TORSION-FREE AND DIVISIBLE MODULES OVER NON-INTEGRAL-DOMAINS

LAWRENCE LEVY

Introduction.* In trying to extend the concept of torsion to rings more general than commutative integral domains the first thing that we notice is that if the definition is carried over word for word, integral domains are the only rings with torsion-free modules. Thus, if m is an element of any right module M over a ring containing a pair of non-zero elements x and y such that $xy = 0$, then either $mx = 0$ or $(mx)y = 0$. A second difficulty arises in the non-commutative case: Does the set of torsion elements of M form a submodule? The answer to this question will not even be "yes" for arbitrary non-commutative integral domains.

The definition we shall use is: An element m of a right R-module is a *torsion* element if $md = 0$ for some regular element (i.e. some non-zero-divisor) d of R. With this definition, which coincides with the ordinary one in the case of integral domains, every ring is a torsion-free module over itself. The question about the existence of a torsion submodule is answered by the following theorem: The torsion elements of each right R-module form a submodule if and only if R has a right ring of quotients (Thm 1.4). (Quotient rings are defined in § 1.) It is easily seen that if T is the torsion submodule of M, then M/T is torsion-free (that is, has no non-zero torsion elements).

We also make the dual definition that a right R-module M is *divisible* if $Md = M$ for every regular element d of R. Again we are forced to restrict d to be regular, because otherwise the ring of quotients would not be a divisible module.

In § 2 we give some examples of rings that will satisfy the hypotheses of the theorems in subsequent sections. The main example is that if R is the direct sum of a finite number of rings, each of which is a full matrix ring over a Dedekind ring, then every one-sided ideal of R is projective and R has a semi-simple two-sided quotient ring. In fact, we show that many properties of modules over an arbitrary ring R carry over to modules over a full matrix ring over R.

In § 3 we study the relation between divisible and injective modules. It is known that, over a commutative integral domain, every injective module is divisible, every torsion-free divisible module is injective; and *every* divisible module is injective if and only if the domain is a Dedekind ring. In our case

Received December 12, 1961.

*Throughout this paper all rings will be understood to have identity elements, all modules to be unitary, and semi-simple rings to satisfy the minimum condition.

we show that over an arbitrary ring every injective module is divisible. If R has a two-sided quotient ring S, then:

(1) Every torsion-free divisible right R-module is injective $\Leftrightarrow S$ is semi-simple.

(2) Every divisible right R-module is injective $\Leftrightarrow R$ is hereditary (every right ideal projective) and S is semi-simple.

We actually show that (1) holds with the weaker hypothesis that S is a right ring of quotients. However, the corresponding question for (2) is left open.

We study the structure of rings satisfying (2) above, showing in § 3 that R is noetherian, and in § 4 the direct sum of a finite number of hereditary rings, each having a simple quotient ring S_i such that $S = \sum_i \oplus S_i$.

In § 5 we study the condition (TF): Every finitely generated torsion-free right module is a submodule of a free module. We show that if R has a two-sided quotient ring S, then R satisfies (TF) if and only if S does. In particular R satisfies (TF) if S is semi-simple. R may satisfy (TF) even though S is not semi-simple. However, in the main result of this section we prove that if R has a right quotient ring S, and R has no non-zero nilpotent ideals and satisfies (TF), then S is semi-simple (with minimum condition) and also the left quotient ring of R.

In § 6 we show that for the ring R of all $n \times n$ matrices over a Dedekind ring (n fixed), every finitely generated right module is the direct sum of a free module $R^{(k)}$, a right ideal J not isomorphic to R, and indecomposable cyclic torsion modules T_i $(i = 1, \ldots, n)$, k being unique and the modules J and T_i being unique up to isomorphism.

1. Preliminaries.

The results of this section, unlike many of the later ones, hold for rings without an identity element. Consequently, for this section only, we drop the requirement that all rings have an identity.

Let R and S be rings. We say that S is a *right quotient ring* of R if: (1) $R \subseteq S$, (2) every regular element of R has a two-sided inverse* in S, and (3) every element of S has the form rd^{-1} for properly chosen r, d in R. It is easily shown that when S exists it is unique up to isomorphism over R. By a *two-sided quotient ring* we shall mean a ring that is both a right and a left quotient ring. The following lemma is an immediate consequence of (3):

LEMMA 1.1. *Let S be a right quotient ring of R. Then*

(1) *For each right ideal J^* of S, $J^* = (J^* \cap R)S$.*

(2) *If J and K are right ideals of R whose sum is direct, then $(J \oplus K)S = JS \oplus KS$.*

It is known that R has a right quotient ring if and only if R has regular elements and satisfies the following *Common Multiple Property* (CM): for every x, d in R with d regular, there exist d_1, y with d_1 regular, such that

*In order for this to make sense, S must have an identity.

https://doi.org/10.4153/CJM-1963-016-1 Published online by Cambridge University Press

$xd_1 = dy$. (This follows from Thm. 1, Chap. 5, § 7 of *Algèbre, Vol. I*, by P. Dubreil, (Paris, 1954), and Lemma 1.2 below.)

LEMMA 1.2. *If R satisfies* (CM) *and $dx = d_1$, where d and d_1 are regular, then x is regular.*

Proof. x is obviously not a left zero divisor. Applying (CM) to d and d_1 we obtain e, f with e regular, such that $de = d_1 f$. Then $de = d_1 f = dxf$ so that $e = xf$. This shows that x is not a right zero divisor.

From the above lemma, it follows that in a ring with (CM), for every pair of regular elements d_1, d_2 there exists another pair of regular elements c_1, c_2 such that $d_1 c_1 = d_2 c_2$. An easy induction then completes the proof of:

LEMMA 1.3. *If R has a right quotient ring S, and if $s_i = r_i d_i^{-1} \in S$ $(i = 1, \ldots, n; r_i, d_i \in R)$ then there exist elements $x_i, d \in R$ such that $s_i = x_i d^{-1}$.*

THEOREM 1.4. *The set of torsion elements of each right R-module forms a submodule if and only if R has a right quotient ring.*

Proof. Suppose R has a right quotient ring; let M be a right R-module, and T the set of torsion elements of M. If $t_1, t_2 \in T$, then $t_1 d_1 = t_2 d_2 = 0$ for regular d_1 and d_2. Then there exist regular c_1, c_2 such that $d_1 c_2 = d_2 c_2$, giving $(t_1 - t_2)(d_1 c_1) = 0$, that is, $t_1 - t_2 \in T$. If $x \in R$, then there exist d_3, y with d_3 regular, such that $xd_3 = d_1 y$. This gives $(t_1 x)d_3 = t_1 d_1 x = 0$ so that $t_1 x \in T$ and T is a submodule (T is not empty since $0 \in T$).

Conversely, suppose that the set of torsion elements of each right R-module forms a submodule. Let $x, d \in R$ be given with d regular. Then by hypothesis the set of torsion elements of dR/d^2R forms a submodule. Now, $d + d^2R$ is a torsion element since $(d + d^2R)d^2 = d^2R$. Hence $dx + d^2R = (d + d^2R)x$ is also a torsion element. Hence for some regular d_1, $dxd_1 \in d^2R$. Suppose $dxd_1 = d^2y$. Then $xd_1 = dy$ and R satisfies (CM). To see that R contains regular elements we observe that the torsion elements of the right R-module R forms a submodule. Hence 0 is a torsion element of R, and this implies that $0d = 0$ for some regular d in R.

