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A Generalized Rao Bound for Ordered Orthogonal Arrays and (t, m, s)-Nets

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Abstract. In this paper, we provide a generalization of the classical Rao bound for orthogonal arrays, which can be applied to ordered orthogonal arrays and (t, m, s)-nets. Application of our new bound leads to improvements in many parameter situations to the strongest bounds (*i.e.*, necessary conditions) for existence of these objects.

1 Introduction

In 1987, Niederreiter [9] introduced the idea of a (t, m, s)-net in base b. (In fact, a restricted class of these objects, having $b = 2$, were studied by Sobol' in 1967 [13].) A (t, m, s)-net is a collection of points, in the s-dimensional unit cube, that satisfies certain desirable uniformity properties which are useful for applications in numerical integration and pseudorandom number generation.

There has been considerable interest in both constructions and bounds for existence of (t, m, s)-nets. (For a recent survey, see [1].) In this paper, we study bounds (necessary conditions) for (t, m, s)-nets. Most previous general bounds for (t, m, s)-nets are derived by using the important fact that the existence of a (t, m, s)-net implies the existence of an orthogonal array with certain parameters. Hence, it follows that any bound on orthogonal arrays yields a bound on (t, m, s)-nets. A general bound of this type is due to Lawrence (see, *e.g.*, [4, Theorem 6.1]); this bound is in fact the strongest general bound for (t, m, s)-nets and is the source of the bounds in [1].

It has been remarked by several researchers that the orthogonal array obtained from a (t, m, s)-net is, in general, a much "weaker" structure than the (t, m, s)-net from which it was derived. Thus, it has been conjectured that the bounds on (t, m, s)-nets that are derived from orthogonal array bounds are, in general, not the strongest possible bounds. This conjecture in fact was verified in one interesting parameter situation by Lawrence [5]. In this paper, we prove a generalization of the classical Rao bound for orthogonal arrays, which can be applied to (t, m, s)-nets. This extends the result of Schmid and Wolf [12, Proposition 1], who proved an identical bound for the special case of *digital* (t, m, s)-nets. Our bound is the first bound for general (t, m, s)-nets that uses the entire "structure" of a (t, m, s)-net. We find many parameter situations where our new bound improves the best known previous bound from [1].

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2 Definitions and Basic Theory

We begin with Niederreiter's definition of a (t, m, s)-net. Let $s \geq 1$ and $b \geq 2$ be integers. An *elementary interval* in base b is an interval of the form

$$E = \prod_{i=1}^{s} \left[a_i b^{-d_i}, (a_i + 1) b^{-d_i}\right),$$

where a_i and d_i are non-negative integers such that $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$. The *volume* of E is

$$\prod_{i=1}^{s} b^{-d_i} = b^{-\sum_{i=1}^{s} d_i}.$$

For integers $0 \leq t \leq m$, a (t, m, s)*-net* in base b is a set $\mathcal{N}$ of b^m points in $[0, 1)^s$ such that every elementary interval E in base b having volume b^{t-m} contains exactly b^t points of $\mathcal{N}$.

As an example,

$$\mathcal{N} = \left\{ (0, 0), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{4}, \frac{1}{4}\right) \right\}$$

is a $(0, 2, 2)$-net in base 2. Each of the following twelve elementary intervals of volume $1/4$ contains exactly one point of $\mathcal{N}$:

$$\begin{array}{llll}
[0, \frac{1}{4}) \times [0, 1), & [\frac{1}{4}, \frac{1}{2}) \times [0, 1), & [\frac{1}{2}, \frac{3}{4}) \times [0, 1), & [\frac{3}{4}, 1) \times [0, 1) \\
[0, \frac{1}{2}) \times [0, \frac{1}{2}), & [\frac{1}{2}, 1) \times [0, \frac{1}{2}), & [0, \frac{1}{2}) \times [\frac{1}{2}, 1), & [\frac{1}{2}, 1) \times [\frac{1}{2}, 1) \\
[0, 1) \times [0, \frac{1}{4}), & [0, 1) \times [\frac{1}{4}, \frac{1}{2}), & [0, 1) \times [\frac{1}{2}, \frac{3}{4}), & [0, 1) \times [\frac{3}{4}, 1).
\end{array}$$

