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WHEN ARE IMMERSIONS DIFFEOMORPHISMS?

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ABSTRACT. It is shown in this paper that an immersion of a connected, closed n-manifold into another connected n-manifold is a diffeomorphism if and only if the induced homomorphism between the fundamental groups is surjective at some point. This is proved as a consequence of a more general assertion about topological spaces.

Let M be a connected n-dimensional closed manifold, and $f: M \to N$ be an immersion of M *into* another connected n-dimensional manifold N. In this note we shall show that f is a diffeomorphism of M onto N if and only if the induced homomorphism $f_*: \pi_1(M, x) \to \pi_1(N, f(x))$ is surjective for some x in M. In fact, we shall prove a more general statement about topological spaces. First we need some terminologies. A map $f: X \to Y$ is called a *local homeomorphism* if each point of X has an open neighborhood which is carried by f homeomorphically onto an open subset of Y, and f is called a *proper* map if $f^{-1}(K)$ is compact for each compact subset K of Y. Our main result is the following.

THEOREM. *Let X and Y be two pathwise connected, Hausdorff spaces. A local homeomorphism $f: X \to Y$ of X into Y is a global homeomorphism if and only if* (1). *f is proper, and* (2). *the induced homomorphism $f_*: \pi_1(X, x) \to \pi_1(Y, f(x))$ is surjective for some $x \in X$.*

The assertion about the immersions and diffeomorphisms above is clearly a consequence of this theorem. The proof of the theorem is based on the ideas of one of the author's previous papers [1].

Proof of the theorem. The "only if" part of the theorem is trivial. Now suppose $f: X \to Y$ is a local homeomorphism of X into Y, which satisfies the two conditions. We first show that f is surjective. By condition (2), $X \neq \emptyset$. We may therefore pick a point $y_0 \in \mathrm{Im}(f)$. Let an arbitrary point $y \in Y$ be given. We first choose a path $\alpha: I \to Y$ connecting y_0 to y, where $I = [0, 1]$ is the unit closed interval. Since $f: X \to Y$ is proper, $f^{-1}(\alpha(I))$ is compact in X. Therefore, $\mathrm{Im}(f) \cap \alpha(I) = f(f^{-1}(\alpha(I)))$ is closed in Y. On the other hand, since f is a local

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homeomorphism, Im(f) is open in Y. Consequently, the set Im(f) $\cap \alpha(I)$ is both open and closed in $\alpha(I)$, and hence, must contain $\alpha(I)$. In particular, $y \in$ Im(f). This shows that f is surjective.

To show that f is injective, we first note that a proper, local homeomorphism from a Hausdorff space onto a pathwise connected Hausdorff space is always a covering projection (see [1, Lemma 3]). Now suppose that there are two points x_1 and x_2 in X such that $f(x_1)=f(x_2)=y$. Let α_1 be a path in X connecting x_1 to x_2. Then $f \circ \alpha_1$ is a loop in Y based at y. By the pathwise-connectedness of X and Y, we may assume that $f_*: \pi_1(X, x_1) \to \pi_1(Y, y)$ is surjective. Hence, there exists a loop α_0 in X based at x_1 such that $f \circ \alpha_0$ is homotopic to $f \circ \alpha_1$ in Y. Let F_t be a homotopy such that $F_0 = f \circ \alpha_0$ and $F_1 = f \circ \alpha_1$. By the homotopy lifting property of covering projections, F_t can be lifted into a homotopy F'_t in X with $F'_0 = \alpha_0$. By the uniqueness of path lifting, it is easy to show that F'_1 must coincide with α_1. In particular, $F'_1(1) = x_2$. But the end points $F'_t(1)$ of the liftings F'_t $(0 \le t \le 1)$ describe a continuous path in X from x_1 to x_2, which is the lifting of the constant path at y in Y. By the uniqueness of the path lifting again, $F'_t(1)$ $(0 \le t \le 1)$ must coincide with the constant path at x_1 in X. In particular, $x_1 = F'_1(1) = x_2$. This shows the injectiveness of f.

A bijective local homeomorphism is clearly a global homeomorphism. This finishes the proof of the theorem.

Reference

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