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# DIFFERENCES OF COMPOSITION OPERATORS BETWEEN WEIGHTED BERGMAN SPACES AND WEIGHTED BANACH SPACES OF HOLOMORPHIC FUNCTIONS

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**Abstract.** We characterise boundedness and compactness of differences of composition operators acting between weighted Bergman spaces $A_{v,p}$ and weighted Banach spaces $H_w^\infty$ of holomorphic functions defined on the open unit disk $D$.

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**1. Introduction.** Let $\phi$ and $\psi$ be analytic self-maps of the open unit disk $D$ in the complex plane. Such maps induce linear composition operators $C_\phi(f) = f \circ \phi$ resp. $C_\psi(f) = f \circ \psi$ acting on the space $H(D)$ of holomorphic functions on $D$.

Moreover, let $v$ and $w$ be strictly positive bounded continuous functions (*weights*) on $D$. We are interested in differences $C_\phi - C_\psi$ of composition operators acting between the weighted Bergman space

$$A_{v,p} = \left\{ f \in H(D);\ \|f\|_{v,p} := \left( \int_D |f(z)|^p v(z)\, dA(z) \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 \le p < \infty,$$

where $dA(z)$ is the area measure on $D$ normalised so that area of $D$ is 1 and the weighted Banach space of holomorphic functions (weighted Bergman space of infinite order),

$$H_w^\infty := \{ f \in H(D);\ \|f\|_w := \sup_{z \in D} w(z)|f(z)| < \infty \}.$$

Composition operators and weighted composition operators have been studied on various spaces of holomorphic functions, see e.g. [**2–4**, **6**]. For more general information on composition operators we refer to the monographs [**7**] and [**15**]. Boundedness and compactness of differences of composition operators on various spaces of analytic functions have been investigated by several authors, see e.g. [**5**, **11**, **13**, **14**]. In this paper we want to characterise boundedness and compactness of differences of composition operators acting between spaces of the type defined above in terms of the weights.

**2. Preliminaries.** First, we need some geometric data of the open unit disk. Fix $\alpha \in D$ and consider the automorphism $\varphi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} z}$, $z \in D$, which interchanges 0 and $\alpha$. Then the pseudohyperbolic metric is defined by

$$\rho(z, \alpha) := |\varphi_\alpha(z)| = \left| \frac{\alpha - z}{1 - \overline{\alpha} z} \right| \quad \text{for every } z, \alpha \in D.$$

Moreover, we use the fact that

$$-\varphi_\alpha'(z) = \frac{1-|\alpha|^2}{(1-\overline{\alpha}z)^2}, \quad z \in D.$$

Next, we need some information on weights and weighted spaces. We denote by $B_w^\infty$ the closed unit ball of $H_w^\infty$. We are interested in *radial* weights, i.e. weights $w$ which satisfy $w(z) = w(|z|)$ for every $z \in D$. The formulation of results concerning weighted spaces of holomorphic functions requires the so-called *associated weights*. For a weight $w$, the associated weight $\tilde{w}$ is defined as follows:

$$\tilde{w}(z) := \frac{1}{\sup\{|f(z)|;\, f \in B_w^\infty\}}, \quad z \in D.$$

The associated weights are also continuous and $\tilde{w} \geq w > 0$ (see **[1]**). Furthermore, for each $z \in D$ there is $f_z \in H_w^\infty$, $\|f_z\|_w \leq 1$, such that $|f_z(z)| = \frac{1}{\tilde{w}(z)}$. A weight $v$ is called *essential* if there is a constant $C > 0$ with

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \quad \text{for every } z \in D.$$

For examples of essential weights and conditions when weights are essential, see **[1–3]**. In this article the following condition $(L1)$ (which is due to Lusky, see **[12]**) plays an important role. We say, a radial weight $v$ satisfies $(L1)$ if

$$\inf_k \frac{v(1-2^{-k-1})}{v(1-2^{-k})} > 0.$$

This condition is equivalent to the following condition (see e.g. **[8]**):

(a) There are $0 < r < 1$ and $1 < C < \infty$ with $\dfrac{v(z)}{v(a)} \leq C$ for every $a, z \in D$ with $\rho(z, a) \leq r$.

