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NOTE ON A HYPERGEOMETRIC INTEGRAL

D. S. JONES
Division of Mathematics, University of Dundee, Dundee DD1 4HN, UK

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Abstract The asymptotic behaviour of a certain integral is investigated. The investigation involves a hypergeometric function of a type for which the asymptotics have not previously been considered.

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1. Introduction

The integral

$$I(a,b,c,z)=\int_0^1 t^a(1-t)^b(1-tz)^c\,\mathrm{d}t \tag{1.1}$$

is well known. It exists provided that $\mathcal{R}(a)>-1$ and $\mathcal{R}(b)>-1$. Also, it defines a regular function of z in the z-plane cut along the real axis from $z=1$ to $+\infty$. Its relation to the standard hypergeometric function, denoted by F, is given by

$$I(a,b,c,z)=\frac{a!b!}{(a+b+1)!}F(-c,a+1;a+b+2;z). \tag{1.2}$$

The aim of this paper is to discuss the asymptotic behaviour of I for various values of the parameters a, b and c. Several results are already known. For example, if a, c and z are fixed and $b\to\infty$,

$$I(a,b,c,z)\sim\frac{a!b!}{(a+b+1)!}\left\{1-\frac{c(a+1)}{a+b+2}z+\cdots\right\}. \tag{1.3}$$

Another formula is

$$I(a+\lambda,b,c-\lambda,z) \sim\frac{b!(1-z)^{b+c+1-\lambda}}{(a+b+\lambda-1)!\lambda^{b+1}}\left[1-\frac{1}{\lambda}(b+1)\{a+1+\tfrac{1}{2}b-(a+b+c+2)z\}+\cdots\right] \tag{1.4}$$

as $\lambda\to\infty$ with a, b, c, z fixed and $\mathcal{R}(z)<1$. Here, and subsequently, the restriction $|\mathrm{ph}(1-z)|<\pi$ is imposed.

Various expansions in terms of Bessel functions can be derived from the corresponding expressions for the hypergeometric function (see [**1**]). One integral that can be estimated in this way is $I(a+\lambda, b+\lambda, c-\lambda, z)$ as $\lambda \to \infty$ with a, b, c and z fixed.

One case that does not seem to have been covered is that in which b and c grow simultaneously. To simplify matters, the investigation will be limited to the case in which $b = c$. Thus, an integral of the form $I(\nu, \mu - \frac{1}{2}, \mu - \frac{1}{2}, z)$ will be considered. The aim is to discover what happens as $\nu + \mu \to \infty$, which involves the cases where either ν or μ or both tend to infinity. The behaviour of other integrals can be deduced by taking advantage of relations such as

$$\frac{\mathrm{d}}{\mathrm{d}z}[z^{-c} I(a,b,c,z)] = -cz^{-c-1} I(a,b,c-1,z), \tag{1.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}[z^{a+b+1} I(a,b,c,z)] = bz^{a+b} I(a,b-1,c,z), \tag{1.6}$$

$$\begin{aligned}(1-z)^{c+a+b+2}\frac{\mathrm{d}}{\mathrm{d}z}[z^{a+b+1}(1-z)^{-c-a-b-1} I(a,b,c,z)]& \\ = az^{a+b} I(a-1,b,c,z),&\end{aligned} \tag{1.7}$$

$$\begin{aligned}(1-z)^{c+b+2}\frac{\mathrm{d}}{\mathrm{d}z}[z^{a+b+c+2}(1-z)^{-c-b-1} I(a,b,c,z)]& \\ = (a+b+c+2)z^{a+b+c+1} I(a,b,c+1,z),&\end{aligned} \tag{1.8}$$

provided that the integrals on the right-hand sides exist. Evidently, there is no loss of generality if it is assumed that $\nu \geqslant 0$ and $\mu \geqslant \frac{1}{2}$ since lower values can be handled by the above relations. From now on, therefore, it will be assumed that $\nu + \mu \to \infty$ subject to the conditions $\nu \geqslant 0$ and $\mu \geqslant \frac{1}{2}$.

In the discussion of the asymptotic behaviour, free use of the theory of Olver (see, for example, [**2**–**4**]) will be made without further specific reference.