An important result of Schmid [11], [7] showed that (t, m, s)-nets are equivalent to a combinatorial object called an *orthogonal orthogonal array*. An equivalent result, shown independently by Lawrence [3], [4], was stated in terms of *generalized orthogonal arrays*. We will present our results in terms of ordered orthogonal arrays.

We use the definition of Edel and Bierbrauer [2], which is equivalent to Schmid's. Let A be an $N \times |C|$ array of v symbols, whose columns are indexed by a set C. Let $D \subseteq C$. We say that A is *balanced* with respect to D if, within the columns of A indexed by D, every $|D|$-tuple of symbols occurs in exactly $N/v^{|D|}$ rows.

For future reference, we record the following lemma, which is simple but useful.

Lemma 2.1 *Suppose that A is balanced with respect to a set of columns D, and suppose $D' \subseteq D$. Then A is balanced with respect to D'.*

We now define ordered orthogonal arrays, using the notation of Edel and Bierbrauer [2]. An $\mathrm{OOA}_\lambda(k, s, \ell, v)$ is a $\lambda v^k \times s\ell$ array of v elements, say A, which satisfies the following properties:

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1. The set of columns, C, is partitioned into s groups of ℓ columns, denoted $C_1, \ldots, C_s$. For $1 \leq i \leq s$, we write $C_i = \{c_{ij} : 1 \leq j \leq \ell\}$.
2. Let $(a_1, \ldots, a_s)$ be an s-tuple of non-negative integers such that $a_i \leq \ell$ for $1 \leq i \leq s$ and $\sum a_i = k$. Then A is balanced with respect to

$$\bigcup_{i=1}^{s} \{c_{ij} : 1 \leq j \leq a_i\}.$$

As an example, we present an $\mathrm{OOA}_1(2, 2, 2, 2)$:

0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	1

To verify that this is in fact an $\mathrm{OOA}_1(2, 2, 2, 2)$, it suffices to check that the array is balanced with respect to the following sets of columns: $\{c_{11}, c_{12}\}$, $\{c_{21}, c_{22}\}$ and $\{c_{11}, c_{21}\}$.

Observe that we can assume without loss of generality that $\ell \leq k \leq s\ell$ when we study $\mathrm{OOA}_\lambda(k, s, \ell, v)$. When $\ell = 1$, the definition reduces to that of a "regular" orthogonal array, *i.e.*, an $\mathrm{OOA}_\lambda(k, s, 1, v)$ is equivalent to an $\mathrm{OA}_\lambda(k, s, v)$. The case of most interest to us in this paper is $k = \ell$, which corresponds to (t, m, s)-nets, as follows:

Theorem 2.2 **([11], [3])** *A (t, m, s)-net in base b is equivalent to an $\mathrm{OOA}_{b^t}(m - t, s, m - t, b)$.*

Note that the $\mathrm{OOA}_1(2, 2, 2, 2)$ and the $(0, 2, 2)$-net in base 2 that we presented above are equivalent structures, in view of Theorem 2.2.

In this paper, we are interested in necessary conditions for the existence of ordered orthogonal arrays and (t, m, s)-nets. In the case of $\mathrm{OOA}_\lambda(k, s, \ell, v)$, we will derive lower bounds on λ as a function of k, s, ℓ and v. In the case of (t, m, s)-nets in base b, we will obtain upper bounds on s as a function of t, m and b. We will prove some Rao-type bounds for these objects that generalize the classical Rao bound for orthogonal arrays. (A Rao bound for digital (t, m, s)-nets was previously proved by Schmid and Wolf [12].)

