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ON CERTAIN RELATIONS BETWEEN PRODUCTS OF BILATERAL HYPERGEOMETRIC SERIES

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1. Introduction. Darling [3] in 1932 and Bailey [2] in 1933 gave certain theorems on products of hypergeometric series. Again in 1948 Sears [**4**] used the relation which expresses the ${}_M\Phi_{M-1}(x)$ series in terms of M other series of the same type to derive transformations between products of both basic and ordinary hypergeometric series. In this paper I give certain general theorems on products of bilateral hypergeometric series together with some of their interesting special cases.

The following notation is used throughout the paper :

$$(a\,;\ n)=(1-a)(1-aq)\dots(1-aq^{n-1}),\quad (a\,;\ 0)=1,$$
$$(a\,;\ -n)=(-1)^n q^{\frac{1}{2}n(n+1)}/a^n(q/a\,;\ n),\quad |\,q\,|<1,$$
$$(a)_n=a(a+1)\dots(a+n-1),\ (a)_0=1,\quad (a)_{-n}=(-1)^n/(1-a)_n,$$
$${}_r\Psi_r\begin{bmatrix}a_1, a_2, \dots, a_r\,;\,z\\ b_1, b_2, \dots, b_r\end{bmatrix}=\sum_{n=-\infty}^{\infty}\frac{(a_1\,;\,n)(a_2\,;\,n)\dots(a_r\,;\,n)}{(b_1\,;\,n)(b_2\,;\,n)\dots(b_r\,;\,n)}z^n,$$
$${}_rH_r\begin{bmatrix}a_1, a_2, \dots, a_r\,;\,z\\ b_1, b_2, \dots, b_r\end{bmatrix}=\sum_{n=-\infty}^{\infty}\frac{(a_1)_n(a_2)_n\dots(a_r)_n}{(b_1)_n(b_2)_n\dots(b_r)_n}z^n,$$
$$\Pi\begin{bmatrix}a_1, a_2, \dots, a_r\\ b_1, b_2 \dots, b_r\end{bmatrix}=\prod_{n=0}^{\infty}\frac{(1-a_1q^n)(1-a_2q^n)\dots(1-a_rq^n)}{(1-b_1q^n)(1-b_2q^n)\dots(1-b_rq^n)},$$
$$\Gamma\begin{bmatrix}a_1, a_2, \dots, a_r\\ b_1, b_2, \dots, b_r\end{bmatrix}=\frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_r)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_n)},$$

and idem $(a\,;\,b)$ means that the preceding expression is repeated with a and b interchanged.

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2. Slater [**6**], in 1952, gave the following relation which expresses the general bilateral ${}_M\Psi_M(x)$ series in terms of M other series of the same type :

$$\Pi\left[\begin{matrix} xA, q/xA, b_1, b_2, \dots, b_M, q/c_1, q/c_2, \dots, q/c_M\,; \\ a_1, a_2, \dots, a_M, q/a_1, q/a_2, \dots, q/a_M \end{matrix}\right] \times {}_M\Psi_M\left[\begin{matrix} c_1, c_2, \dots, c_M\,; x \\ b_1, b_2, \dots, b_M \end{matrix}\right]$$
$$= \frac{q}{a_1}\,\Pi\left[\begin{matrix} a_1xA/q, q^2/Axa_1, a_1/c_1, a_1/c_2, \dots, a_1/c_M, qb_1/a_1, qb_2/a_1, \dots, qb_M/a_1\,; \\ a_1, q/a_1, a_1/a_2, a_1/a_3, \dots, a_1/a_M, qa_2/a_1, qa_3/a_1, \dots, qa_M/a_1 \end{matrix}\right]$$
$$\times {}_M\Psi_M\left[\begin{matrix} qc_1/a_1, qc_2/a_1, \dots, qc_M/a_1\,; x \\ qb_1/a_1, qb_2/a_1, \dots, qb_M/a_1 \end{matrix}\right] + \text{idem}\,(a_1\,; a_2, a_3, \dots, a_M), \quad \dots(2.1)$$

where $$A = \frac{c_1c_2 \dots c_M}{a_1a_2 \dots a_M}, \quad |x| < 1, \quad \left|\frac{b_1b_2 \dots b_M}{c_1c_2 \dots c_Mx}\right| < 1.$$