Because of the following proposition, we shall use the phrase, "consider M to be a submodule of $M \otimes_R S$" to mean, "identify m and $m \otimes 1$." When this identification is permissible, we shall then have $MS = M \otimes S$.

PROPOSITION 1.5. *Let R have a right quotient ring S and let M be a right R-module. Then M is an R-submodule of some S-module if and only if M is torsion-free. When the condition holds, every element of* MS *has the form md^{-1} $(m \in M, d \in R)$ and $MS \cong M \otimes_R S$ under the correspondence $ms \to m \otimes s$.*

Proof. Suppose M is torsion-free. The map $m \to m \otimes 1$ is an R-homomorphism. We wish to show that it is one-to-one. Let F be the free abelian group whose generators are the ordered pairs $(m, s) \in M \times S$, and let f be the map of F onto $M \otimes S$ given by $f(\sum \pm (m_i, s_i)) = \sum \pm m_i \otimes s_i$. Then $\ker f$ is generated by the elements of the forms $(m_1 + m_2, s) - (m_1, s) - (m_2, s)$,

$(m, s_1 + s_2) - (m, s_1) - (m, s_2)$, and $(mr, s) - (m, rs)$, $(r \in R)$. If for some n, $n \otimes 1 = 0$, then

$$(n, 1) = \sum_{i=1}^{t} \pm(m_i, s_i),$$

where the terms on the right, when properly grouped, are among the generators of $\ker f$ (or their negatives). Let d be a common right denominator for the elements s_i (Lemma 1.3). Then $n \otimes 1 = 0$ in $M \otimes Rd^{-1}$. But $M \otimes Rd^{-1} \cong M \otimes R \cong M$ (as additive groups) under the correspondence $m \otimes rd^{-1} \to m \otimes r \to mr$. Hence $0 = n \otimes 1 = n \otimes dd^{-1} \to nd$. Since d is invertible in S, and hence regular in R, $n = 0$. Hence M is contained isomorphically in $M \otimes S$ with the imbedding $m \to m \otimes 1$.

Conversely, if M is contained in some S-module and $md = 0$ ($m \in M$, d regular in R), then $0 = mdd^{-1} = m$ so that M is torsion-free.

Every element of MS has the form $\sum m_i s_i$. If we write $s_i = r_i d^{-1}$ (Lemma 1.3), then $\sum m_i s_i = (\sum m_i r_i)d^{-1}$, which is of the form md^{-1}. Similarly every element of $M \otimes S$ can be written in the form $m \otimes d^{-1}$. Hence, by the elementary properties of tensor products, the map $m \otimes s \to ms$ of $M \otimes S$ onto MS is well-defined. It is one-to-one since $m \otimes d^{-1} \to 0 = md^{-1}$ implies $m = 0$, and hence $m \otimes d^{-1} = 0$.

COROLLARY 1.6. *Let R have a right quotient ring S, let M and N be R-submodules of right S modules, and let f be an R-homomorphism of M into N. Then $f^*: ms \to f(m)s$ extends f to an S-homomorphism of MS into NS. If f is one-to-one or onto, so is f^*.*

Finally, we summarize some results from other sources which will be needed later. Let S be semi-simple (with minimum condition) and a right quotient ring of R. Write

$$S = \sum_{i=1}^{n} \oplus S_i,$$

where each S_i is a simple ring. Let R_i be the projection of R in S_i. Then each S_i is a right quotient ring of R_i (**10**, Cor. 6.5), R is a semi-prime ring (has no nilpotent ideals $\neq 0$) and each R_i is a prime ring (the product of two nonzero ideals is non-zero) (**6**, Thms. 4.4, 5.5).

In a semi-prime ring the left and right annihilators of a two-sided ideal M coincide with each other and have zero intersection with M (if L is the left annihilator of M, then $(L \cap M)^2 = 0$ so that $L \cap M = 0$. Hence $ML \subseteq L \cap M = 0$ so that L is contained in the right annihilator of M). We denote this annihilator by ann M. If T is a subset of a ring R and several rings are being discussed we shall use the more complete notation r. $\text{ann}_R\, T$ for the right annihilator, in R, of T.

Returning to the notation of the paragraph above the previous one, we remark that $\{R \cap R_i : i = 1, \ldots, n\}$ is the set of minimal annihilator ideals of R, and $\text{ann}_R(R \cap R_i) = \{x \in R : x_i = 0\}$ (**10**, Prop. 3.8 with $M = R$).

2. Matrix rings. Let D be a given ring and let R be the ring of all $n \times n$ matrices over D for some n. Since right multiplication of a matrix c by a matrix d corresponds to performing column operations on c, a right ideal J of R is completely known when the first column of every element of J is known. The following lemma formalizes a more general version of this statement.

Throughout this paper e_{ij} will denote the matrix with 1 in the (i, j) position and zeros elsewhere.

LEMMA 2.1. *Let R be the ring of $n \times n$ matrices over a ring D; let M and N be right R-modules, and let f' be a D-homomorphism of Me_{11} into Ne_{11}. Then f' can be extended in one and only one way to an R-homomorphism of M into N. This is given by*

$$f(m) = \sum_{j=1}^{n} f'(me_{j1})e_{1j}.$$

If f' is one-to-one or onto, so is f.

Proof. f as defined above is obviously a D-homomorphism of M into N. To see that it is an R-homomorphism, it is sufficient to show that $f(me_{ij}) = f(m)e_{ij}$ for every (i, j):

$$f(me_{ij}) = \sum_{k=1}^{n} f'(me_{ij}e_{k1})e_{1k} = f'(me_{i1})e_{1j} = \sum_{k=1}^{n} f'(me_{k1})e_{1k}e_{ij} = f(m)e_{ij}.$$

If g is any extension of f' to an R-homomorphism of M into N, then

$$g(m) = \sum_{j=1}^{n} g(m)e_{jj} = \sum_{j=1}^{n} g(me_{j1})e_{1j} = \sum_{i=1}^{n} f'(me_{j1})e_{1j}$$

so that $g = f$.

Now suppose that f' is one-to-one and that $f(m) = 0$. Then

$$\sum_{j=1}^{n} f'(me_{j1})e_{1j} = 0.$$

Multiplying by e_{i1} and remembering that e_{11} acts as the identity on Me_{11} we get $f'(me_{i1}) = 0$ $(i = 1, 2, \ldots, n)$. This gives $me_{i1} = 0$ for all i so that

$$m = \sum_{i=1}^{n} me_{i1}e_{1i} = 0.$$

Therefore f is one-to-one.

Finally, suppose that f' is onto and let h be a given element of N. Then for each j we can find $m_j \in M$ such that $f'(m_je_{11}) = he_{j1}$. Set

$$m = \sum_{j=1}^{n} m_je_{1j}.$$

Then

$$f(m) = \sum_{j=1}^{n} f'(me_{j1})e_{1j} = \sum_{j=1}^{n} f'(m_je_{11})e_{1j} = \sum_{j=1}^{n} he_{jj} = h$$

so that f is onto.

https://doi.org/10.4153/CJM-1963-016-1 Published online by Cambridge University Press

LEMMA 2.2. *Let R be the ring of $n \times n$ matrices over a ring D and let M, L, P be right R-modules. Then any commutative diagram of D-homomorphisms*

$$\begin{tikzcd} Me_{11} \arrow[r, "f_2'"] \arrow[d, "f_1'"'] & Pe_{11} \arrow[dl, "f_3'"] \\ Le_{11} & \end{tikzcd}$$

can be extended in one and only one way to a commutative diagram of R-homomorphisms

$$\begin{tikzcd} M \arrow[r, "f_2"] \arrow[d, "f_1"'] & P \arrow[dl, "f_3"] \\ L & \end{tikzcd}$$

For each i, f_i is one-to-one or onto if and only if f_i' has the same property.