For a proof of this equivalence, we refer the reader to **[8]** or **[11]**. By **[1]** radial weights satisfying $(L1)$ are essential.

In the sequel we consider the following weights. Let $\nu$ be a holomorphic function on $D$, non-vanishing, strictly positive on $[0, 1]$ and satisfying $\lim_{r\to 1} \nu(r) = 0$. Then we define the weight $v$ as follows: $v(z) := \nu(|z|^2)$ for every $z \in D$.

Next, we give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1-z)^\alpha$, $\alpha \geq 1$. Then the corresponding weight is the so-called standard weight $v(z) = (1-|z|^2)^\alpha$.

(ii) Select $\nu(z) = e^{-\frac{1}{(1-z)^\alpha}}$, $\alpha \geq 1$. Then we obtain the weight $v(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$.

(iii) Choose $\nu(z) = \sin(1-z)$ and the corresponding weight is given by $v(z) = \sin(1-|z|^2)$.

Examples (i)–(iii) also satisfy condition $(L1)$ (see **[12]**). Hence, the class of weights we introduce here contains the classical examples, which have been studied before as well as some other weights. Thus, in our studies the Bergman space, which is weighted with the standard weight, is included as well as the Bergman space weighted with an exponential weight.

For a fixed point $a \in D$ we want to introduce a function $v_a(z) := \nu(\overline{a}z)$ for every $z \in D$. Since $\nu$ is holomorphic on $D$, the function $v_a$ is also holomorphic on $D$.

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**3. Results.** We first need the following auxiliary result. The following lemma is well known for standard weights (see **[9]** or **[10]**), but to the best of our knowledge not known for the weights described above. The following results are valid for all $1 \leq p < \infty$.

LEMMA 1. *Let $v$ be a radial weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in D$) such that $\sup_{a \in D} \sup_{z \in D} \frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \leq C < \infty$. Then*

$$|f(z)| \leq \frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\|f\|_{v,p}$$

*for all $z \in D$ and all $f \in A_{v,p}$.*

*Proof.* Let $\alpha \in D$ be an arbitrary point. Consider the map

$$T_\alpha : A_{v,p} \to A_{v,p}, \;\; T_\alpha(f(z)) = f(\varphi_\alpha(z))\varphi_\alpha'(z)^{\frac{2}{p}}v_\alpha(\varphi_\alpha(z))^{\frac{1}{p}}.$$

Then a change of variables yields

$$\begin{aligned}
\|T_\alpha f\|_{v,p}^p &= \int_D v(z)|f(\varphi_\alpha(z))|^p|\varphi_\alpha'(z)|^2|v_\alpha(\varphi_\alpha(z))|\, dA(z),\\
&= \int_D \frac{v(z)|v_\alpha(\varphi_\alpha(z))|}{v(\varphi_\alpha(z))}|f(\varphi_\alpha(z))|^p|\varphi_\alpha'(z)|^2 v(\varphi_\alpha(z))\, dA(z)\\
&\leq \sup_{z \in D}\frac{v(z)v_\alpha(\varphi_\alpha(z))}{v(\varphi_\alpha(z))}\int_D |f(\varphi_\alpha(z))|^p|\varphi_\alpha'(z)|^2 v(\varphi_\alpha(z))\, dA(z)\\
&\leq C\int_D v(t)|f(t)|^p\, dA(t) = C\|f\|_{v,p}^p.
\end{aligned}$$

Now put $g(z) := T_\alpha(f(z))$. By the mean-value property we obtain

$$v(0)|g(0)|^p \leq \int_D v(z)|g(z)|^p\, dA(z) = \|g\|_{v,p}^p \leq C\|f\|_{v,p}^p.$$

Hence,

$$v(0)|g(0)|^p = v(0)|f(\alpha)|^p(1-|\alpha|^2)^2 v(\alpha) \leq C\|f\|_{v,p}^p.$$

Thus, $|f(\alpha)| \leq C^{\frac{1}{p}}\frac{\|f\|_{v,p}}{v(0)^{\frac{1}{p}}(1-|\alpha|^2)^{\frac{2}{p}}v(\alpha)^{\frac{1}{p}}}$. Since $\alpha$ was arbitrary, the claim follows. □

Next, we need an estimate for the difference. In the case of weighted Banach spaces of holomorphic functions a similar lemma was proved in **[8]**.