By a transformation of the hypergeometric function, it follows from (1.2) that

$$I(\nu, \mu-\tfrac{1}{2}, \mu-\tfrac{1}{2}, z) = \frac{\nu!}{\lambda!}(\mu-\tfrac{1}{2})!(1-z)^{\mu-1/2} F\left(\tfrac{1}{2}-\mu, \mu+\tfrac{1}{2}; \lambda+1; \frac{z}{z-1}\right), \tag{1.9}$$

where $\lambda = \mu + \nu + \frac{1}{2}$.

Now make the transformation

$$z/(z-1) = \tfrac{1}{2}(1-w). \tag{1.10}$$

Since (1.10) is a bilinear transformation, circles in one plane are mapped into circles in the other plane. In particular, the cut for the integral in the z-plane becomes a cut along the real axis from $-\infty$ to -1 in the w-plane. Also, the imaginary axis of the w-plane maps into the unit circle of the z-plane.

If $\lambda \gg 1$ while μ and $|w|$ are bounded, it is standard that

$$F(\tfrac{1}{2}-\mu, \mu+\tfrac{1}{2}; \lambda+1; \tfrac{1}{2}(1-w)) \sim 1 + \tfrac{1}{2}(\tfrac{1}{4}-\mu^2)\frac{1-w}{\lambda+1} \tag{1.11}$$

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to a first approximation. For $0 \leqslant \mu \leqslant 2$ and $|1-w| < 1$, the error in approximation (1.11) is less than 1% for $\lambda > 7$. But the error grows rapidly with μ. Even a moderate increase in μ to 5 (say) requires λ to exceed 80 to achieve an accuracy of 1%. Thus, (1.11) is not very satisfactory for practical purposes unless μ/λ is relatively small.

2. Behaviour of the hypergeometric function

The hypergeometric function in (1.11) satisfies

$$(1-w^2)\frac{\mathrm{d}^2F}{\mathrm{d}w^2} - 2(\lambda + w)\frac{\mathrm{d}F}{\mathrm{d}w} + (\mu^2 - \tfrac{1}{4})F = 0. \tag{2.1}$$

On making the substitution $F = (1-w)^{-(\lambda+1)/2}(1+w)^{(\lambda-1)/2}p$, the differential equation (2.1) becomes

$$\frac{\mathrm{d}^2p}{\mathrm{d}w^2} = -\frac{p}{(1-w^2)^2}[\lambda^2\{\sigma^2(1-w^2) - 1\} + 1 - \tfrac{1}{4}(1-w^2)], \tag{2.2}$$

where $\sigma = \mu/\lambda > 0$. This differential equation has turning points on the imaginary axis at $w = \pm\mathrm{i}(1-\sigma^2)^{1/2}/\sigma$ since $\sigma < 1$. The turning points can be placed on the real axis by putting $w = \mathrm{i}v$, with the result that

$$\frac{\mathrm{d}^2p}{\mathrm{d}v^2} = \frac{p}{(1+v^2)^2}[\lambda^2\{\sigma^2(1+v^2) - 1\} + \tfrac{3}{4} - \tfrac{1}{4}v^2]. \tag{2.3}$$

Due to the substitution in (2.1), there are cuts in the v-plane from i to i∞ and from $-$i∞ to $-$i.

A new variable ζ is now introduced via

$$\frac{\mathrm{d}v}{\mathrm{d}\zeta} = (1+v^2)\left\{\frac{\zeta^2 - \alpha^2}{\sigma^2(1+v^2) - 1}\right\}^{1/2}, \tag{2.4}$$

where α is a real non-negative constant selected so that the points $\zeta = \pm\alpha$ and $v = \pm(1-\sigma^2)^{1/2}/\sigma$ correspond. Integration of (2.4), starting from $\zeta = \alpha$, yields

$$\begin{aligned}\sigma \ln[\sigma v + \{\sigma^2(1+v^2) - 1\}^{1/2}] \\ - \ln[v + \{\sigma^2(1+v^2) - 1\}^{1/2}] + \tfrac{1}{2}\ln(1+v^2) + \tfrac{1}{2}(1-\sigma)\ln(1-\sigma^2) \\ = \tfrac{1}{2}\zeta(\zeta^2 - \alpha^2)^{1/2} - \tfrac{1}{2}\alpha^2 \ln[\{\zeta + (\zeta^2 - \alpha^2)^{1/2}\}/\alpha].\end{aligned} \tag{2.5}$$