Proof. Define f_1, f_2, and f_3 by Lemma 2.1. Then the only thing we have to prove is that the second diagram commutes. (Recall that $e_{1k}e_{1j} = 0$ for $k \neq 1$, and $e_{11}e_{11}e_{1j} = e_{1j}$):

$$\begin{aligned} f_3f_2(m) = \sum_{j=1}^{n} [f_3f_2'(me_{j1})]e_{1j} &= \sum_{j=1}^{n} \sum_{k=1}^{n} f_3'[f_2'(me_{j1})e_{k1}]e_{1k}e_{1j} \\ &= \sum_{j=1}^{n} f_3'f_2'(me_{j1})e_{1j} = \sum_{j=1}^{n} f_1'(me_{j1})e_{1j} = f_1(m). \end{aligned}$$

It is easily established that if f_i is one-to-one or onto, then f_i' also has this property. The converse of this statement is part of Lemma 2.1.

COROLLARY 2.3. *Me_{11} is projective or injective as a D-module if and only if M has the corresponding property as an R-module.*

Proof. If Me_{11} is injective or projective use the fact that any R-homomorphism $M \to N$ can be restricted to a D-homomorphism $Me_{11} \to Ne_{11}$, and then the above lemma.

Let H be any D-module. Let $\{f_i : i = 1, 2, \ldots, n\}$ be a set of isomorphisms of H onto other modules, but let f_1 be the identity map on H. Let $H^* = f_1(H) \oplus f_2(H) \oplus \ldots \oplus f_n(H)$. Then we can make H^* into an R-module by defining $f_i(h)e_{ij} = f_j(h)$, and $f_i(h)e_{kj} = 0$ if $i \neq k$. Then $H^* e_{11} = H$.

Now suppose that M^* is an injective R-module, and let a diagram of D-homomorphisms be given

$$\begin{array}{ccccc} 0 & \longrightarrow & H & \longrightarrow & L \\ & & \downarrow & & \\ & & M^*e_{11} & & \end{array}$$

in which the row is exact. Construct H^* and L^* as in the above paragraph so that the row becomes $0 \to H^*e_{11} \to L^*e_{11}$. By Lemma 2.1 or 2.2 this can be extended to a diagram of R-homomorphisms

$$\begin{array}{ccccc} 0 & \longrightarrow & H^* & \longrightarrow & L^* \\ & & \downarrow & & \\ & & M^* & & \end{array}$$

in which the row is still exact. Since M^* is injective, there is a homomorphism $L^* \to M^*$ such that the above diagram commutes. The restriction of this map to $L^* e_{11} \to M^* e_{11}$ shows that $M^* e_{11}$ is injective as a D-module.

The case in which M^* is projective is handled in the same fashion.

Recall, now, that a ring is (right) *semi-hereditary* if every finitely generated right ideal is projective.

PROPOSITION 2.4. *Let R be the ring of $n \times n$ matrices over a ring D. Then R is hereditary or semi-hereditary if and only if D has the same property.*

Proof. Suppose that D is hereditary or semi-hereditary. Let J be a right ideal of R, and if D is only semi-hereditary suppose that J is finitely generated. In the latter case Je_{11} is a finitely generated right D-module. (If J is generated by p elements as an R-module, then it is generated by np elements as a D-module, and Je_{11} is a direct summand of the D-module J.)

R is a free right D-module, and it is known that for hereditary rings every submodule of a free module is projective, while for semi-hereditary rings every finitely generated submodule of a free module is projective (**2**, Chap. I, Props. 5.3 and 6.2). Thus, in either case Je_{11} is a projective D-module and by Corollary 2.3, J is a projective R-module.

Conversely, suppose that R is hereditary or semi-hereditary, and let J be a right ideal of D (which is finitely generated if R is only semi-hereditary). Let J^* be the right ideal of R whose elements have arbitary entries from J in the first row and zeros elsewhere. Then $J^* e_{11} \cong_D J$, and by the two propositions from (**2**) quoted above, J^* is a projective R-module. Then by Corollary 2.3, J is a projective D-module.

PROPOSITION 2.5. *Let R be the ring of $n \times n$ matrices over a ring D, and let M be an R-module such that the D-module Me_{11} is the direct sum of a free D-module and a right ideal of D. Then M is the direct sum of a free R-module and a right ideal of R.*

Proof. Suppose that $Me_{11} \cong F' \oplus J_D{}'$ where F' is a direct sum of k' copies of D and J_D' is a right ideal of D. Write $k' = kn + r$ where $0 \leqslant r < n$; let F be the R-module which is equal to the direct sum of k copies of R; and let J be the right ideal of R whose elements have arbitrary entries in their first r rows, arbitrary entries from J_D in the next row, and zeros in the remaining rows, if any remain. Then $Me_{11} \cong_D Fe_{11} \oplus Je_{11}$. By Lemma 2.1, $M \cong_R F \oplus J$.

PROPOSITION 2.6. *Let D be any ring such that every non-finitely generated projective right module is free* (e.g. a Dedekind ring (**9**, Thm. 2b; and **2**, Chap. I, Thm. 5. 3)). *Then the ring of $n \times n$ matrices over D has the same property.*

The proof is the same as that of Proposition 2.5 except that since k' is infinite, $k' = kn + r$ can be replaced by $k' = k$ and $r = 0$. We can also prove the following proposition using these methods.

PROPOSITION 2.7. *Let D be any ring such that every finitely generated right module is a direct sum of cyclic modules. Then the ring of $n \times n$ matrices over D has the same property.*

LEMMA 2.8. *Let S be a semi-simple ring. Then an element x of S is a left zero divisor if and only if it is a right zero divisor. Every non-zero-divisor is a unit.*

Proof. First assume that S is simple with minimum condition. Then we can consider its elements to be linear transformations of a finite-dimensional vector space over a division ring. The lemma then follows from the facts that such a transformation is one-to-one if and only if it is onto, and that every subspace is a direct summand. If S is semi-simple, then we write S as a direct sum of simple rings and apply the above result to the simple components of S.

PROPOSITION 2.9. *Let a semi-simple ring K be a right quotient ring of D. Then the ring of $n \times n$ matrices over K is a right quotient ring of the ring of $n \times n$ matrices over D.*

Proof. Let R be the ring of $n \times n$ matrices over D and S the ring of $n \times n$ matrices over K, and let s be any given element of S. Let the (i, j) entry of the matrix s be s_{ij} $(s_{ij} \in K)$. Then $s_{ij} = r_{ij} d_{11}^{-1}$ for properly chosen elements $r_{ij}, d_{11} \in D$. If d is the matrix whose (i, i) entry is d_{11} for all i and whose other entries are zero, then $s = rd^{-1}$ with $r, d \in R$.