If we put $x = \dfrac{a_1a_2 \dots a_M}{qc_1c_2 \dots c_M}$ in (2.1), the expression on the left vanishes and we get

$$a_1\Pi\left[\begin{matrix} a_1/c_1, a_1/c_2, \dots, a_1/c_M, qb_1/a_1, qb_2/a_1, \dots, qb_M/a_1\,; \\ a_1/a_2, a_1/a_3, \dots, a_1/a_M, qa_2/a_1, qa_3/a_1, \dots, qa_M/a_1 \end{matrix}\right]$$
$$\times {}_M\Psi_M\left[\begin{matrix} qc_1/a_1, qc_2/a_1, \dots, qc_M/a_1\,; \dfrac{a_1a_2 \dots a_M}{qc_1c_2 \dots c_M} \\ qb_1/a_1, qb_2/a_1, \dots, qb_M/a_1 \end{matrix}\right] + \text{idem}\,(a_1\,; a_2, \dots, a_M) = 0, \quad \dots(2.2)$$

where $$\left|\frac{a_1a_2 \dots a_M}{qc_1c_2 \dots c_M}\right| < 1 \text{ and } \left|\frac{qb_1b_2 \dots b_M}{a_1a_2 \dots a_M}\right| < 1.$$

Now, replacing M by $M+N$ in (2.2) and then putting $b_{M+1} = a_{M+1}, b_{M+2} = a_{M+2}, \dots, b_{M+N} = a_{M+N}$, we get

$$a_1\Pi\left[\begin{matrix} a_1/c_1, a_1/c_2, \dots, a_1/c_{M+N}, qb_1/a_1, qb_2/a_1, \dots, qb_M/a_1\,; \\ a_1/a_2, a_1/a_3, \dots, a_1/a_{M+N}, qa_2/a_1, qa_3/a_1, \dots, qa_M/a_1 \end{matrix}\right]$$
$$\times {}_{M+N}\Psi_{M+N}\left[\begin{matrix} qc_1/a_1, \dots, qc_M/a_1, qc_{M+1}/a_1, \dots, qc_{M+N}/a_1\,; \dfrac{a_1a_2 \dots a_{M+N}}{qc_1c_2 \dots c_{M+N}} \\ qb_1/a_1, \dots, qb_M/a_1, qa_{M+1}/a_1, \dots, qa_{M+N}/a_1 \end{matrix}\right]$$
$$+ \text{idem}\,(a_1\,; a_2, a_3, \dots, a_M)$$
$$+ a_{M+1}\Pi\left[\begin{matrix} a_{M+1}/c_1, \dots, a_{M+1}/c_M, a_{M+1}/c_{M+1}, \dots, a_{M+1}/c_{M+N}, qb_1/a_{M+1}, \dots, qb_M/a_{M+1}, q\,; \\ a_{M+1}/a_1 \dots, a_{M+1}/a_M, a_{M+1}/a_{M+2}, \dots, a_{M+1}/a_{M+N}, qa_1/a_{M+1}, \dots, qa_M/a_{M+1} \end{matrix}\right]$$
$$\times {}_{M+N}\Phi_{M+N-1}\left[\begin{matrix} qc_1/a_{M+1}, \dots, qc_M/a_{M+1}, qc_{M+1}/a_{M+1}, \dots, qc_{M+N}/a_{M+1}\,; \dfrac{a_1a_2 \dots a_{M+N}}{qc_1c_2 \dots c_{M+N}} \\ qb_1/a_{M+1}, \dots, qb_M/a_{M+1}, qa_{M+2}/a_{M+1}, \dots, qa_{M+N}/a_{M+1} \end{matrix}\right]$$
$$+ \text{idem}\,(a_{M+1}\,; a_{M+2}, \dots, a_{M+N}) = 0 \quad \dots(2.3)$$

If we again put $a_{M+1} = c_1q^{-n}, a_{M+2} = c_2q^{-n}, \dots, a_{M+N} = c_Nq^{-n}$ and $c_{M+1} = b_1q^{-1-n}$, $c_{M+2} = b_2q^{-1-n}, \dots, c_{M+N} = b_Nq^{-1-n}$ in (2.3), we obtain

$$a_1\Pi\left[\begin{matrix} a_1/c_1, \dots, a_1/c_M, a_1q/b_1, \dots, a_1q/b_N, qb_1/a_1, \dots, qb_M/a_1\,; \\ a_1/a_2, \dots, a_1/a_M, a_1/c_1, \dots, a_1/c_N, qa_2/a_1, \dots, qa_M/a_1 \end{matrix}\right]$$
$$\times \frac{(a_1/c_1\,; n)(a_1/c_2\,; n) \dots (a_1/c_N\,; n)}{(a_1q/b_1\,; n)(a_1q/b_2\,; n) \dots (a_1q/b_N\,; n)}$$
$$\times {}_{M+N}\Psi_{M+N}\left[\begin{matrix} qc_1/a_1, \dots, qc_M/a_1, q^{-n}b_1/a_1, \dots, q^{-n}b_N/a_1\,; \dfrac{a_1a_2 \dots a_Mc_1c_2 \dots c_N}{c_1c_2 \dots c_Mb_1b_2 \dots b_N}q^{N-1} \\ qb_1/a_1 \dots, qb_M/a_1, q^{1-n}c_1/a_1, \dots, q^{1-n}c_N/a_1 \end{matrix}\right]$$
$$+ \text{idem}\,(a_1\,; a_2, a_3, \dots, a_M) = 0, \quad \dots(2.4)$$