The square roots in (2.5) are defined to be positive on the real axis where $\zeta > \alpha$ or $v > (1-\sigma^2)^{1/2}/\sigma$ and elsewhere by continuity. In the interval $-\alpha < \zeta < \alpha$, it is more convenient to use trigonometric functions instead of logarithms. There is no difficulty in seeing that, for $-\alpha < \zeta < \alpha$, (2.5) becomes

$$\sigma \cos^{-1}\frac{\sigma v}{(1-\sigma^2)^{1/2}} - \cos^{-1}\frac{v}{(1-\sigma^2)^{1/2}(1+v^2)^{1/2}} = \tfrac{1}{2}\zeta(\alpha^2 - \zeta^2)^{1/2} - \tfrac{1}{2}\alpha^2\cos^{-1}\frac{\zeta}{\alpha}, \tag{2.6}$$

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where $\cos^{-1}$ is taken to lie in the interval $(0,\pi)$. The substitution $\zeta = -\alpha$, $v = -(1-\sigma^2)^{1/2}/\sigma$ in (2.6) leads to

$$\alpha^2 = 2(1-\sigma), \tag{2.7}$$

which specifies α. Note also that (2.6) implies that $\zeta = 0$ corresponds to $v = 0$ by virtue of (2.7).

Although σ is never strictly unity it can be close to this value. So it is worth observing that, when $\sigma = 1$ and $\alpha = 0$, (2.5) reduces to

$$\zeta^2 = \ln(1+v^2). \tag{2.8}$$

The transformation

$$p = \left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)^{1/2} q(\zeta) \tag{2.9}$$

changes (2.3) to

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\zeta^2} = \{\lambda^2(\zeta^2-\alpha^2)+\psi\}q, \tag{2.10}$$

where

$$\psi = \frac{2\alpha^2+3\zeta^2}{4(\zeta^2-\alpha^2)^2} + \frac{(\zeta^2-\alpha^2)(1+v^2)}{4\{\sigma^2(1+v^2)-1\}^3}[(\sigma^2-4)\{\sigma^2(1+v^2)-1\}+5\sigma^2-5]. \tag{2.11}$$

Thus

$$F(\tfrac{1}{2}-\mu,\mu+\tfrac{1}{2};\lambda+1;\tfrac{1}{2}(1-\mathrm{i}v)) = (1-\mathrm{i}v)^{-(\lambda+1)/2}(1+\mathrm{i}v)^{(\lambda-1)/2}\left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)^{1/2} q(\zeta), \tag{2.12}$$

where q is a suitable solution of (2.10). Once q has been determined, the integral $I(\nu,\mu-\frac{1}{2},\mu-\frac{1}{2},(\mathrm{i}v-1)/(\mathrm{i}v+1))$ may be deduced from (1.9).

At first sight it appears that ψ has singularities at $\zeta = \pm\alpha$. However, it may be checked that the contributions of the various singular terms cancel, so that ψ is bounded at $\zeta = \pm\alpha$. Furthermore, as $v \to \infty$ along the real axis,

$$\zeta^2 \sim 2\sigma \ln v,$$

so $\psi = O(1/\zeta^2)$ as $\zeta \to \infty$.

3. The first approximation

Since λ is large, a first approximation to q can be obtained by neglecting ψ in (2.10). Solutions can then be expressed in terms of parabolic cylinder functions that satisfy

$$\frac{\mathrm{d}^2 U}{\mathrm{d}z^2} = (\tfrac{1}{4}z^2 + a)U. \tag{3.1}$$

Appropriate solutions of (3.1) are

$$U(a,z) = y_1 \cos\pi(\tfrac{1}{4}+\tfrac{1}{2}a) - y_2 \sin\pi(\tfrac{1}{4}+\tfrac{1}{2}a), \tag{3.2}$$

$$V(a,z) = y_1 \sin\pi(\tfrac{1}{4}+\tfrac{1}{2}a) + y_2 \cos\pi(\tfrac{1}{4}+\tfrac{1}{2}a), \tag{3.3}$$