Next we observe that S is semi-simple: Let J be a right ideal of S. Then Je_{11} is a module over the semi-simple ring K and therefore injective (**2**, Chap. 1, Thm. 4.2). Therefore by Corollary 2.3, J is an injective R-module, and this shows that S is semi-simple (**2**, same theorem). (Of course, this could have been done directly in terms of the classical Wedderburn theory.) Let b be a regular element of R. If $bs = 0$ for some $s \in S$, then writing $s = rd^{-1}$ as in the above paragraph, we get $br = 0$ so that $r = 0$ and therefore $s = 0$. Therefore, by Lemma 2.8, b has a two-sided inverse in S and S is a right quotient ring of R.

We now observe that it is possible to construct many examples of right hereditary rings having right quotient rings which are semi-simple: Let D be a Dedekind ring and R the ring of $n \times n$ matrices over D. Then by Propositions 2.4 and 2.9, R has the required properties. Direct sums of such rings obviously have these properties also.

Goldie (**6**, § 6) has given an example of a right principal ideal domain D which has a right quotient ring, but which has no left quotient ring. The ring of $n \times n$ matrices over D gives a one-sided example of a non-integral domain which is right hereditary and which has a simple right ring of quotients.

3. Divisible and injective modules. Over a commutative integral domain every injective module is divisible, every torsion-free divisible module is injective; and the domain is a Dedekind ring if and only if *every* divisible module is injective (**2**, Chap. VII). We now consider these ideas in the case of rings with right rings of quotients. The proofs given here are based on the corresponding proofs in (**2**) whenever these can be adapted to our case.

THEOREM 3.1. *Let R be an arbitrary ring. Then every injective R-module is divisible.*

Proof. Let M be an injective right R-module, and let $m \in M$ and a regular element $d \in R$ be given. Then the function $dr \rightarrow mr$ of dR into M is well-defined since d is not a zero divisor. It is obviously an R-homomorphism and therefore can be extended to an R-homomorphism ϕ of R into M. Suppose $\phi(1) = m_1$. Then $m_1 d = \phi(1 \cdot d) = \phi(d \cdot 1) = m \cdot 1 = m$ so that $Md = M$ and M is divisible.

COROLLARY 3.2. *Every module is a submodule of a divisible module.*

Proof. Every module is a submodule of an injective module (**2**, Chap. I, Thm. 3.3).

THEOREM 3.3. *Let R have a right quotient ring S. Then the following statements are equivalent:*

(1) *Every torsion-free divisible right R-module is injective.*

(2) *S is semi-simple.*

Proof. First we recall that if S is a right quotient ring of any ring R, then S is torsion-free as an R-module (Prop. 1.5). Also, if M is a *torsion-free*, divisible right R-module, then M is also an S-module: if $m \in M$ and d is regular in R, then the solution m_1 of $m_1 d = m$ is unique, and we can define $md^{-1} = m_1$.

Now suppose that (1) holds, and let J^* be a right ideal of S. Then J^* is a torsion-free, divisible right R-module, and therefore injective. Therefore $S = J^* \oplus K$ (direct sum of R-modules). Since a homomorphic image of a divisible module is obviously divisible, K is a divisible R-module. Therefore, by the above paragraph K is an S-module, and therefore a right ideal of S. But it is known that if every right ideal of a ring with unit is a direct summand, then the ring is semi-simple (**2**, Chap. I, Thm. 4.2).

Conversely, suppose that S is semi-simple, that M is a torsion-free, divisible right R-module, and that f is a homomorphism of a right ideal J of R into M. Then (Cor. 1.6) f can be extended to an S-homomorphism f^* of JS into $MS = M$. Since JS is a direct summand of S, f^* can be further extended to a homomorphism of S into M (for example, define f^* to be zero on a complement of JS). Then restriction of this last map to R gives an extension of f to R, showing that M is injective.

THEOREM 3.4. *Let R have a two-sided quotient ring S. Then the following statements are equivalent:*

(1) *Every divisible right R-module is injective.*

(2) *S is semi-simple and R is (right) hereditary.*

The proof of this theorem will be broken up into several parts. The hypothesis that S is a two-sided ring of quotients, instead of being only a right ring of quotients, is needed at only one point: the part following the proof of Lemma 3.10. I do not know whether the proof can be improved to eliminate this hypothesis, or whether the one-sided version of the theorem is false. We begin the proof of Theorem 3.4 with a proof that (1) $\Rightarrow$ (2).

PROPOSITION 3.5. *Let R have a right quotient ring S. If every divisible right R-module is injective, then R is hereditary and S is semi-simple.*

Proof. By Theorem 3.3, S is semi-simple. Let M be an injective right R-module. Then by Theorem 3.1, M is divisible. Therefore every homomorphic image of M is divisible. By hypothesis, this implies that every homomorphic image of M is injective. But if every homomorphic image of every injective right R-module is injective, R must be right hereditary (**2**, Chap. I, Thm. 5.4).

LEMMA 3.6. *Let R have a semi-simple right quotient ring S. Then every homomorphism of a right ideal f of R into S is given by $f(x) = \alpha x$ for some α in S. If f is one-to-one then α can be chosen to be invertible.*

Proof. Suppose f maps J onto K. Then f can be extended to an S-homomorphism f^* of JS onto KS and if f is one-to-one, so is f^* (Cor. 1.6). Suppose $S = JS \oplus J_1^* = KS \oplus K_1^*$. Then we can extend f^* to all of S by defining $f^*(J_1^*) = 0$. However, if f^* is one-to-one, then $JS \cong_S KS$ so that $J_1^* \cong_S K_1^*$. Therefore f^* can be extended to an automorphism of S by defining it properly on J_1^*. In either case let $f^*(1) = \alpha$. Then $f(x) = f^*(x) = \alpha x$ for all $x \in J$. If f^* is an automorphism of S, then α must be invertible ($\alpha x = 0$ implies that $x = 0$ so that, by Lemma 2.8, α is a unit).

LEMMA 3.7. *Let R have a semi-simple right quotient ring S. Then any element of R which is a zero divisor of S is a left zero divisor of R.*

Proof. Suppose $x \in R$ is a zero divisor in S. Then by Lemma 2.8, x is a left zero divisor in S. Suppose that $x(rd^{-1}) = 0$ for $r, d \in R$. Then $xr = 0$.

LEMMA 3.8. *Let R have a semi-simple right quotient ring S, and let J be a projective right ideal of R which contains a regular element. Then there are elements $\{a_i : i = 1, 2, \ldots, n\}$ in J and elements $\{\alpha_i : i = 1, 2, \ldots, n\}$ in S such that $\alpha_i J \subseteq R$ for all i, and $\sum a_i \alpha_i = 1$. (In the commutative case this means that J is invertible in S.)*

Proof. Write $F = J \oplus M$, where F is a free R-module. Let $\{u_i : i \in I\}$ be a basis for F and write $u_i = a_i + m_i$ where $a_i \in J$ and $m_i \in M$. Then every element of J has the form $b = \sum a_i b_i$ ($b_i \in R$). The projections $b \rightarrow b_i$ are homomorphisms of a right ideal of R into S. Therefore, by Lemma 3.6 there are elements α_i in S such that $b_i = \alpha_i b$ for all $b \in J$. This shows that $\alpha_i J \subseteq R$. Since F is a free module, if b is fixed, we must have $\alpha_i b = b_i = 0$ for all but a finite number of subscripts i. Let b be an element of J which is a regular element of R. Then b is a regular element of S by Lemma 3.7, and $\alpha_i b = 0$ implies that $\alpha_i = 0$. Hence there are only finitely many non-zero α_i's. Since b is regular,

$$b = \sum a_i b_i = \sum_{i=1}^{n} a_i \alpha_i b$$

implies that $1 = \sum a_i \alpha_i$.