LEMMA 2. *Let $v$ be a radial weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in D$) such that $\sup_{a \in D} \sup_{z \in D} \frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \leq C < \infty$. Moreover, assume that the weight $v$ satisfies condition $(L1)$. Then there exist $0 < r < 1$ and a constant $M > 0$ such that for $f \in A_{v,p}$*

$$|f(z) - f(a)| \leq \frac{4MC^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}}\frac{\|f\|_{v,p}}{r(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\rho(z,a)$$

*for every $z, a \in D$ with $\rho(z,a) \leq \frac{r}{2}$.*

*Proof.* By hypothesis $v$ has condition $(L1)$ and, moreover, we know that $(L1)$ is equivalent to condition $(A)$. Since the weight $u(z) = 1 - |z|^2$ also satisfies condition $(L1)$, we can find $0 < r < 1$ and constants $M_1 < \infty$ and $M_2 < \infty$ such that

$$\frac{v(z)}{v(a)} \leq M_1 \text{ and } \frac{1-|z|^2}{1-|a|^2} \leq M_2 \text{ for every } z, a \in D \text{ with } \rho(z, a) \leq r.$$

Let $a \in D$ be fixed. Since $\varphi_a(\varphi_a(z)) = z$ and $\varphi_a(0) = a$, we get

$$|f(z) - f(a)| = |f(\varphi_a(\varphi_a(z)) - f(\varphi_a(\varphi_a(a))|.$$

For $|z| = \rho(\varphi_a(z), a) \leq r$ we obtain by using Lemma 1

$$\begin{aligned} |f(\varphi_a(z))| &\leq \frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}(1-|\varphi_a(z)|^2)^{\frac{2}{p}}v(\varphi_a(z))^{\frac{1}{p}}}\|f\|_{v,p} \\ &= \frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}}\frac{\|f\|_{v,p}}{(1-|a|^2)^{\frac{2}{p}}v(a)^{\frac{1}{p}}}\frac{v(a)^{\frac{1}{p}}(1-|a|^2)^{\frac{2}{p}}}{(1-|\varphi_a(z)|^2)^{\frac{2}{p}}v(\varphi_a(z))^{\frac{1}{p}}} \\ &\leq \frac{C^{\frac{1}{p}}M_1^{\frac{1}{p}}M_2^{\frac{2}{p}}}{v(0)^{\frac{1}{p}}}\frac{\|f\|_{v,p}}{(1-|a|^2)^{\frac{2}{p}}v(a)^{\frac{1}{p}}}. \end{aligned}$$

Let us now consider $g_a(z) := f(\varphi_a(z))$. Thus, for $\rho(z, a) = |\varphi_a(z)| \leq \frac{r}{2}$ we can find $\theta \in D$ with $|\theta| \leq |\varphi_a(z)| \leq \frac{r}{2}$ such that

$$\begin{aligned} |f(z) - f(a)| &= |g_a(\varphi_a(z)) - g_a(0)| \\ &\leq |\varphi_a(z)| \left| \int_0^1 \left[\frac{\partial}{\partial t} g_a\right](t\varphi_a(z))dt \right| \\ &\leq \left| \frac{\partial}{\partial z} g_a(\theta) \right| |\varphi_a(z)| \\ &= |\varphi_a(z)| \frac{1}{2\pi} \left| \int_{|\xi|=r} \frac{g_a(\xi)}{(\xi-\theta)^2}\, d\xi \right|. \end{aligned}$$

Finally,

$$\begin{aligned} |f(z) - f(a)| &\leq \frac{C^{\frac{1}{p}}M_1^{\frac{1}{p}}M_2^{\frac{2}{p}}}{v(0)^{\frac{1}{p}}}|\varphi_a(z)| r \frac{\|f\|_{v,p}}{(r-|\varphi_a(z)|)^2}\frac{1}{(1-|a|^2)^{\frac{2}{p}}v(a)^{\frac{1}{p}}} \\ &\leq \frac{4C^{\frac{1}{p}}M_1^{\frac{1}{p}}M_2^{\frac{2}{p}}\|f\|_{v,p}}{v(0)^{\frac{1}{p}}r(1-|a|^2)^{\frac{2}{p}}v(a)^{\frac{1}{p}}}\rho(z, a). \end{aligned}$$

