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On probabilities of large deviations

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Let $\{X_n\}$ be a sequence of independent identically distributed random variables and let $S_n = \sum_{k=1}^{n} X_k$. The rate of convergence of probabilities $P\left\{|S_n| > (n\log n)^{1/r}\right\}$, where $2 > r > 1$, is studied.

1. Introduction

Let $\{X_n : n \geq 1\}$ denote a sequence of independent identically distributed random variables with common distribution function F. Write $S_n = \sum_{k=1}^{n} X_k$. If F belongs to the domain of normal attraction of a stable law $V(x)$ with characteristic exponent α $(1 < \alpha < 2)$ then for some $a > 0$ and some A_n

$$\lim_{n\to\infty} P\left\{a^{-1}n^{-1/\alpha}S_n - A_n \leq x\right\} = V(x) .$$

(See, for example [3, p. 181].) If, moreover, $EX_1 = 0$ then the constants A_n may be taken to be zero and it follows that $a^{-1}n^{-1/\alpha}(\log n)^{-1/\alpha}S_n \to 0$ in probability. Clearly

$$P\left\{|S_n| > \varepsilon(n\log n)^{1/\alpha}\right\} \geq P\left\{|S_n| > \varepsilon(n\log n)^{1/r}\right\}$$

for $0 < r < \alpha$ and it follows that $(n\log n)^{-1/r}S_n \to 0$ in probability.

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The probability $P\left\{|S_n| > \varepsilon(n\log n)^{1/r}\right\}$ or either of its one sided components is called a probability of large deviation [7]. If X has a finite variance the probability $P\left\{|S_n| > \varepsilon(n\log n)^{1/2}\right\}$ is called a probability of moderate deviation (see [1], [5], [6]). We remark that we do not assume that $EX_2^1 < \infty$ but only that $E|X_1|^r < \infty$ for $1 < r < 2$.

In what follows we will assume that $\{X_n\}$ is a sequence of independent identically distributed random variables with common distribution function F. A median for the random variable X is denoted by $\text{med}(X)$ and $\lg x$ is the function defined by $\lg x = \log_e x$ for $x > 1$, $= 0$ otherwise. c denotes a generic (positive) constant and $1 < r < 2$.

2. Results

Lemmas 1 and 2 are stated here for completness. For proofs we refer to [1].

LEMMA 1. *For* $r \geq 1$, $E|X|^r < \infty$ *if and only*

$$\sum n^{r-1}(\log n)^r P\{|X| > n\log n\} < \infty .$$

LEMMA 2. *Let* $\{A_n\}$ *be a sequence of independent events. If* $\sum PA_n < \infty$ *then*

$$P\{\cup_n A_n\} \geq \sum_n PA_n - \sum_n PA_n \sum_{j=n+1}^{\infty} PA_j .$$

THEOREM. *For* $1 < r < 2$ *the following statements are equivalent:*

(a) $EX_1 = 0$, $E|X_1|^r < \infty$;

(b) $\sum n^{-1}\log n P\left\{|S_n| > \varepsilon(n\log n)^{1/r}\right\} < \infty$ *for all* $\varepsilon > 0$;

(c) $\sum n^{-1}\log n P\left\{\max_{1\leq k\leq n} |S_k| > \varepsilon(n\lg n)^{1/r}\right\} < \infty$ *for all* $\varepsilon > 0$;

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(d) $\sum n^{-1}P\left\{\sup_{k\ge n}\left|S_k/(k\lg k)^{1/r}\right| > \varepsilon\right\} < \infty$ *for all* $\varepsilon > 0$.

Proof. The methods of proof parallel those used in [*1*] and are fairly standard.

Equivalence of *(a)* and *(b)*

It is convenient to make the proofs for symmetrized random variables X_n^s , $n = 1, 2, \ldots$ and then use the weak symmetrization inequalities [*4*, p. 245] to transfer to the required results.