LEMMA 3.9. (Goldie **6**, Thms. 3.9, 4.4). *Let R have a semi-simple right quotient ring. Then a right ideal J of R contains a regular element if and only if $J \cap M \neq 0$ for every non-zero right ideal M of R.*

LEMMA 3.10. *Let R have a semi-simple right quotient ring S. Then every right ideal J of R is a direct summand of a right ideal containing a regular element.*

Proof. This follows immediately from Goldie's concept of *right dimension* (**6**) and the above lemma. It can also be proved as follows: Suppose $S = JS \oplus K^*$, and let M be any right ideal of R. Then if $[J \oplus (K^* \cap R)] \cap M = 0$, we have (by Lemma 1.1), $[J \oplus (K^* \cap R) \oplus M]S = JS \oplus K^* \oplus MS = S$ so that $M = 0$. Therefore by the above lemma $J \oplus (K^* \cap R)$ contains a regular element.

Proof that (2) $\Rightarrow$ (1). We now assume the full hypothesis of Theorem 4.3: R is hereditary and has a two-sided quotient ring S which is semi-simple. Let D be a divisible right R-module, and let J be a right ideal of R. To show that D is injective we have to show that every homomorphism $f : J \rightarrow D$ can be extended to a homomorphism of R into D. By Lemma 3.10, J is a direct summand of a right ideal K which contains a regular element. f can obviously be extended from J to K, so we shall assume that J itself contains a regular element.

Choose $\{a_i\}$ and $\{\alpha_i\}$ according to Lemma 3.8. Since S is a *left* quotient ring of R we can find a common left denominator for $\{\alpha_i\}$, that is we can find elements r_i, $d \in R$ such that $\alpha_i = d^{-1} r_i$ ($i = 1, 2, \ldots, n$). (The corresponding property of right quotient rings is stated in Lemma 1.3.) Since D is divisible, we can choose elements m_i in D such that $f(a_i) = m_i d$ $(i = 1, 2, \ldots, n)$.

Now let $x \in J$ be given. Recalling that $\alpha_i x \in R$ and $d\alpha_i = r_i$ we get $\sum_i m_i r_i x = \sum_i m_i(d\alpha_i x) = \sum(m_i d)(\alpha_i x) = \sum_i f(a_i)(\alpha_i x) = f(\sum a_i \alpha_i x) = f(x)$. Therefore if $z \in R$, the map $z \to (\sum_i m_i r_i)z$ is an extension of f and this shows that D is injective.

THEOREM 3.11. *Let R be a hereditary ring with a semi-simple right quotient ring. Then R is right noetherian.*

Proof. Let K be a right ideal of R. Then K is a direct summand of a right ideal J which contains a regular element of R (Lemma 3.10). By Lemma 3.8 we can choose $\{a_i : i = 1, 2, \ldots, n\} \subseteq J$ and $\{\alpha_i : i = 1, 2, \ldots, n\} \subseteq S$ (the right quotient ring of R) such that $\alpha_i J \subseteq R$ and $\sum_i a_i \alpha_i = 1$. Then if $b \in J$, $b = \sum_i a_i(\alpha_i b)$ so that J is finitely generated. Therefore K is also finitely generated and R is right noetherian.

THEOREM 3.12. *Let R be a hereditary ring with a two-sided, semi-simple quotient ring. Then every divisible right R-module M is a direct sum of indecomposable, divisible R-modules, and the number of summands of each isomorphism type is a complete set of invariants for M.*

Proof. E. Matlis (**11**, Thm. 2.5 and Prop. 2.7) has shown that over a right noetherian ring every injective module is a direct sum of indecomposable, injective modules, and the number of summands of each isomorphism type is a complete set of invariants for M. Theorem 3.11 shows that this result applies to the rings of this theorem, while Theorems 3.4 and 3.1 show that we can replace "injective" by "divisible" in Matlis's result.

4. Structure of hereditary rings with semi-simple right quotient rings. We recall that a ring is *semi-prime* if it has no non-zero nilpotent ideals, that in such a ring the right and left annihilators of a two-sided ideal A coincide. We denote this annihilator ideal by ann A.

LEMMA 4.1. *Let R be a semi-prime ring, let M be a two-sided ideal and N a right ideal such that $R = M \oplus N$. Then N is a two-sided ideal, and $N =$ ann M.*

Proof. $NM \subseteq N \cap M = 0$. Therefore $N \subseteq$ ann M. Suppose that $xM = 0$ and write $x = m + n$ with $m \in M, n \in N$. Then $x - n = m \in (\text{ann } M) \cap M$. But in any semi-prime ring $(\text{ann } M) \cap M = 0$ for every two-sided ideal M (§ 1). Therefore $x = n \in N$.

LEMMA 4.2. (Goldie, **6**, Thms. 3.7, 4.4). *Let J be an annihilator right ideal in a ring R which has a semi-simple right quotient ring. Then for some $t \in R$, $J = \text{r.ann}\, t$ (the right annihilator of t).*

THEOREM 4.3. *Let R be a (right) hereditary* ring which has a semi-simple right quotient ring S. Then R is the direct sum of hereditary rings $\{R_i : i = 1, 2, \ldots, n\}$ which have simple right quotient rings. When considered as a set of ideals of R, $\{R_i\}$ constitutes the set of minimal annihilator ideals of R (hence the decomposition is unique) and the quotient rings of the R_i's are the simple components of S.*

Proof. Write $S = S_1 \oplus \ldots \oplus S_n$ where each S_i is a simple ring and let R_i be the projection of R in S_i. According to § 1 $\{R \cap R_i\}$ is the set of minimal annihilator ideals of the semi-prime ring R. Therefore by Lemma 4.2, $R \cap R_i = r.\mathrm{ann}\, t$ for some $t \in R$. Consider the homomorphism $x \to tx$ of R into itself. By hypothesis, the image tR is a projective module. Therefore the kernel $R \cap R_i$ is a direct summand of R. By Lemma 4.1, this gives $R = R \cap R_i \oplus \mathrm{ann}\,(R \cap R_i)$. According to § 1, $\mathrm{ann}(R \cap R_i) = \{x \in R : x_i = 0\}$. Therefore taking the projection in S_i of the above decomposition of R we get $R_i = R \cap R_i$. This shows that R contains each R_i and hence their sum, which is direct. Since every element of R is a sum $r = \sum r_i$ where $r_i \in R_i$ we have $R = \sum_i \oplus R_i$. Each R_i is now obviously hereditary.

Every commutative prime ring is an integral domain. Therefore, in the commutative case, each R_i is an integral domain in which every ideal is projective, that is, a Dedekind ring. Since every commutative ring has a ring of quotients this gives:

COROLLARY 4.4. *Every commutative hereditary ring whose quotient ring is the direct sum of a finite number of fields is the direct sum of a finite number of Dedekind rings.*

5. Torsion-free modules. In this section we shall study the condition: (TF). *Every finitely generated torsion-free right module is a submodule of a free module.*

LEMMA 5.1. *Let R have a right quotient ring S. If R satisfies* (TF), *then every finitely generated right S-module is a submodule of a free S-module.*

Proof. Let $M = \sum m_i S$ be a finitely generated S-module. Then M is torsion-free as an R-module, since every S-module is torsion-free as both an R- and an S-module (regular elements of both R and S are invertible in S). Let $M_1 = \sum m_i R$. Since R satisfies (TF), M_1 is a submodule of a free R-module M_2. Consider M_2 to be a submodule of $M_2 \otimes_R S$ (which, by Prop. 1.5, then equals $M_2 S$). Since $R \otimes_R S \cong_S S$, and since tensor products preserve direct sums, $M_2 S$ is a free S-module containing $M = M_1 S$.