Virtually all previous bounds for general (t, m, s)-nets are based on the observation that a (t, m, s)-nets in base b, which is equivalent to an $\mathrm{OOA}_{b^t}(m - t, s, m - t, b)$, implies the existence of an $\mathrm{OOA}_{b^t}(m - t, s, 1, b)$, which is in turn equivalent to an $\mathrm{OA}_{b^t}(m - t, s, b)$. (In general, the existence of an $\mathrm{OOA}_\lambda(k, s, \ell, v)$ implies the existence of an $\mathrm{OOA}_\lambda(k, s, \ell', v)$ for all ℓ' such that $1 \leq \ell' \leq \ell$: it suffices to erase all but the first ℓ' columns in each group.) Hence, any bound on orthogonal arrays gives rise to a bound on (t, m, s)-nets. This is discussed in more detail in [1], where various bounds on orthogonal arrays are also reviewed. The drawback of this approach is that an $\mathrm{OA}_{b^t}(m - t, s, b)$ is a much "weaker" structure than a (t, m, s)-net in base b: much information has been lost by throwing away all but one column in each group of the related ordered orthogonal array. The approach we take is to modify the classical Rao bound to apply to ordered orthogonal arrays, in such a way that all $s\ell$ columns contribute to the computation of the bound.

The remainder of the paper is organized as follows. In Section 3, we derive our general bound for ordered orthogonal arrays. In Section 4, we discuss the application of this bound to orthogonal arrays and (t, m, s)-nets. Our bound, when specialized to orthogonal arrays, reduces to the classical Rao bound. When we consider the application of our bound to (t, m, s)-nets, we make use of some observations that simplify the computations required. Finally, in Section 5, we compile some tables of bounds, which provide numerous improvements to the best previous bounds from [1].

3 A New Bound for Ordered Orthogonal Arrays

Suppose A is an $\mathrm{OOA}_\lambda(k, s, \ell, v)$, where C denotes the set of columns of A. Suppose D is a function, $D\colon C \to \mathbb{Z}_v$. Given $D_1, D_2\colon C \to \mathbb{Z}_v$, we define the function $D_1 - D_2$ in the usual way, by the rule $(D_1 - D_2)(c_{ij}) = D_1(c_{ij}) - D_2(c_{ij}) \bmod v$ for all $c_{ij} \in C$.

Define the *profile* of a function $D\colon C \to \mathbb{Z}_v$ to be

$$\mathrm{profile}(D) = (d_1, \dots, d_s),$$

where

$$d_i = \begin{cases} 0 & \text{if } D(c_{ij}) = 0 \text{ for } 1 \le j \le \ell \\ \max\{j : D(c_{ij}) \ne 0\} & \text{otherwise,} \end{cases}$$

for $1 \le i \le s$. Note that $0 \le d_i \le \ell$ for $1 \le i \le s$.

Define

$$\mathrm{height}(D) = \sum_{i=1}^{s} d_i,$$

and define

$$\mathrm{width}(D) = |\{i : d_i \ne 0\}|.$$

Note that $\mathrm{width}(D) \le \mathrm{height}(D)$ for any $D \subseteq C$.

Finally, define the *support* of D to be

$$\mathrm{supp}(D) = \{c_{ij} : D(c_{ij}) \ne 0\}.$$

Observe that, if A is an $\mathrm{OOA}_\lambda(k, s, \ell, v)$, then it follows from Lemma 2.1 that A is balanced with respect to $\mathrm{supp}(D)$ provided that $\mathrm{height}(D) \le k$.

We use the following important lemma.