where $$\left|\frac{a_1a_2 \dots a_Mc_1c_2 \dots c_Nq^{N-1}}{c_1c_2 \dots c_Mb_1b_2 \dots b_N}\right| < 1 \text{ and } \left|\frac{b_1b_2 \dots b_Mq}{a_1a_2 \dots a_M}\right| < 1.$$

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Hence we obtain the following general theorem on products of bilateral hypergeometric series, which follows because of (2.4), when we compare the coefficients of z^n.

THEOREM.

$$a_1 \Pi \begin{bmatrix} a_1/c_1, \ldots, a_1/c_M, a_1q/b_1, \ldots, a_1q/b_N, qb_1/a_1, \ldots, qb_M/a_1; \\ a_1/a_2, \ldots, a_1/a_M, a_1/c_1, \ldots, a_1/c_N, qa_2/a_1, \ldots, qa_M/a_1 \end{bmatrix}$$
$$\times {}_M\Psi_M \begin{bmatrix} qc_1/a_1, \ldots, qc_M/a_1; zt \\ qb_1/a_1, \ldots, qb_M/a_1 \end{bmatrix} {}_N\Psi_N \begin{bmatrix} a_1/c_1, \ldots, a_1/c_N; z \\ a_1q/b_1 \ldots, a_1q/b_N \end{bmatrix}$$
$$+ \text{idem } (a_1; a_2, a_3, \ldots, a_M) = 0, \quad \ldots\ldots(2.5)$$

where
$$t = \frac{a_1a_2 \ldots a_M}{qc_1c_2 \ldots c_M}, \quad |z| < 1, \; |zt| < 1,$$
$$\left|\frac{q^Nc_1c_2 \ldots c_N}{b_1b_2 \ldots b_N z}\right| < 1, \; \left|\frac{qb_1b_2 \ldots b_M}{a_1a_2 \ldots a_M z}\right| < 1, \quad M > 1 \text{ and } N > 0.$$

Special cases. (i) If we take $N = M$ and put $b_1 = a_1, b_2 = a_2, \ldots, b_M = a_M$ in (2.5), we get the following theorem:

$$a_1 \left[\left(1 - \frac{a_1}{a_2}\right) \ldots \left(1 - \frac{a_1}{a_M}\right)\right]^{-1} {}_M\Phi_{M-1} \begin{bmatrix} qc_1/a_1, \ldots, qc_M/a_1; zt \\ qa_2/a_1, \ldots, qa_M/a_1 \end{bmatrix} {}_M\Phi_{M-1} \begin{bmatrix} a_1/c_1, \ldots, a_1/c_M; z \\ a_1q/a_2, \ldots, a_1q/a_M \end{bmatrix}$$
$$+ \text{idem } (a_1; a_2, a_3, \ldots, a_M) = 0, \quad \ldots\ldots(2.6)$$

where
$$t = \frac{a_1a_2 \ldots a_M}{qc_1c_2 \ldots c_M}, \qquad |z| < 1 \text{ and } |zt| < 1.$$

Now if we replace zt by z, qc_r/a_1 by A_r and qa_r/a_1 by B_{r-1} in (2.6), it becomes Sears' theorem [**4**, § 6.2] on products of basic hypergeometric series. Hence (2.5) is a basic bilateral generalisation of Sears' theorem.

(ii) If we let $q \to 1$ in (2.5) in the usual manner, we get the following transformation between products of ordinary bilateral hypergeometric series:

$$\Gamma \begin{bmatrix} a_1 - a_2, a_1 - a_3, \ldots, a_1 - a_M, a_1 - c_1, \ldots, a_1 - c_N, 1 + a_2 - a_1, \ldots, 1 + a_M - a_1; \\ a_1 - c_1, \ldots, a_1 - c_M, 1 + a_1 - b_1, \ldots, 1 + a_1 - b_N, 1 + b_1 - a_1, \ldots, 1 + b_M - a_1 \end{bmatrix}$$
$$\times {}_MH_M \begin{bmatrix} 1 + c_1 - a_1, \ldots, 1 + c_M - a_1; z \\ 1 + b_1 - a_1, \ldots, 1 + b_M - a_1 \end{bmatrix} {}_NH_N \begin{bmatrix} a_1 - c_1, \ldots, a_1 - c_N; z \\ 1 + a_1 - b_1, \ldots, 1 + a_1 - b_N \end{bmatrix}$$
$$+ \text{idem } (a_1; a_2, a_3, \ldots, a_M) = 0, \quad \ldots\ldots(2.7)$$

where $|z| = 1$ for convergence.