We select $M := M_1^{\frac{1}{p}}M_2^{\frac{2}{p}}$ and obtain the claim. □

LEMMA 3. *Let $v$ be a weight as defined in the previous section (i.e. $v(z) := v(|z|^2)$ for every $z \in D$) such that $\sup_{a\in D}\sup_{z\in D}\frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \leq C < \infty$ and $v$ satisfies condition $(L1)$.*

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*Then for every $f \in A_{v,p}$ there is $C_v > 0$ such that*

$$|f(z) - f(a)| \leq C_v \|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}}, \frac{1}{(1-|a|^2)^{\frac{2}{p}} v(a)^{\frac{1}{p}}}\right\} \rho(z,a)$$

*for every $z, a \in D$.*

*Proof.* By Lemma 2. we can find $0 < s < 1$ and a constant $M < \infty$ such that

$$|f(z) - f(a)| \leq \frac{4MC^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}} \frac{\|f\|_{v,p}}{s(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}} \rho(z,a)$$

for every $z, a \in D$ with $\rho(z,a) \leq \frac{s}{2}$. Next, if $\rho(z,a) > \frac{s}{2}$, then

$$\begin{aligned} |f(z) - f(a)| &\leq 2\frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}} \|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}}, \frac{1}{(1-|a|^2)^{\frac{2}{p}} v(a)^{\frac{1}{p}}}\right\} \\ &\leq \frac{4}{s} \frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}} \|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}}, \frac{1}{(1-|a|^2)^{\frac{2}{p}} v(a)^{\frac{1}{p}}}\right\} \rho(z,a). \end{aligned}$$

Hence with $C_v := \max\left\{\frac{4MC^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}} \frac{\|f\|_{v,p}}{s}, \frac{4}{s}\frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}} \|f\|_{v,p}\right\}$ we conclude

$$|f(z) - f(a)| \leq C_v \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}}, \frac{1}{(1-|a|^2)^{\frac{2}{p}} v(a)^{\frac{1}{p}}}\right\} \rho(z,a)$$

for every $z, a \in D$ and the claim follows. □

THEOREM 4. *Let $w$ be an arbitrary weight and $v$ be a weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in D$) such that $\sup_{a \in D} \sup_{z \in D} \frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \leq C < \infty$ and such that $v$ has condition $(L1)$. Moreover, let $\phi$ and $\psi$ be analytic self-maps of $D$. Then the difference $C_\phi - C_\psi : A_{v,p} \to H_w^\infty$ is bounded if and only if*

$$\sup_{z \in D} w(z) \max\left\{\frac{1}{v(\phi(z))^{\frac{1}{p}}(1-|\phi(z)|^2)^{\frac{2}{p}}}, \frac{1}{v(\psi(z))^{\frac{1}{p}}(1-|\psi(z)|^2)^{\frac{2}{p}}}\right\} \rho(\phi(z), \psi(z)) < \infty.$$

*Proof.* By **[1]** we know that under the given assumptions $v$ and $\tilde{v}$ are equivalent, i.e. we can find a constant $k > 0$ such that $v(z) \leq \tilde{v}(z) \leq kv(z)$ for every $z \in D$. First, we suppose that the difference is bounded and want to show that $\sup_{z \in D} \frac{w(z)}{v(\phi(z))^{\frac{1}{p}}(1-|\phi(z)|^2)^{\frac{2}{p}}} < \infty$ indirectly. We can assume that there is a sequence $(z_n)_n \subset D$ such that $|\phi(z_n)| \to 1$ and

$$\frac{w(z_n)}{v(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{\frac{2}{p}}} \geq n$$