Lemma 3.1 *Let λ, k, s, ℓ and v be positive integers where $s \ge 2$ and $v \ge 2$, and suppose that an* $\mathrm{OOA}_\lambda(k, s, \ell, v)$ *exists. Let $\mathcal{D}$ be a set of functions such that* $\mathrm{height}(D_1 - D_2) \le k$ *for all $D_1, D_2 \in \mathcal{D}$. Then $\lambda v^k \ge |\mathcal{D}|$.*

Proof Let $1, \omega, \dots, \omega^{v-1}$ be the complex v-th roots of unity, and suppose that A is an $\mathrm{OOA}_\lambda(k, s, \ell, v)$, defined on symbol set $1, \omega, \dots, \omega^{v-1}$. Let $N = \lambda v^k$, and think of each

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column $c \in C$ as a vector $v_c \in \mathbb{C}^N$. For any function $D\colon C \to \mathbb{Z}_v$, take $D(c)$ copies of every vector v_c, $c \in C$, and define $v_D \in \mathbb{C}^N$ to be the componentwise product of these vectors.

It is easy to see that $\langle v_{D_1}, v_{D_2}\rangle = 0$ provided that $D_1 \neq D_2$ and A is balanced with respect to $\operatorname{supp}(D_1 - D_2)$, where $\langle\cdot,\cdot\rangle$ denotes the Hermitian product of two vectors. It follows that the vectors v_D, $D \in \mathcal{D}$ are mutually orthogonal, and hence linearly independent. ■

We will construct a particular set $\mathcal{D}_{k,s,\ell,v}$ satisfying the conditions of Lemma 3.1. When k is even, we define

$$\mathcal{D}_{k,s,\ell,v} = \left\{D\colon C \to \mathbb{Z}_v : \operatorname{height}(D) \le \frac{k}{2}\right\}.$$

Let $D_1, D_2 \in \mathcal{D}_{k,s,\ell,v}$. Clearly,

$$\operatorname{height}(D_1 - D_2) \le \operatorname{height}(D_1) + \operatorname{height}(D_2) \le k.$$

Therefore $\mathcal{D}_{k,s,\ell,v}$ satisfies the conditions of Lemma 3.1.

When k is odd, we define $\mathcal{D}_{k,s,\ell,v}$ slightly differently:

$$\mathcal{D}_{k,s,\ell,v} = \mathcal{D}_{k-1,s,\ell,v} \cup \left\{D\colon C \to \mathbb{Z}_v : \operatorname{height}(D) = \frac{k+1}{2} \quad \text{and} \quad d_1 \ge 1\right\}.$$

Let $D_1, D_2 \in \mathcal{D}_{k,s,\ell,v}$, k odd. By the argument above, $\operatorname{height}(D_1 - D_2) \le k$ if $\operatorname{height}(D_1) \le (k-1)/2$ or if $\operatorname{height}(D_2) \le (k-1)/2$. The only remaining case is when $\operatorname{height}(D_1) = \operatorname{height}(D_2) = (k+1)/2$. But in this case, we obtain

$$\operatorname{height}(D_1 - D_2) \le \operatorname{height}(D_1) + \operatorname{height}(D_2) - 1 \le k.$$

We will obtain a bound on $\mathrm{OOA}_\lambda(k, s, \ell, v)$ from Lemma 3.1 if we can compute $|\mathcal{D}_{k,s,\ell,v}|$. Before doing this, we state two useful lemmas.

Lemma 3.2 *Suppose that $D\colon C \to \mathbb{Z}_v$, $\operatorname{height}(D) = h$ and $\operatorname{width}(D) = w$. Then there exist exactly $(v-1)^w v^{h-w}$ functions $D'\colon C \to \mathbb{Z}_v$ such that $\operatorname{profile}(D') = \operatorname{profile}(D)$.*

Let w and h be integers such that $1 \le w \le h$. Let $N_{h,w,\ell}$ denote the number of integral solutions to the equation

$$\sum_{i=1}^{w} x_i = h$$

such that $1 \le x_i \le \ell$ for $1 \le i \le w$.