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where

$$y_1 = \frac{(-\frac{1}{2}a-\frac{3}{4})!}{\pi^{1/2}2^{a/2+1/4}}\mathrm{e}^{-z^2/4}\,{}_1F_1(\tfrac{1}{2}a+\tfrac{1}{4};\tfrac{1}{2};\tfrac{1}{2}z^2),$$
$$y_2 = \frac{(-\frac{1}{2}a-\frac{1}{4})!}{\pi^{1/2}2^{a/2-1/4}}z\mathrm{e}^{-z^2/4}\,{}_1F_1(\tfrac{1}{2}a+\tfrac{3}{4};\tfrac{3}{2};\tfrac{1}{2}z^2),$$

and ${}_1F_1$ is the usual confluent hypergeometric function. Connection formulae follow immediately from (3.2) and (3.3); they are

$$U(a,-z) = V(a,z)\cos\pi a - U(a,z)\sin\pi a, \tag{3.4}$$
$$V(a,-z) = U(a,z)\cos\pi a + V(a,z)\sin\pi a. \tag{3.5}$$

They permit results in one half of the z-plane to be carried over to the remainder of the z-plane. One particular relation is

$$U(a,-z) + \mathrm{i}V(a,-z) = \mathrm{i}\mathrm{e}^{\mathrm{i}\pi a}\{U(a,z) - \mathrm{i}V(a,z)\}. \tag{3.6}$$

It can be seen from the formulae for y_1 and y_2 that

$$U(a,0) = \frac{(-\frac{1}{2}a-\frac{3}{4})!}{\pi^{1/2}2^{a/2+1/4}}\cos\pi(\tfrac{1}{4}+\tfrac{1}{2}a) = \frac{\pi^{1/2}2^{-a/2-1/4}}{(\frac{1}{2}a-\frac{1}{4})!}, \tag{3.7}$$
$$V(a,0) = \frac{(-\frac{1}{2}a-\frac{3}{4})!}{\pi^{1/2}2^{a/2+1/4}}\sin\pi(\tfrac{1}{4}+\tfrac{1}{2}a). \tag{3.8}$$

Moreover, if $\mathrm{d}U(a,z)/\mathrm{d}z$ is denoted by $U'(a,z)$,

$$U'(a,0) = -\frac{(-\frac{1}{2}a-\frac{1}{4})!}{\pi^{1/2}2^{a/2-1/4}}\sin\pi(\tfrac{1}{4}+\tfrac{1}{2}a) = -\frac{\pi^{1/2}2^{1/4-a/2}}{(\frac{1}{2}a-\frac{3}{4})!}, \tag{3.9}$$
$$V'(a,0) = \frac{(-\frac{1}{2}a-\frac{1}{4})!}{\pi^{1/2}2^{a/2-1/4}}\cos\pi(\tfrac{1}{4}+\tfrac{1}{2}a). \tag{3.10}$$

Consequently,

$$U(a,z)V'(a,z) - U'(a,z)V(a,z) = (-a-\tfrac{1}{2})!\left(\frac{2}{\pi}\right)^{1/2}. \tag{3.11}$$

When a is fixed and $|z| \to \infty$ with $|\mathrm{ph}\, z| < 3\pi/4$,

$$U(a,z) \sim z^{-a-1/2}\mathrm{e}^{-z^2/4}\sum_{s=0}(-2)^s(\tfrac{1}{2}a+\tfrac{1}{4})_s\frac{(\frac{1}{2}a+\frac{3}{4})_s}{s!z^{2s}}, \tag{3.12}$$

where, for arbitrary $(b)_s$, we denote by $(b)_s$ the Pochhammer symbol defined by

$$(b)_s = \frac{(b+s-1)!}{(b-1)!}.$$

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When $\pi/4 < \operatorname{ph} z < 5\pi/4$,

$$U(a,z) \sim z^{-a-1/2}\mathrm{e}^{-z^2/4} + \frac{(2\pi)^{1/2}}{(a-\frac{1}{2})!}\mathrm{e}^{\mathrm{i}\pi(1/2-a)}z^{a-1/2}\mathrm{e}^{z^2/4} \tag{3.13}$$

as $|z| \to \infty$. For $-5\pi/4 < \operatorname{ph} z < -\pi/4$, change the sign of i in the exponential in the second term on the right-hand side of (3.13).