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for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and choose $f_n^p \in B_v^\infty$ such that $|f_n(\phi(z_n))|^p = \frac{1}{\tilde{v}(\phi(z_n))}$. Now put $g_n(z) := f_n(z)\varphi'_{\phi(z_n)}(z)^{\frac{2}{p}}\varphi_{\psi(z_n)}(z)$ for every $z \in D$. Then a change of variables yields

$$\begin{aligned}\|g_n\|_{v,p}^p &= \int_D |g_n(z)|^p v(z)\, dA(z) = \int_D |f_n(z)|^p|\varphi'_{\phi(z_n)}(z)|^2|\varphi_{\psi(z_n)}(z)|^p v(z)\, dA(z)\\ &\leq \sup_{z\in D} v(z)|f_n(z)|^p \sup_{z\in D}|\varphi_{\psi(z_n)}(z)|^p \int_D |\varphi'_{\phi(z_n)}(z)|^2\, dA(z) = \int_D dA(t) = 1.\end{aligned}$$

Obviously, $(g_n)_n$ belongs to the closed unit ball of $A_{v,p}$ and by boundedness of the difference we can find a constant $c > 0$ such that

$$c \geq w(z_n)|g_n(\phi(z_n)) - g_n(\psi(z_n))| = w(z_n)\frac{\rho(\phi(z_n), \psi(z_n))}{\tilde{v}(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{\frac{2}{p}}} \geq n$$

for every $n \in \mathbb{N}$, which is a contradiction. We get $\sup_{z\in D}\frac{w(z)}{v(\psi(z))^{\frac{1}{p}}(1-|\psi(z)|^2)^{\frac{2}{p}}} < \infty$ analogously.

For the converse, we apply Lemma 3. and can conclude that there exists a constant $C_v > 0$ such that

$$\begin{aligned}\|C_\phi - C_\psi\| &= \sup_{z\in D} w(z)\sup\{|f(\phi(z)) - f(\psi(z))|;\ f \in A_{v,p},\ \|f\|_{v,p} \leq 1\}\\ &\leq \sup_{z\in D} C_v \max\left\{\frac{w(z)}{v(\phi(z))^{\frac{1}{p}}(1-|\phi(z)|^2)^{\frac{2}{p}}}, \frac{w(z)}{v(\psi(z))^{\frac{1}{p}}(1-|\psi(z)|^2)^{\frac{2}{p}}}\right\}\rho(\phi(z), \psi(z)) < \infty.\end{aligned}$$

Hence the difference is bounded. □

THEOREM 5. *Let $w$ be an arbitrary weight and $v$ be a weight as defined in the previous section (i.e. $v(z) := \nu(|z|^2)$ for every $z \in D$) such that $\sup_{a\in D}\sup_{z\in D}\frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \leq C < \infty$ and $v$ satisfies condition $(L1)$. Moreover, let $\phi$ and $\psi$ be analytic self-maps of $D$ with* $\max\{\|\phi\|_\infty, \|\psi\|_\infty\} = 1$. *Then the difference $C_\phi - C_\psi : A_{v,p} \to H_w^\infty$ is compact if and only if the following conditions are satisfied:*

(i) $\limsup_{|\phi(z)|\to 1}\frac{w(z)}{(1-|\phi(z)|^2)^{\frac{2}{p}}v(\phi(z))^{\frac{1}{p}}}\rho(\phi(z), \psi(z)) = 0$,
(ii) $\limsup_{|\psi(z)|\to 1}\frac{w(z)}{(1-|\psi(z)|^2)^{\frac{2}{p}}v(\psi(z))^{\frac{1}{p}}}\rho(\phi(z), \psi(z)) = 0$.

*Proof.* Note that under the given assumptions $v$ and $\tilde{v}$ are equivalent. First, we assume that (i) and (ii) hold. Let $(f_n)_n$ be a bounded sequence in $A_{v,p}$ that converges to zero uniformly on compact subsets of $D$. Let $M = \sup_n \|f_n\|_{v,p} < \infty$. Given $\varepsilon > 0$, there is $r > 0$ such that if $|\phi(z)| \geq r$, then

$$\frac{w(z)}{(1-|\phi(z)|^2)^{\frac{2}{p}}v(\phi(z))^{\frac{1}{p}}}\rho(\phi(z), \psi(z)) < \frac{\varepsilon}{3MC_v}$$

and if $|\psi(z)| \geq r$, then

$$\frac{w(z)}{(1-|\psi(z)|^2)^{\frac{2}{p}}v(\psi(z))^{\frac{1}{p}}}\rho(\phi(z), \psi(z)) < \frac{\varepsilon}{3MC_v}.$$

On the other hand, since $f_n \to 0$ uniformly on $\{u;\ |u| \le r\}$ there is an $n_0 \in \mathbb{N}$ such that, if $|\phi(z)| \le r$ and $n \ge n_0$, then $|f_n(\phi(z))| < \frac{\varepsilon}{3N}$ and if $|\psi(z)| \le r$ and $n \ge n_0$, then $|f_n(\psi(z))| < \frac{\varepsilon}{3N}$, where $N = \sup_{z\in D} w(z)$.