Lemma 3.3 *Let w and h be integers such that $1 \le w \le h$. Then*

$$N_{h,w,\ell} = \sum_{j=0}^{\lfloor \frac{h-w}{\ell} \rfloor} (-1)^j \binom{w}{j} \binom{h - \ell j - 1}{w-1}.$$

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Further,

$$N_{h,w,\ell} = \binom{h-1}{w-1}$$

if $\ell > h - w$, *and*

$$N_{h,w,1} = \delta_{hw} = \begin{cases} 1 & \text{if } h = w \\ 0 & \text{otherwise.} \end{cases}$$

Proof The formula for $N_{h,w,\ell}$ is a standard exercise in many combinatorics textbooks. If $\ell > h - w$, then the sum contains only one term, as indicated. Finally, the fact that $N_{h,w,1} = \delta_{hw}$ follows immediately from the definition. ■

Lemma 3.4 $|\mathcal{D}_{k,s,\ell,v}| = \mathrm{GR}(k, s, \ell, v)$, *where*

$$\mathrm{GR}(k,s,\ell,v) = \begin{cases} 1 + \sum_{h=1}^{k/2} \sum_{w=1}^{h} \binom{s}{w} N_{h,w,\ell} (v-1)^w v^{h-w} & \text{if } k \text{ is even} \\ \mathrm{GR}(k-1,s,\ell,v) + \sum_{w=1}^{(k+1)/2} \binom{s-1}{w-1} N_{\frac{k+1}{2},w,\ell} (v-1)^w v^{\frac{k+1}{2}-w} & \text{if } k \text{ is odd.} \end{cases}$$

Proof Suppose k is even. Given integers h and w such that $1 \le w \le h \le k/2$, we apply Lemma 3.3 to show that there are $\binom{s}{w} N_{h,w,\ell}$ profiles P of height h and width w. Then, from Lemma 3.2, there are $(v-1)^w v^{h-w}$ functions having any given profile P of height h and width w.

The proof for k odd is similar. ■

Summarizing the above results, we have our main bound for ordered orthogonal arrays.

Theorem 3.5 *If an* $\mathrm{OOA}_\lambda(k, s, \ell, v)$ *exists, then* $\lambda v^k \ge \mathrm{GR}(k, s, \ell, v)$.

4 Applications

4.1 The Classical Rao Bound

We remarked earlier that an $\mathrm{OOA}_\lambda(k, s, 1, v)$ is equivalent to an $\mathrm{OA}_\lambda(k, s, v)$. In view of Lemma 3.3, the formula for $\mathrm{GR}(k, s, \ell, v)$ given in Lemma 3.4 can be simplified, when $\ell = 1$, as follows:

$$\mathrm{GR}(k,s,1,v) = \begin{cases} 1 + \sum_{h=1}^{k/2} \binom{s}{h} (v-1)^h & \text{if } k \text{ is even} \\ \mathrm{GR}(k-1,s,1,v) + \binom{s-1}{\frac{k-1}{2}} (v-1)^{\frac{k+1}{2}} & \text{if } k \text{ is odd.} \end{cases}$$

Applying Theorem 3.5, we obtain the classical Rao bound for orthogonal arrays, first proved in [10].

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4.2 A Bound for (t, m, s)-Nets

As mentioned earlier, (t, m, s)-nets correspond to ordered orthogonal arrays with $k = \ell$. Appealing to Theorem 2.2, we have the following bound for (t, m, s)-nets in base b.

Theorem 4.1 *If there exists a (t, m, s)-net in base b, then*

$$b^m \geq \mathrm{GR}(m-t, s, m-t, b).$$

We now make some observations on $\mathrm{GR}(k, s, \ell, v)$ that hold whenever $\ell \geq \lceil \frac{k}{2} \rceil$. First, observe that if $h \leq \lceil \frac{k}{2} \rceil$, $w \geq 1$ and $\ell \geq \lceil \frac{k}{2} \rceil$, then $\ell > h - w$, so $N_{h,w,\ell} = \binom{h-1}{w-1}$ from Lemma 3.3. Therefore the formula in Lemma 3.4 can be simplified as follows.