Suppose that *(a)* holds and write $S_n^s = \sum_1^n X_k^s$. Note that $E|X_1^s|^r < \infty$. (See [*4*, p. 246].) Define

$$X_{kn}^s = \begin{cases} X_k^s & \text{if } |X_k^s| < \varepsilon(n\lg n)^{1/r} \\ 0 & \text{otherwise,} \end{cases} \qquad 1 \le k \le n$$

and let $S_{nn}^s = \sum_1^n X_{kn}^s$. Since

$$P\left\{|S_n^s| > \varepsilon(n\lg n)^{1/r}\right\} \le nP\left\{|X_1^s| > \varepsilon(n\lg n)^{1/r}\right\} + P\left\{|S_{nn}^s| > \varepsilon(n\lg n)^{1/r}\right\} ,$$

and from Lemma 1, $E|X_1^s|^r < \infty$ implies $\sum \lg n P\left\{|X_1^s| > \varepsilon(n\lg n)^{1/r}\right\} < \infty$, it just remains to show that $\sum \frac{\lg n}{n} P\left\{|S_{nn}^s| > \varepsilon(n\lg n)^{1/r}\right\} < \infty$.

From Markov's inequality [*4*, p. 148] we have

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$$\sum \frac{\lg n}{n} P\left\{\left|S^s_{nn}\right| > \varepsilon(n\lg n)^{1/r}\right\}$$
$$\le c \sum (n\lg n)^{-2/r} \lg n E\left(X^s_{1n}\right)^2$$
$$\le c \sum_n n^{-2/r}(\lg n)^{1-2/r} \sum_k (k\lg k)^{2/r} P\{(k-1)\lg(k-1) \le |X^s|^r < k\lg k\}$$
$$= c \sum_k (k\lg k)^{2/r} P\{(k-1)\lg(k-1) \le |X^s|^r < k\lg k\} \sum_{n=k}^{\infty} n^{-2/r}(\lg n)^{1-2/r}$$
$$\le c \sum_k (k\lg k) P\{(k-1)\lg(k-1) \le |X^s|^r < k\lg k\}$$
$$< \infty .$$

The last series converges because $E|X^s_1|^r < \infty$.

Now note that $E|X_1|^r < \infty$,

$$EX_1 = 0 \Rightarrow n^{-1/r} S_n \xrightarrow{P} 0 \Rightarrow (n\lg n)^{-1/r} S_n \xrightarrow{P} 0 \Rightarrow \operatorname{med}\left(\frac{S_n}{(n\log n)^{1/r}}\right) \to 0 .$$

It is now easy to complete the proof of *(a)* $\Rightarrow$ *(b)* by a simple use of weak symmetrization inequalities.

Next suppose that *(b)* holds. By the symmetrization inequalities $\sum n^{-1} \lg n P\left\{S^s_n > \varepsilon(n\lg n)^{1/r}\right\} < \infty$ where S^s_n , as before, is the sum of the symmetrized random variables. We first show that $(n\lg n)^{-1/r} S^s_n \xrightarrow{P} 0$. If not, there exists an $\varepsilon_0 > 0$ such that either $P\left\{S^s_{n_k} > \varepsilon_0 \left(n_k \lg n_k\right)^{1/r}\right\} > \varepsilon_0$ or $P\left\{S^s_{n_k} < -\varepsilon_0 \left(n_k \lg n_k\right)^{1/r}\right\} > \varepsilon_0$ for infinitely many k . For the sake of argument assume $P\left\{S^s_{n_k} > \varepsilon_0 \left(n_k \lg n_k\right)^{1/r}\right\} > \varepsilon_0$ for infinitely many k and choose $n_{k+1} > 2n_k$. Then for each j , $n_k \le j < 2n_k$, we have

$$(j\lg j)^{1/r} < \left[2n_k \lg\left(2n_k\right)\right]^{1/r} \le 2^{2/r}\left(n_k \lg n_k\right)^{1/r}$$

and

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$$P\left\{S_j^s > \varepsilon_0 \frac{(j\lg j)^{1/r}}{2^{2/r}}\right\} \ge P\left\{S_j^s > \varepsilon_0 \left(n_k \lg n_k\right)^{1/r}\right\}$$

$$\ge \frac{1}{2} P\left\{S_{n_k}^s > \varepsilon_0 \left(n_k \lg n_k\right)^{1/r}\right\}$$

$$\ge \frac{\varepsilon_0}{2} .$$

It follows that

$$\sum \frac{\lg n}{n} P\left\{S_n^s > \frac{\varepsilon_0}{2^{2/r}} (n\lg n)^{1/r}\right\} \ge \sum_k \sum_{i=n_k}^{2n_k} \frac{\lg i}{i} P\left\{S_i^s > \frac{\varepsilon_0}{2^{2/r}} (i\lg i)^{1/r}\right\}$$

$$\ge \frac{\varepsilon_0}{2} \sum_k \sum_{i=n_k}^{2n_k} \frac{\lg i}{i} = \infty .$$