THEOREM 5.2. *Let R have a two-sided quotient ring S. Then R satisfies* (TF) *if and only if every finitely generated right S-module is a submodule of a free S-module.*

*The proof of this theorem only requires that every *principal* right ideal be projective. Then each R_i, instead of being hereditary, has each principal right ideal projective.

Proof. We first observe that since every regular element of S is invertible, every S-module is torsion-free, so that the theorem could have been stated in the alternative form: R satisfies (TF) if and only if S satisfies (TF).

Now suppose that S satisfies (TF) and let

$$M = \sum_{i=1}^{n} m_i R$$

be a finitely generated torsion-free right R-module. Consider M to be a submodule of $M \otimes_R S$. Then $MS = \sum m_i S$ is a submodule of a free S-module. Since each of the m_i's can be written as a combination of a finite number of basis elements of this free module, we can assume that the free module is finitely generated. Suppose the free module is isomorphic to the direct sum $S^{(k)}$ of k copies of S. In the S-isomorphism $MS \rightarrow S^{(k)}$ suppose that $m_i \rightarrow (s_{1i}, s_{2i}, \ldots, s_{ki})$. Then $M = \sum m_i R \cong_R \sum_i (s_{1i}, s_{2i}, \ldots, s_{ki})R$. Let d be a common left denominator for the nk elements s_{ji}. That is, let $s_{ji} = d^{-1}r_{ji}$ with d, $r_{ji} \in R$ (Lemma 1.3). Then $M \cong_R \sum (d^{-1}r_{1i}, \ldots, d^{-1}r_{ki})R \cong_R \sum_i (r_{1i}, \ldots, r_{ki})R \subseteq R^{(k)}$.

The converse is a special case of Lemma 5.1.

A natural question to ask now is: What restriction does the condition that R satisfy (TF) place on S? It clear is from the above theorem that R will satisfy (TF) whenever S is semi-simple. However, S does not have to be semi-simple, as the following example shows. Let R be the ring generated by the integers and an element x such that $x^2 = 0$. Then S is the ring generated by the rationals Q and x, and $qx = xq$ for every $q \in Q$. S is clearly not semi-simple. The only proper ideal of S is xS. Therefore S is a principal ideal ring with minimum condition. Over any such ring every finitely generated module is a direct sum of cyclic modules (**7**, Chap. 4, Thm. 4.3). However, the only cyclic S-modules are S and S/xS. Since xS is the kernel of the mapping $s \rightarrow xs$ of S onto xS (nothing else can be the kernel since xS is the only proper ideal) we have $S/xS \cong xS$. Hence every finitely generated S-module is isomorphic to a submodule of a free module, and by Theorem 5.2, R satisfies (TF).

I do not know whether or not S must satisfy the minimum condition if R satisfies (TF), or whether S must be, in every case, the two-sided quotient ring of R in order for Theorem 5.2 to hold. However, if R is semi-prime (has no non-zero nilpotent ideals) we can answer these questions.

THEOREM 5.3.* *Let R be a semi-prime ring with a right quotient ring S. If R satisfies* (TF), *then S is semi-simple,*† *and also the left quotient ring of R.*

Before proving this, we establish the following proposition, which is of interest in itself.

---

*The second assertion of this theorem was proved by E. Gentile (**4**) for the case of integral domains with a right quotient division ring.

†Recall that "semi-simple" means "semi-simple with minimum condition."

PROPOSITION 5.4. *Let R be a semi-prime ring in which every maximal (two-sided) ideal has a non-zero annihilator. Then R is the direct sum of a finite number of simple rings (not necessarily with minimum condition).*

Proof. R has maximal ideals, since it has an identity. Let M be a maximal ideal. By § 1 we have $M \cap \operatorname{ann} M = 0$. But, by hypothesis, $\operatorname{ann} M \neq 0$. Therefore $M \oplus \operatorname{ann} M = R$. Thus every maximal ideal is a direct summand of R, and its complement $\operatorname{ann} M$ must therefore be a minimal ideal. This shows that R has minimal ideals. Let $\{R_i\}$ be a family of minimal ideals which is maximal relative to the property that the sum $\sum_i R_i$ is direct (such a family exists, by Zorn's lemma). If $\sum_i \oplus R_i \neq R$, then $\sum_i \oplus R_i$ is contained in a maximal ideal, M, and M is a direct summand of R. Then the sum $(\sum_i \oplus R_i) \oplus \operatorname{ann} M$ contradicts the maximality of $\{R_i\}$. Hence $R = \sum_i \oplus R_i$.

Renumber the R_i, if necessary, so that

$$1 = \sum_{i=1}^{n} e_i$$

with $e_i \in R_i$. Then

$$R = \sum_{i=1}^{n} R_i$$

so that the index set $\{i\}$ is finite. Finally, each R_i is a simple ring: since R_i is a direct summand of R, an ideal of R_i is also an ideal of R.

We remark that Proposition 5.2 actually characterizes finite direct sums of simple rings, the proof in the other direction being trivial.

Proof of semi-simplicity of S in Theorem 5.3. Let J^* be an ideal of S such that $(J^*)^p = 0$ for some integer $p > 0$. Then $(J^* \cap R)^p = 0$, so that since R is semi-prime, $J^* \cap R = 0$. Therefore, by Lemma 1.1, $J^* = 0$, showing that S is semi-prime. We still have to show that S satisfies the minimum condition.

By Lemma 5.1 every finitely generated right S-module is a submodule of a free module. Let M be a maximal (two-sided) ideal of S. Then the right module S/M is generated by $1 + M$ and is therefore a submodule of a free module. Therefore there exists a non-zero homomorphism f of S/M into S. Let $f(S/M) = J$. Since M is two-sided, $(S/M)M = 0$ so that $JM = f(0) = 0$.

Thus every maximal ideal of S has a non-zero annihilator, so that by Proposition 5.4, S is the direct sum of a finite number of simple rings S_i. The proof will be complete if we can show that each S_i satisfies the minimum condition. We observe that every S_i-module is, in a natural way, an S-module. Conversely, every S-module can be written as a direct sum of modules which are also S_i-modules: if e_i is the identity element of S_i, and if H is a right S-module, then

$$H = \sum_{i=1}^{n} \oplus He_i.$$

Since every finitely generated right S-module is a submodule of a free module, the same is therefore true for each S_i. Therefore we shall assume for the rest of the proof that S is simple.

Let M be a maximal right ideal of S. Then the cyclic S-module S/M is a submodule of a free module, so that there exists a non-zero homomorphism f of S/M into S. Since S/M has no submodules other than itself and zero, f must be one-to-one so that $f(S/M)$ is a minimal right ideal of S. However, any simple ring with identity which contains a minimal right ideal must satisfy the minimum condition (**8**, Chap. IV, § 15, p. 88 (the paragraph preceding Def. 1)).

Before proving that S is a two-sided quotient ring we need several lemmas. We shall make frequent use of the definitions and results of § 1.