Lemma 4.2 *Suppose that $\ell \geq \lceil \frac{k}{2} \rceil$. Then*

$$\mathrm{GR}(k, s, \ell, v) = \begin{cases} 1 + \sum_{h=1}^{k/2} \sum_{w=1}^{h} \binom{s}{w} \binom{h-1}{w-1} (v-1)^w v^{h-w} & \text{if } k \text{ is even} \\ \mathrm{GR}(k-1, s, \ell, v) + \sum_{w=1}^{(k+1)/2} \binom{s-1}{w-1} \binom{\frac{k-1}{2}}{w-1} (v-1)^w v^{\frac{k+1}{2}-w} & \text{if } k \text{ is odd.} \end{cases}$$

Further simplification can be obtained from the following result which generalizes [3, Lemma 4.3.2].

Lemma 4.3 *Suppose that $\ell \geq \lceil \frac{k}{2} \rceil$ and k is odd. Then $\mathrm{GR}(k, s, \ell, v) = v \times \mathrm{GR}(k-1, s, \ell, v)$.*

Proof Suppose k is odd. We define a function ϕ, where

$$\phi\colon \left\{ D\colon C \to \mathbb{Z}_v : \mathrm{height}(D) = \frac{k+1}{2} \text{ and } d_1 \geq 1 \right\} \to \mathcal{D}_{k-1,s,\ell,v},$$

by the following rule:

$$\phi(D)(c_{ij}) = \begin{cases} 0 & \text{if } i = 1 \text{ and } j = d_1 \\ D(c_{ij}) & \text{otherwise.} \end{cases}$$

Since $d_1 \geq 1$, it follows that $\mathrm{height}\big(\phi(D)\big) \leq \mathrm{height}(D) - 1 \leq (k-1)/2$. Hence, $\phi(D) \in \mathcal{D}_{k-1,s,\ell,v}$.

We will show that ϕ is a surjective mapping, and for any $D' \in \mathcal{D}_{k-1,s,\ell,v}$, there are exactly $v - 1$ functions D such that $\phi(D) = D'$. This proves the desired result.

Let $D' \in \mathcal{D}_{k-1,s,\ell,v}$ have profile $(d'_1, \ldots, d'_s)$. We will proceed to determine the inverse images of D' under ϕ. Define

$$x = \frac{k+1}{2} + d'_1 - \mathrm{height}(D').$$

Since $\mathrm{height}(D') \leq (k-1)/2$, we have that $x \geq d'_1 + 1$. Since $d'_1 \leq \mathrm{height}(D')$, we have that $x \leq (k+1)/2 \leq \ell$.

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Now, for $1 \leq h \leq v-1$, define the function

$$D_h(c_{ij}) = \begin{cases} h & \text{if } i = 1 \text{ and } j = x \\ D'(c_{ij}) & \text{otherwise.} \end{cases}$$

Then it is easy to see that $\text{height}(D_h) = (k+1)/2$ and $\phi(D_h) = D'$ for $1 \leq h \leq v-1$. It is also easy to check that there are no other functions D such that $\text{height}(D) = (k+1)/2$ and $\phi(D) = D'$. This completes the proof. ■

In the case where $m-t$ is odd, the condition $b^m \geq \text{GR}(m-t, s, m-t, b)$ is equivalent to the condition $b^{m-1} \geq \text{GR}(m-1-t, s, m-1-t, b)$, in view of Lemma 4.3. Hence, if we are making a table of upper bounds on s, given t, m and b, then we can restrict our attention to the cases where $m-t$ is even.