This contradiction shows that $(n\lg n)^{-1/r} S_n^s \to 0$ in probability. By the degenerate convergence criterion [4, p. 317] we get $nP\left\{X_n^s > \varepsilon(n\lg n)^{1/r}\right\} \to 0$. Following Erdös [2] we write $A_k = \left\{X_k^s > \varepsilon(n\lg n)^{1/r}\right\}$ and $B_k = \left\{\sum_{j\ne k}^{n} X_j^s \ge 0\right\}$ and see that

$$P\left\{S_n^s > \varepsilon(n\lg n)^{1/r}\right\} \ge P\left\{\bigcup_{k=1}^{n} \left(A_k \cap B_k\right)\right\}$$

$$\ge \sum^{n} PA_i \left[PB_i - nPA_i\right]$$

$$\ge nPA_i \left|\frac{1}{2} - nPA_i\right| .$$

Thus for $\delta > 0$ and large n we have

$$P\left\{S_n^s > \varepsilon(n\lg n)^{1/r}\right\} \ge \left(\frac{1}{2} - \delta\right) nPA_i .$$

It follows that

$$\infty > \sum n^{-1} \lg n P\left\{S_n^s > \varepsilon(n\lg n)^{1/r}\right\} \ge c \sum \lg n P\left\{X_1^s > \varepsilon(n\lg n)^{1/r}\right\} .$$

By Lemma 1, we obtain $E\left|X_1^s\right|^r < \infty$ and thus $E\left|X_1\right|^r < \infty$ by Corollary 2 [4,

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p. 246].

To show that $EX_1 = 0$ we only have to note that

$$E|X_1|^r < \infty \Rightarrow n^{-1}S_n \xrightarrow{a.s.} EX_1 \quad \text{and} \quad (n\lg n)^{-1/r}S_n \xrightarrow{P} 0 \ .$$

Equivalence of *(b)* and *(c)*

The *(c)* $\Rightarrow$ *(b)* part is trivial and the *(b)* $\Rightarrow$ *(c)* part follows from Lévy's inequality [4, p. 247] and the fact that *(b)* $\Rightarrow (n\lg n)^{-1/r}S_n \xrightarrow{P} 0$.

Equivalence of *(d)* and *(a)*

We first show that *(a)* and *(b)* $\Rightarrow$ *(d)*. Choose i such that $2^i \le n < 2^{i+1}$ and again consider the symmetrized random variables X_k^s and S_n^s . We have

$$\begin{aligned} P\left\{S_n^s > \varepsilon(n\lg n)^{1/r}\right\} &\ge P\left\{S_n^s > \varepsilon\left(2^{i+1}\lg 2^{i+1}\right)^{1/r}\right\} \\ &\ge \frac{1}{2} P\left\{S_{2^i}^s > \varepsilon\left(2^{i+1}\lg 2^{i+1}\right)^{1/r}\right\} \\ &\ge \frac{1}{2} P\left\{S_{2^i}^s > \varepsilon 2^{2/r}\left(2^i\lg 2^i\right)^{1/r}\right\} . \end{aligned}$$

Using once again the symmetrization inequalities, we have

$$\begin{aligned} \infty > \sum n^{-1}\lg n P\left\{S_n^s > \varepsilon(n\lg n)^{1/r}\right\} &= \sum_i \sum_{n=2^i}^{2^{i+1}-1} \frac{\lg n}{n} P\left\{S_n^s > \varepsilon(n\lg n)^{1/r}\right\} \\ &\ge \frac{1}{4} \sum_i \lg 2^i P\left\{S_{2^i}^s > 2^{2/r}\varepsilon\left(2^i\lg 2^i\right)^{1/r}\right\} , \end{aligned}$$

so that

$$\sum_n n^{-1}P\left\{\sup_{k\geq n} \frac{S_k^s}{(k\lg k)^{1/r}} > \varepsilon\right\}$$