LEMMA 5.5. *In a prime ring the product of two non-zero right ideals is non-zero, and the intersection of a non-zero right ideal with a non-zero two-sided ideal is non-zero.*

Proof. Let J and K be non-zero right ideals and choose non-zero elements $x \in J$, $y \in K$. Then $xRy \neq 0$ since R is a prime ring. But $(xR)y \subseteq JK$ so that $JK \neq 0$. If K is a two-sided ideal, then $J \cap K \supseteq JK \neq 0$.

LEMMA 5.6. *Let R have a semi-simple right quotient ring S, let M be a finitely generated submodule of a free right R-module, and let J^* be a minimal right ideal of S. If $M \otimes_R S \cong_S J^*$, then M is R-isomorphic to a right ideal of R which is contained in J^*.*

Proof. Since each of the generators of M is a combination of a finite number of the basis elements of the free module containing M, M is contained in a finitely generated free module. Let $\{u_i : i = 1, 2, \ldots, k\}$ be a basis for this finitely generated free module, and suppose the free module to be chosen so that k has the smallest possible value. Then for each i, $M \cap u_iR \neq 0$. To see this, suppose that $M \cap u_kR = 0$ and let $m_1 = \sum_i u_i r_i$ be an arbitrary element of M. Let $m_2 = \sum_{i<k} u_i r_i + u_k r_k'$ be any other element of M whose first $k - 1$ components are the same as those of m_1. Then $m_1 - m_2 = u_k(r_k - r_k') \in M \cap u_kR = 0$ so that $m_1 = m_2$. Therefore the mapping $m_1 \to \sum_{i<k} u_i r_i$ imbeds M isomorphically in $\sum_{i<k} \oplus u_iR$, contradicting the minimality of k. Therefore $M \cap u_kR \neq 0$. We proceed similarly for $i = 1, 2, \ldots, k - 1$.

M is torsion-free. Therefore we can consider it to be a submodule of $M \otimes_R S = MS$. Let M_1 and M_2 be non-zero R-submodules of M and choose a non-zero element $m_1 \in M_1$. Since MS is S-isomorphic to a minimal right ideal of S, $M_1S = MS = M_2S$. But every element of M_2S can be written in the form m_2d^{-1} for $m_2 \in M_2$, $d \in R$ (Prop. 1.5). In particular we can write m_1 in this form. This gives $m_1d = m_2 \in M_1 \cap M_2$ so that $M_1 \cap M_2 \neq 0$.

Now we can show that the basis $\{u_i\}$ contains only one element. For if

https://doi.org/10.4153/CJM-1963-016-1 Published online by Cambridge University Press

$k \geqslant 2$, then by the first paragraph of this proof we would have $M \cap u_1R \neq 0$ and $M \cap u_2R \neq 0$. The above paragraph would therefore give

$$0 \neq (M \cap u_1R) \cap (M \cap u_2R) \subseteq u_1R \cap u_2R = 0,$$

a contradiction. Therefore $M \subseteq u_1R$, and we can find a right ideal J_1 of R such that $M \cong_R J_1$. Let $J_1{}^* = J_1S$. Then $J_1{}^* \cong_S J^*$. We shall now show that there is an S-isomorphism f of $J_1{}^*$ onto J^* such that $f(J_1) \subseteq R$. Then $f(J_1)$ will be the required right ideal of R.

Let S be written as a direct sum of simple rings. Then since J^* and $J_1{}^*$ are S-isomorphic, they are contained in the same simple component, say S_1, of S. Now let R_1 be the projection of R in S_1. Then by § 1, S_1 is a right quotient ring of R_1, and therefore $J^* \cap R_1 \neq 0$ and $J_1{}^* \cap R_1 \neq 0$ (Lemma 1.1). $R \cap R_1$ is a non-zero two-sided ideal of the prime ring R_1. Therefore by Lemma 5.5, $J^* \cap R \cap R_1 = (J^* \cap R_1) \cap (R \cap R_1)$ and $J_1{}^* \cap R \cap R_1$ are non-zero. Therefore $(J^* \cap R \cap R_1)(J_1{}^* \cap R \cap R_1) \neq 0$. Let $v \in J^* \cap R \cap R_1$ be chosen such that $v(J_1{}^* \cap R \cap R_1) \neq 0$. Then $vJ_1{}^* \neq 0$ and $vJ_1{}^* \subseteq J^*$. By the minimality of J^* this gives

$$vJ_1{}^* = J^*.$$

The map $x \to vx$ of $J_1{}^*$ onto J^* is one-to-one since $J_1{}^*$ is minimal.

If we let $f(x) = vx$ for $x \in J_1{}^*$, then since $v \in J^* \cap R \cap R_1 \subseteq R$, we have $f(J_1) = vJ_1 \subseteq R$ and $f(J_1{}^*) = J^*$, completing the proof.

LEMMA 5.7. *Let R satisfy* (TF) *and have a semi-simple right quotient ring S. Then every finitely generated right R-submodule of S is isomorphic to a right ideal of R.*

Proof. Let $S = J_1{}^* \oplus \ldots \oplus J_n{}^*$ be a decomposition of S into a direct sum of minimal right ideals, and let M be a finitely generated right R-submodule of S. Then every element m of M has a unique expression in the form $m = m_1 + m_2 + \ldots + m_n$ with $m_i \in J_i{}^*$. Let M_i be the projection of M in $J_i{}^*$. Then since M is finitely generated, so is each M_i. Each M_i is torsion-free by Proposition 1.5. Therefore, by hypothesis, M_i is a submodule of a free R-module. If $M_i \neq 0$, then $M_iS = J_i{}^*$ since $J_i{}^*$ is a minimal right ideal. Therefore by Lemma 5.6 there is a right ideal J_i of R such that $M_i \cong_R J_i$ and $J_i \subseteq J_i{}^*$. If $M_i = 0$ we set $J_i = 0$.

Since the sum $J_1{}^* \oplus \ldots \oplus J_n{}^*$ is direct, so is the sum $J_1 \oplus \ldots \oplus J_n$. Suppose that in the isomorphism $M_i \to J_i$ we have $m_i \to r_i$. Then $m = m_1 + \ldots + m_n \to r_1 + \ldots + r_n$ gives an R-isomorphism of M onto a right subideal of $J_1 \oplus \ldots \oplus J_n$.

Completion of the proof of Theorem 5.3. We still have to prove that S is a left quotient ring of R. That is, if $s \in S$ we must show that $s = d^{-1}r$ for properly chosen $d, r \in R$. By Lemma 5.7 the right R-module sR is isomorphic to a right ideal of R. By Lemma 3.6 the isomorphism $sR \to R$ is given by left multiplication by an invertible element v of S. From $vsR \subseteq R$ we conclude that

$$(1) \qquad sR \subseteq v^{-1}R.$$

Let $v^{-1} = d_1 d_2^{-1}$ with $d_1, d_2 \in R$. Since v is invertible in S, d_1 must be a regular element of R. Again, by Lemma 5.7, $d_1^{-1}R + d_2^{-1}R$ is isomorphic to a right ideal of R. Suppose that this isomorphism is given by left multiplication by the invertible element w of S. Then $w(d_1^{-1}R + d_2^{-1}R) \subseteq R$ so that $wd_1^{-1} = c_1 \in R$ and $wd_2^{-1} = c_2 \in R$. Since w is invertible in S, so is c_1, and $d_1 w^{-1} = c_1^{-1}$. Combining the last two equations we get $v^{-1} = d_1 d_2^{-1} = c_1^{-1}c_2$. Substituting this into (1) gives $sR \subseteq c_1^{-1}c_2 R$ so that for some $r \in R$, $s = c_1^{-1}(c_2 r)$ and this completes the proof of Theorem 5.3.