For future reference, we compute $\text{GR}(k, s, k, 2)$ for some small even values of k:

$$\text{GR}(2, s, 2, 2) = 1 + s$$

$$\text{GR}(4, s, 4, 2) = 1 + 3s + \binom{s}{2}$$

$$\text{GR}(6, s, 6, 2) = 1 + 7s + 5\binom{s}{2} + \binom{s}{3}$$

$$\text{GR}(8, s, 8, 2) = 1 + 15s + 17\binom{s}{2} + 7\binom{s}{3} + \binom{s}{4}$$

$$\text{GR}(10, s, 10, 2) = 1 + 31s + 49\binom{s}{2} + 31\binom{s}{3} + 9\binom{s}{4} + \binom{s}{5}.$$

If we write

$$\text{GR}(2n, s) = \sum_{i=0}^{n} a_{n,i}\binom{s}{i},$$

then it is not hard to prove that

$$a_{n,0} = 1$$

$$a_{n,n} = 1, \quad \text{and}$$

$$a_{n,i} = a_{n-1,i-1} + 2a_{n-1,i}.$$

An interesting test case is that of a $(1, 5, 6)$-net in base 2, which has been shown not to exist by Lawrence [5] by an ad hoc argument (non-existence of this object does not follow from any previously known general bound). If we let $t = 1$, $m = 5$ and $s = 6$ in Theorem 4.1, we get

$$32 \geq \text{GR}(4, 6) = 1 + 3 \times 6 + \binom{6}{2} = 34.$$

So Theorem 4.1 is sufficient to rule out the existence of a $(1,5,6)$-net in base 2.

There is one other result that is useful in computing bounds for (t,m,s)-nets. Niederreiter [9] has proven that the existence of a (t,m,s)-net in base b implies the existence of a (t,n,s)-net in base b for all integers n such that $t+2 \le n \le m$. In fact, we prove the following slightly more general result.

Theorem 4.4 *Suppose there exists an* $\mathrm{OOA}_\lambda(k,s,\ell,v)$. *Then there exists an* $\mathrm{OOA}_\lambda(k-1,s,\ell-1,v)$.

Proof Suppose that A is an $\mathrm{OOA}_\lambda(k,s,\ell,v)$. Let x be any symbol. Delete all rows of A in which the entry in column c_{11} is not an x. Then delete column c_{11}, and delete columns $c_{i\ell}$ for $2 \le i \le s$. The resulting array can be shown to be an $\mathrm{OOA}_\lambda(k-1,s,\ell-1,v)$. ■

Hence, we have the following strengthening of Theorem 4.1.

Theorem 4.5 *If there exists a* (t,m,s)*-net in base* b*, then*

$$b^n \ge \mathrm{GR}(n-t,s,n-t,b)$$

for all integers n *such that* $t+2 \le n \le m$.

5 Numerical Results

Theorem 4.5 allows us to compute an upper bound on s as a function of m and t. Suppose we define

$$S^*(t,m,b) = \max\{s : b^n \ge \mathrm{GR}(n-t,s,n-t,b) \text{ for } t+2 \le n \le m\}.$$

Then $s \le S^*(t,m,b)$ if a (t,m,s)-net in base b exists. Applying Lemma 4.3, we have the following more efficient way of computing $S^*(t,m,b)$:

$$S^*(t,m,b) = \max\{s : b^n \ge \mathrm{GR}(n-t,s,n-t,b) \text{ for } n-t \text{ even}, t+2 \le n \le m\}.$$

We tabulate $S^*(t,m,2)$ for $1 \le t \le 11$, $t+2 \le m \le 15$, $m-t$ even, in Table 1, where we also record the best previously known upper bounds on s from [1] (as given in the Tables on the web site http://www.emba.uvm.edu/jcd/toappear.html in the version of September 12, 1997). (As mentioned before, we can restrict our attention to $m-t$ odd since $S^*(t,m,b) = S^*(t,m-1,b)$ if $m-t$ is odd.)

For the values of m and t considered in Table 1, our new bound is at least as good as the bound from [1], except for $(t,m) = (2,6)$ and $(5,13)$. There are numerous cases where our bound improves the bound from [1].