Hence, applying Lemma 3. we obtain by setting $X := \{z \in D; |\phi(z)| \le r\}$ and $Y := \{z \in D; |\psi(z)| \le r\}$

$$\begin{aligned}
&\sup_{z\in D} w(z)|C_\phi f_n(z) - C_\psi f_n(z)| \\
&\quad = \sup_{z\in D} w(z)|f_n(\phi(z)) - f_n(\psi(z))| \\
&\quad \le \sup_{z\in X\cap Y} w(z)|f_n(\phi(z)) - f_n(\psi(z))| + \sup_{z\in D\setminus(X\cap Y)} w(z)|f_n(\phi(z)) - f_n(\psi(z))| \\
&\quad \le \sup_{z\in D\setminus X\cap Y} w(z)|f_n(\phi(z)) - f_n(\psi(z))| + \sup_{z\in X} w(z)|f_n(\phi(z))| + \sup_{z\in Y} w(z)|f_n(\psi(z))| \\
&\quad \le \sup_{z\in D\setminus X\cap Y} \max\left\{ \frac{C_v\|f_n\|_{v,p} w(z)}{(1-|\phi(z)|^2)^{\frac{2}{p}} v(\phi(z))^{\frac{1}{p}}}, \frac{C_v\|f_n\|_{v,p} w(z)}{(1-|\psi(z)|^2)^{\frac{2}{p}} v(\psi(z))^{\frac{1}{p}}} \right\} \rho(\phi(z), \psi(z)) \\
&\qquad + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \le \varepsilon
\end{aligned}$$

for every $n \ge n_0$. Conversely, suppose $C_\phi - C_\psi : A_{v,p} \to H_w^\infty$ is compact and $(i)$ does not hold, then there are $\delta > 0$ and $(z_n)_n \subset D$ with $|\phi(z_n)| \to 1$ such that

$$\frac{w(z_n)}{(1-|\phi(z_n)|^2)^{\frac{2}{p}} \tilde{v}(\phi(z_n))^{\frac{1}{p}}} \rho(\phi(z_n), \psi(z_n)) \ge \delta$$

for all $n$. Since $|\phi(z_n)| \to 1$, there exist natural numbers $\alpha(n)$ with $\lim_{n\to\infty} \alpha(n) = \infty$ and such that $|\phi(z_n)|^{\alpha(n)} \ge \frac{1}{2}$ for all $n$. Next, for every $n \in \mathbb{N}$ we consider the function $g_n$

$$g_n(z) := f_n(z)\varphi'_{\phi(z_n)}(z)^{\frac{2}{p}} z^{\alpha(n)},$$

where $f_n$ is chosen as in the proof of Theorem 4., i.e. we select $f_n^p \in B_v^\infty$ such that $|f_n(\phi(z_n))|^p = \frac{1}{\tilde{v}(\phi(z_n))}$. Then $(g_n)_n$ is norm bounded and $g_n \to 0$ pointwise because of the factor $z^{\alpha(n)}$. Thus, it follows that a subsequence of $((C_\phi - C_\psi)g_n)_n$ tends to 0 in $H_w^\infty$. On the other hand

$$\begin{aligned}
\|(C_\phi - C_\psi)g_n\|_w &\ge w(z_n)|(C_\phi - C_\psi)g_n(z_n)| = w(z_n)|g_n(\phi(z_n)) - g_n(\psi(z_n))| \\
&= \frac{w(z_n)|\phi(z_n)|^{\alpha(n)}}{(1-|\phi(z_n)|^2)^{\frac{2}{p}} v(\phi(z_n))^{\frac{1}{p}}} \rho(\phi(z_n), \psi(z_n)) \ge \frac{1}{2}\delta,
\end{aligned}$$

which is a contradiction.

We can prove condition $(ii)$ analogously. □

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