\big), \tag{20}$$

$$2\|A^*XB + B^*XA\|_p^p \leq \|AA^*X + XBB^*\|_p^p + \|AA^*X^* + X^*BB^*\|_p^p \tag{21}$$

for all p with $1 \leq p < \infty$.

Remark 3 In view of the inequalities (17) and (18), it is reasonable to conjecture that if A, B and X are operators in $B(H)$ with X positive, then

$$|||A^*XB + B^*XA||| \leq |||A^*AX + XB^*B|||, \tag{22}$$

$$|||A^*XB + B^*XA||| \leq |||AXA^* + BXB^*||| \tag{23}$$

for every unitarily invariant norm. However, this conjecture is refuted by the example $A = X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, for which $\|A^*XB + B^*XA\|_1 = 2$ and $\|A^*AX + XB^*B\|_1 = \|AXA^* + BXB^*\|_1 = 1$.

Remark 4 It has been remarked in [5] that if A, B are operators in $B(H)$ with AB normal, then

$$|||AB||| \leq |||BA||| \tag{24}$$

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for every unitarily invariant norm. Moreover, it has been mentioned in [5] that another proof of the case $r = 1$ of (6) could be given based on (24), the polar decomposition of X, and (4).

It should be remarked here that the argument used in [5] to derive (24) (see also Theorem 8.1 in [25]) yields the following generalization of (24). If A, B are operators in $B(H)$ with AB normal, then

$$|||\,|AB|^r\,||| \leq |||\,|BA|^r\,||| \tag{25}$$

for every $r > 0$ and for every unitarily invariant norm. We conclude this section by presenting a short proof of (6) based on (5) and (25). The proof goes like this:

$$\begin{aligned} |||\,|A^*XB|^r\,|||^2 &= |||(B^*X^*AA^*XB)^{r/2}|||^2 \\ &\leq |||\,|X^*AA^*XBB^*|^{r/2}\,|||^2 \quad \text{(by (25))} \\ &\leq |||\,|AA^*X|^r\,|||\;|||\,|XBB^*|^r\,||| \quad \text{(by (5))}. \end{aligned}$$

Remark 5 As an immediate consequence of the arithmetic-geometric mean inequality (3), it has been shown in [21] that if R, S and T are operators in $B(H)$ with R, S invertible, then

$$2|||T||| \leq |||R^*TS^{-1} + R^{-1}TS^*||| \tag{26}$$

for every unitarily invariant norm. See also [1] and references therein.

In the same mould, the Cauchy-Schwarz inequality (6) leads to the related inequality

$$|||\,|T|^r\,|||^2 \leq |||\,|R^*TS^{-1}|^r\,|||\;|||\,|R^{-1}TS^*|^r\,||| \tag{27}$$

for every $r > 0$ and for every unitarily invariant norm.

The inequality (27) can be derived from (6) in the same way that (26) was derived from (3). Just substitute $A = R^*$, $B = S$ and $X = R^{-1}TS^{-1}$.

4 On the Heinz Inequalities

The arithmetic-geometric mean inequality (3) and the Cauchy-Schwarz inequality (6) (the case $r = 1$) play a central role in the proofs of the following famous Heinz-McIntosh inequalities. If A, B and X are operators in $B(H)$ with A, B positive, then

$$|||A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha||| \leq |||AX + XB|||, \tag{28}$$

$$|||A^\alpha XB^{1-\alpha} - A^{1-\alpha}XB^\alpha||| \leq |2\alpha - 1|\;|||AX - XB|||, \tag{29}$$

$$|||A^\alpha XB^{1-\alpha}||| \leq |||AX|||^\alpha\;|||XB|||^{1-\alpha} \tag{30}$$

for every α with $0 \leq \alpha \leq 1$ and for every unitarily invariant norm.

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For the usual operator norm the inequalities (28) and (29) have been proved by Heinz [12] using a somewhat complicated complex analysis proof. However, for the same norm, elegant convexity and induction arguments have been used by McIntosh [23] to obtain (28), (29) from (3) and (30) from the case $r = 1$ of (6). See also [5], [9] and [20].