The reader may have noticed that we have omitted the cases $m = t$ and $t+1$. This is because a (t,m,s)-net in base b exists for any s if $m = t$ or $m = t+1$ (see [9]).

We should also mention the cases $m = t+2$ and $m = t+3$, where it was previously known (see Mullen and Whittle [8]) that

$$s \le \frac{b^{t+2}-1}{b-1}$$

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t	m	$S^*(t,m,2)$	[1]	t	m	$S^*(t,m,2)$	[1]
1	3	7	7	1	5	5	5
1	7	5	5	1	9	5	5
1	11	5	5	1	13	5	5
1	15	5	5	2	4	15	15
2	6	9	8	2	8	8	8
2	10	7	8	2	12	7	8
2	14	7	8	3	5	31	31
3	7	13	15	3	9	10	11
3	11	10	11	3	13	9	11
3	15	9	11	4	6	63	63
4	8	20	20	4	10	14	15
4	12	12	14	4	14	12	14
5	5	127	127	5	9	29	29
5	11	19	23	5	13	16	15
5	15	14	15	6	8	255	255
6	10	42	42	6	12	25	26
6	14	20	20	7	9	511	511
7	11	61	63	7	13	32	34
7	15	24	25	8	10	1023	1023
8	12	88	89	8	14	42	44
9	11	2047	2047	9	13	125	127
9	15	54	56	10	12	4095	4095
10	14	178	180	11	13	8191	8191
11	15	253	255				

Table 1: Upper Bounds on s for Existence of (t,m,s)-Nets in Base 2

if a (t,m,s)-net in base b exists (actually, it is shown in [8] that this bound holds for any $m \geq t+2$). Our bound is the same as the Mullen-Whittle bound when $m = t+2$ or $m = t+3$; it is easily verified that

$$S^*(t,t+2,b) = S^*(t,t+3,b) = \frac{b^{t+2}-1}{b-1}.$$

In these cases, it is also often the case that the bound is tight; see [4] for more details.

In a similar manner, we tabulate $S^*(t,m,3)$ for $1 \leq t \leq 11$, $t+2 \leq m \leq 15$, $m-t$ even, in Table 2, comparing it to the best previously known bounds upper bounds on s from [1]. As was the case in Table 1, we find a significant number of improvements.

6 Comments

We have derived a generalized Rao bound that can be applied to ordered orthogonal arrays and (t,m,s)-nets. This raises the question if other bounds for orthogonal arrays could be generalized in a similar fashion. We have pursued this theme in [6], where we give a version

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t	m	$S^*(t,m,3)$	[1]	t	m	$S^*(t,m,3)$	[1]
1	3	13	13	1	5	9	10
1	7	9	8	1	9	9	8
1	11	9	8	1	13	9	8
1	15	9	8	2	4	40	40
2	6	17	18	2	8	14	16
2	10	14	13	2	12	14	12
2	14	14	12	3	5	121	121
3	7	31	32	3	9	22	24
3	11	19	20	3	13	18	17
3	15	18	16	4	6	364	364
4	8	55	56	4	10	32	34
4	12	26	30	4	14	24	24
5	7	1093	1093	5	9	97	98
5	11	48	50	5	13	35	39
5	15	30	34	6	8	3280	3280
6	10	170	171	6	12	71	73
6	14	48	52	7	9	9841	9841
7	11	296	297	7	13	103	106
7	15	64	68	8	10	29524	29524
8	12	513	515	8	14	150	153
9	11	88573	88573	9	13	891	892
9	15	218	221	10	12	265720	265720
10	14	1544	≥ 998	11	13	797161	797161
11	15	2677	≥ 998				

Table 2: Upper Bounds on s for Existence of (t,m,s)-Nets in Base 3

of Delsarte's linear programming bound which applies to ordered orthogonal arrays and (t,m,s)-nets.

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