6. Finitely generated modules over semi-hereditary rings. For the study of divisible modules (§ 3) hereditary rings seemed to be the natural generalization of Dedekind rings. The best generalization for the case of finitely generated modules is not so clear, even in the case of commutative integral domains. (For example, see **9**, Thm. 2a.) This section presents two preliminary results towards finding this generalization.

Recall that a ring is *semi-hereditary* if every finitely generated right ideal is projective.

THEOREM 6.1. *Let R be a semi-hereditary ring having a semi-simple two-sided quotient ring. Then every finitely generated right R-module is the direct sum of its torsion submodule and a finite set of right ideals of R.*

Proof. Let M be the module and T its torsion submodule (see Theorem 1.4). Then M/T is finitely generated and torsion-free. Therefore, by Theorem 5.2 (S is semi-simple) M/T is a submodule of a free module. However, for semi-hereditary rings, every finitely generated submodule of a free module is isomorphic to the direct sum of a finite number of right ideals (**2**, Chap. I, Prop. 6.1). Thus $M/T \cong J_1 \oplus \ldots \oplus J_n$ (external direct sum) where each J_i is a right ideal of R. Since R is semi-hereditary, each J_i, and therefore M/T, is projective. This shows that T is a direct summand of M, completing the proof.

THEOREM 6.2. *Let R be the ring of $n \times n$ matrices over a Dedekind ring. Then every finitely generated right R-module M is isomorphic to the direct sum of a free module $R^{(k)}$ (k copies of R), a right ideal J of R (J not isomorphic to R), and a finite number t of indecomposable (and cyclic) torsion modules T_i. In any other such decomposition*

$$M = R^{(k')} \oplus J' \oplus \sum_{i=1}^{t'} \oplus T'_i : k = k', t = t', J \cong J',$$

and for a suitable renumbering of $\{T_i'\}$ we have $T_i \cong T_i'$.

Before proving the theorem we establish the following lemma.

LEMMA 6.3. *Let R be the ring of $n \times n$ matrices over a ring D and let M be a right R-module. If the D-module Me_{11} admits a direct decomposition $Me_{11} = \sum_{i\epsilon I} \oplus C_i$, then $M = \sum_{i\epsilon I} \oplus C_iR$.*

Proof. Clearly $M = Me_{11}R$ so that we only have to show that the sum $\sum_{i\epsilon I} C_iR$ is direct. Therefore suppose that for some finite subset J of I, $\sum_{i\epsilon J} m_i = 0$ with $m_i \in C_iR = C_ie_{11}R$. Then for each j, $\sum_{i\epsilon J} m_ie_{j1} = 0$. But $m_ie_{j1} \in C_iRe_{j1} = C_ie_{11}Re_{j1} = C_ie_{11}Re_{11} = C_ie_{11}D = C_i$. Therefore for every pair (i, j), $m_ie_{j1} = 0$. Multiplying by e_{1j} and summing gives

$$0 = \sum_{j=1}^{n} m_ie_{j1}e_{1j} = m_i,$$

proving the lemma.

Proof of Theorem 6.2. Let the given Dedekind ring be D. Then the ring S of $n \times n$ matrices over the quotient field of D is a two-sided quotient ring of R (Proposition 2.9). We observe that every torsion element of the R-module M is also a torsion element of the D-module M. Suppose $md = 0$ with $m \in M$ and d regular in R. The matrix d^{-1} has entries in the quotient field of D. Let $\alpha \in D$ be a common denominator for these entries so that $d^{-1}\alpha \in R$. Then $m\alpha = m \cdot 1\alpha = (md)(d^{-1}\alpha) = 0$.

Since M is a finitely generated R-module, and R is a finitely generated right D-module, M is a finitely generated D-module. Therefore the direct summand Me_{11} of the D-module M is finitely generated. According to the theory of finitely generated modules over Dedekind rings (see **3** or **9**)

$$Me_{11} = P \oplus \sum_{i=1}^{t} \oplus m_iD,$$

where P is the direct sum of a free D-module and an ideal of D, and each m_iD is an indecomposable, cyclic torsion D-module. By Lemma 6.3,

$$M = PR \oplus \sum_{i=1}^{t} \oplus m_iR.$$

Since $PRe_{11} = Pe_{11}Re_{11} = P$ $(Pe_{11} = P)$, $PR \cong_R R^{(k)} \oplus J$ for some k and some right ideal J not isomorphic to R (Proposition 2.5). Since each m_i is a torsion element of M, each m_iR is a torsion module (by Theorem 1.4, the torsion elements of M form a submodule). Each m_iR is indecomposable as an R-module because of Lemma 6.3 $(m_iRe_{11} = m_iD)$. This establishes the existence of the decomposition.

To establish the uniqueness, we first note that the decomposition of the torsion submodule T is unique, for by the theory of modules over a Dedekind ring, the decomposition of Te_{11} is unique, and then we use Lemma 2.1. To show the uniqueness of k and J we consider M/T. That is, suppose $R^{(k)} \oplus J \cong R^{(k')} \oplus J'$. Then multiplication by e_{11} gives

$$D^{(nk)} \oplus Je_{11} \cong_D D^{(nk')} \oplus J'e_{11}.$$

Suppose, without loss of generality, that $k \leqslant k'$. Then

$$Je_{11} \cong_D D^{(nk'-nk)} \oplus J'e_{11}$$

since both sides are finitely generated torsion-free modules over a Dedekind ring. The rank of the left side (as a D-module) cannot exceed n. Therefore $nk' - nk = 0$ or n. If the second alternative holds, then $J'e_{11} = 0$ since otherwise the rank of the right-hand side would exceed n. This would give $Je_{11} \cong_D D^{(n)} \cong_D Re_{11}$ so that $J \cong_R R$ (Lemma 2.1), a contradiction. Therefore $nk = nk'$ and $Je_{11} \cong_D J'e_{11}$, giving $J \cong_R J'$.

References

1. E. Artin, C. Nesbitt, and R. Thrall, *Rings with minimum condition* (Ann Arbor, 1944).
2. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton, 1956).
3. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (New York, 1962).
4. E. Gentile, *On rings with one-sided field of quotients*, Proc. Amer. Math. Soc., *11* (1960), 380–384.
5. A. W. Goldie, *The structure of prime rings under ascending chain conditions*, Proc. London Math. Soc., Third Ser., *8*, no. 32 (1958), 589–608.
6. ——— *Semi-prime rings with maximum condition*, Proc. Amer. Math. Soc., Third Ser., *10*, no. 38 (1960), 201–220.
7. N. Jacobson, *The theory of rings* (New York, 1943).
8. ——— *Structure of rings* (Providence, 1956).
9. I. Kaplansky, *Modules over Dedekind and valuation rings*, Trans. Amer. Math. Soc., *72* (1952), 327–340.
10. L. S. Levy, *Unique subdirect sums of prime rings*, to appear in Trans. Amer. Math. Soc.
11. E. Matlis, *Injective modules over noetherian rings*, Pac. J. Math., *8* (1958), 511–528.

University of Illinois
and
University of Wisconsin