3. In this section I deduce another general relation between the products of ordinary bilateral hypergeometric series. Slater [**6**] gave the following relation between M series of the type ${}_MH_M$ (1):

$$\Gamma \begin{bmatrix} 1 + a_2 - a_1, 1 + a_3 - a_1, \ldots, 1 + a_M - a_1, a_1 - a_2, a_1 - a_3, \ldots, a_1 - a_M; \\ 1 + b_1 - a_1, 1 + b_2 - a_1, \ldots, 1 + b_M - a_1, a_1 - c_1, a_1 - c_2, \ldots, a_1 - c_M \end{bmatrix}$$
$$\times {}_MH_M \begin{bmatrix} 1 + c_1 - a_1, 1 + c_2 - a_1, \ldots, 1 + c_M - a_1; \\ 1 + b_1 - a_1, 1 + b_2 - a_1, \ldots, 1 + b_M - a_1 \end{bmatrix}$$
$$+ \text{idem } (a_1; a_2, a_3, \ldots, a_M) = 0. \quad \ldots\ldots(3.1)$$

If we first replace M by $M + N$ in (3.1) and then $c_{M+1}, c_{M+2}, \ldots, c_{M+N}, b_{M+1}, b_{M+2}, \ldots, b_{M+N}$ by $c_{M+1} - n, c_{M+2} - n, \ldots, c_{M+N} - n, b_{M+1} - n, b_{M+2} - n, \ldots, b_{M+N} - n$, respectively, we get,

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$$\Gamma\begin{bmatrix}1+a_2-a_1, \dots, 1+a_{M+N}-a_1, a_1-a_2, \dots, a_1-a_{M+N}\\ 1+b_1-a_1, \dots, 1+b_{M+N}-a_1, a_1-c_1, \dots, a_1-c_{M+N}\end{bmatrix}$$
$$\times\frac{(a_1-b_{M+1})_n(a_1-b_{M+2})_n \dots (a_1-b_{M+N})_n}{(a_1-c_{M+1})_n(a_1-c_{M+2})_n \dots (a_1-c_{M+N})_n}$$
$$\times {}_{M+N}H_{M+N}\begin{bmatrix}1+c_1-a_1, \dots, 1+c_M-a_1, 1+c_{M+1}-a_1-n, \dots, 1+c_{M+N}-a_1-n;\\ 1+b_1-a_1, \dots, 1+b_M-a_1, 1+b_{M+1}-a_1-n, \dots, 1+b_{M+N}-a_1-n\end{bmatrix}$$
$$+\text{idem } (a_1; a_2, a_3, \dots, a_{M+N})=0. \qquad \text{.......................(3.2)}$$

Hence we get the following transformation between products of ordinary bilateral series which can be proved with the help of (3.2) when we compare the coefficients of z^n :

$$\Gamma\begin{bmatrix}1+a_2-a_1, \dots, 1+a_{M+N}-a_1, a_1-a_2, \dots, a_1-a_{M+N};\\ 1+b_1-a_1, \dots, 1+b_{M+N}-a_1, a_1-c_1, \dots, a_1-c_{M+N}\end{bmatrix}$$
$$\times {}_MH_M\begin{bmatrix}1+c_1-a_1, \dots, 1+c_M-a_1; z\\ 1+b_1-a_1, \dots, 1+b_M-a_1\end{bmatrix} {}_NH_N\begin{bmatrix}a_1-b_{M+1}, \dots, a_1-b_{M+N}; z\\ a_1-c_{M+1}, \dots, a_1-c_{M+N}\end{bmatrix}$$
$$+\text{idem } (a_1; a_2, a_3, \dots, a_{M+N})=0, \qquad \text{................................(3.3)}$$

where $|z|=1$.

4. Finally, it may be noted that it has not been found possible to deduce a relation corresponding to (3.3) for basic bilateral series.

Also, the transformation (2.4) can be directly obtained from the basic bilateral integral given by Slater [**7**, § 10 (24)], if we make the same substitutions as required for deducing (2.4).

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