It has been shown in [20] that the inequality (30) is equivalent to the form

$$\| |A^\alpha X B^\alpha| \| \leq \| |X| \|^{1-\alpha} \, \| |AXB| \|^\alpha \tag{31}$$

which, for the usual operator norm, is essentially due to Kato [17]. In the same spirit, we have the following theorem.

Theorem 3 *If A, B and X are operators in $B(H)$ with A, B positive, then*

$$\| |A^\alpha X B^\alpha + A^{1-\alpha} X B^{1-\alpha}| \| \leq \| |AXB + X| \|, \tag{32}$$

$$\| |A^\alpha X B^\alpha - A^{1-\alpha} X B^{1-\alpha}| \| \leq |2\alpha - 1| \, \| |AXB - X| \|, \tag{33}$$

for every α with $0 \leq \alpha \leq 1$ and for every unitarily invariant norm. Moreover, the inequalities (32) and (33) are equivalent to (28) and (29), respectively.

Proof We only prove that (28) implies (32). The other conclusions can be accomplished by similar arguments. For every $\varepsilon > 0$, let $A_\varepsilon = A + \varepsilon I$. Then A_ε is invertible for every $\varepsilon > 0$. Using (28), we have

$$\begin{aligned} \| |A_\varepsilon^\alpha X B^\alpha + A_\varepsilon^{1-\alpha} X B^{1-\alpha}| \| &= \| |(A_\varepsilon^{-1})^\alpha (A_\varepsilon X) B^{1-\alpha} + (A_\varepsilon^{-1})^{1-\alpha} (A_\varepsilon X) B^\alpha| \| \\ &\leq \| |A_\varepsilon^{-1}(A_\varepsilon X) + (A_\varepsilon X) B| \| \\ &= \| |X + A_\varepsilon X B| \|. \end{aligned}$$

Letting $\varepsilon \to 0$ and invoking continuity arguments, we obtain

$$\| |A^\alpha X B^\alpha + A^{1-\alpha} X B^{1-\alpha}| \| \leq \| |AXB + X| \|,$$

as desired.

As an application of the inequality (31) we can show that if A, B and X are operators in $B(H)$ such that $(A^*A)^2 \leq A^{*^2} A^2$ and $(BB^*)^2 \leq B^2 B^{*^2}$ (in particular, if A and B^* are hyponormal), then

$$\| |AXB| \|^n \leq \| |X| \|^{n-1} \, \| |A^n X B^n| \| \tag{34}$$

for every integer $n \geq 1$ and for every unitarily invariant norm. This result, which is a considerable improvement of Theorem 3 in [20], can be proved by using Lemmas 6 and 8

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in [20] together with an induction argument.

In [20, Theorem 3] the inequality (34) has been derived under the stronger assumption that $(A^*A)^n \le A^{*^n}A^n$ and $(BB^*)^n \le B^nB^{*^n}$ for every integer $n \ge 1$. It should be noticed here that in the finite-dimensional case (or, more generally, if A is compact), the conditions $(A^*A)^2 \le A^{*^2}A^2$ and $(A^*A)^n \le A^{*^n}A^n$ for every integer $n \ge 1$, are equivalent. In fact, each of these conditions is equivalent to the normality of A. However, for noncompact operators the situation is different. To see this, assume H is an infinite-dimensional Hilbert space with an orthonormal basis$\{e_j\}$. Let $A = U + I$, where U is the unilateral shift operator defined on H by $Ue_j = e_{j+1}$ for $j = 1, 2, \ldots$. Then $A^{*^2}A^2 - (A^*A)^2 = I - UU^*$, which is positive. On the other hand, $A^{*^3}A^3 - (A^*A)^3 = 6(I - UU^*) + U(I - UU^*) + (I - UU^*)U^*$ is not positive. In fact, the compression of $A^{*^3}A^3 - (A^*A)^3$ to the two-dimensional subspace spanned by e_1, e_2 has the matrix representation $\left[\begin{smallmatrix} 6 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, which is clearly not positive.

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