$$\leq c \sum_i P\left\{\sup_{k\geq 2^i} \frac{S_k^s}{(k\lg k)^{1/r}} > \varepsilon\right\}$$

$$\leq c \sum_i \sum_{j=i}^{\infty} P\left\{\max_{2^j\leq k<2^{j+1}} \frac{S_k^s}{(k\lg k)^{1/r}} > \varepsilon\right\}$$

$$\leq c \sum_i \sum_{j=i}^{\infty} P\left\{S^s_{2^{j+1}} > \varepsilon\left(2^j\lg 2^j\right)^{1/r}\right\} \quad \text{(Lévy's inequality)}$$

$$= c \sum_{j=1}^{\infty} jP\left\{S^s_{2^{j+1}} > \varepsilon\left(2^j\lg 2^j\right)^{1/r}\right\}$$

$$\leq c \sum_j \lg 2^{j+1}P\left\{S^s_{2^{j+1}} > \varepsilon 2^{-2/r}\left(2^{j+1}\lg 2^{j+1}\right)^{1/r}\right\}$$

$$\leq c \sum_n n^{-1}\lg nP\left\{S_n^s > \varepsilon \,.\, 2^{-4/r}(n\lg n)^{1/r}\right\}$$

$$< \infty .$$

It just remains to use the weak symmetrization inequalities and the fact that $(n\lg n)^{-1/r}S_n \xrightarrow{P} 0$ to see that *(a)* and *(b)* ⇒ *(d)*.

Next we show that *(d)* ⇒ *(a)*.

Let $A_k = \left\{|X_k| > \varepsilon(k\lg k)^{1/r}\right\}$. Then the events A_k are independent and satisfy

$$\bigcup_{k=m+1}^{\infty} A_k \subset \bigcup_{k=m}^{\infty} \left\{|S_k| > \frac{\varepsilon}{2}(k\lg k)^{1/r}\right\} ,$$

so that

$$P\left\{\bigcup_{m+1}^{\infty} A_k\right\} \leq P\left\{\bigcup_{k=m}^{\infty} \left\{|S_k| > \frac{\varepsilon}{2}(k\lg k)^{1/r}\right\}\right\}$$

$$= P\left\{\sup_{k\geq m} \left|\frac{S_k}{(k\lg k)^{1/r}}\right| > \frac{\varepsilon}{2}\right\}.$$

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which $\to 0$ as $m \to \infty$ because of the fact that the sequence $P\left\{\sup_{k \geq m} \left|\frac{S_k}{(k \lg k)^{1/r}}\right| > \frac{\varepsilon}{2}\right\}$ is non-increasing in m and *(d)* holds. An application of the Borel zero-one criterion [4, p. 228] now shows that

$$\sum_{k=1}^{\infty} P\left\{|X_1| > \varepsilon(k \lg k)^{1/r}\right\} = \sum_{k=1}^{\infty} PA_k < \infty .$$

By Lemma 2 therefore

$$\begin{aligned} \infty &> \sum_{n=2}^{\infty} n^{-1} P\left\{\sup_{k \geq n} \left|\frac{S_k}{(k \lg k)^{1/r}}\right| > \frac{\varepsilon}{2}\right\} \\ &\geq \sum_{n=2}^{\infty} n^{-1} P\left\{\bigcup_{k=0}^{\infty} \left\{|X_{n+k}| > \varepsilon\big((n+k)\lg(n+k)\big)^{1/r}\right\}\right\} \\ &\geq c \sum n^{-1} \sum_{k=0}^{\infty} P\left\{|X_{n+k}| > \varepsilon\big((n+k)\lg(n+k)\big)^{1/r}\right\}(1-\eta) \\ &\geq c \sum_{k=2}^{\infty} P\left\{|X_1| > \varepsilon(k \lg k)^{1/r}\right\} \sum_{n=2}^{k} \frac{1}{n} \\ &\geq c \sum_{k} \lg k P\left\{|X_1| > \varepsilon(k \lg k)^{1/r}\right\} . \end{aligned}$$

It follows by Lemma 1 that $E|X_1|^r < \infty$. From *(d)* we see that $n^{-1}S_n \xrightarrow{a.s} 0$ so that we must have $EX_1 = 0$. Thus *(d)* ⇒ *(a)*.

This completes the proof of the theorem.

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