The analogous formula for V is

$$V(a,z) \sim \epsilon_0 \mathrm{i}U(a,z) + (-a-\tfrac{1}{2})!\left(\frac{2}{\pi}\right)^{1/2} z^{a-1/2}\mathrm{e}^{z^2/3}\sum_{s=0} 2^s(\tfrac{1}{4}-\tfrac{1}{2}a)_s\frac{(\frac{3}{4}-\frac{1}{2}a)_s}{s!z^{2s}}, \tag{3.14}$$

where $\epsilon_0 = 1$ for $-\pi/4 < \operatorname{ph} z < 3\pi/4$ and $\epsilon_0 = -1$ for $-3\pi/4 < \operatorname{ph} z < \pi/4$. There are apparently two different formulae for V when $|\operatorname{ph} z| < \pi/4$ but they are consistent because here U is negligible compared with the rest of the expression. The asymptotic behaviour of U and V for other regions of $\operatorname{ph} z$ can be deduced from (3.12), (3.13) and (3.14) by means of the connection formulae (3.4) and (3.5).

After the ψ in (2.10) is dropped, comparison with (3.1) shows that the first approximation to q is given by

$$q(\zeta) = AU(-\tfrac{1}{2}\lambda\alpha^2, (2\lambda)^{1/2}\zeta) + BV(-\tfrac{1}{2}\lambda\alpha^2, (2\lambda)^{1/2}\zeta),$$

where A and B are constants to be determined. These constants can be found from the values of q and its derivative at $\zeta = 0$.

From (2.12),

$$\left(\frac{\alpha^2}{1-\sigma^2}\right)^{1/4}\{AU(-\tfrac{1}{2}\lambda\alpha^2,0) + BV(-\tfrac{1}{2}\lambda\alpha^2,0)\} = F(\tfrac{1}{2}-\mu,\mu+\tfrac{1}{2};\lambda+1;\tfrac{1}{2}) = p_1 \quad \text{(say)}. \tag{3.15}$$

For the derivative let

$$p_2 = \left[\frac{\mathrm{d}}{\mathrm{d}v}\{(1-\mathrm{i}v)^\lambda F(\tfrac{1}{2}-\mu,\mu+\tfrac{1}{2};\lambda+1;\tfrac{1}{2}-\tfrac{1}{2}\mathrm{i}v)\}\right]_{v=0}.$$

Then

$$\left(\frac{1-\sigma^2}{\alpha^2}\right)^{1/4}(2\lambda)^{1/2}\{AU'(-\tfrac{1}{2}\lambda\alpha^2,0) + BV'(-\tfrac{1}{2}\lambda\alpha^2,0)\} = p_2. \tag{3.16}$$

On account of (3.11), (3.15) and (3.16) imply that

$$(\tfrac{1}{2}\lambda\alpha^2-\tfrac{1}{2})!\left(\frac{2}{\pi}\right)^{1/2}A = \left(\frac{1-\sigma^2}{\alpha^2}\right)^{1/4}p_1V'(-\tfrac{1}{2}\lambda\alpha^2,0) - \left(\frac{\alpha^2}{1-\sigma^2}\right)^{1/4}\frac{p_2}{(2\lambda)^{1/2}}V(-\tfrac{1}{2}\lambda\alpha^2,0), \tag{3.17}$$

$$(\tfrac{1}{2}\lambda\alpha^2-\tfrac{1}{2})!\left(\frac{2}{\pi}\right)^{1/2}B = \left(\frac{\alpha^2}{1-\sigma^2}\right)^{1/4}\frac{p_2}{(2\lambda)^{1/2}}U(-\tfrac{1}{2}\lambda\alpha^2,0) - \left(\frac{1-\sigma^2}{\alpha^2}\right)^{1/4}p_1U'(-\tfrac{1}{2}\lambda\alpha^2,0). \tag{3.18}$$

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Since

$$p_1 = \frac{\lambda!\pi^{1/2}2^{-\lambda}}{(\frac{1}{2}\lambda - \frac{1}{2}\mu - \frac{1}{4})!(\frac{1}{2}\lambda + \frac{1}{2}\mu - \frac{1}{4})!} \tag{3.19}$$

and

$$p_2 = -\frac{\lambda!\pi^{1/2}2^{1-\lambda}\mathrm{i}}{(\frac{1}{2}\lambda - \frac{1}{2}\mu - \frac{3}{4})!(\frac{1}{2}\lambda + \frac{1}{2}\mu - \frac{3}{4})!}, \tag{3.20}$$

it follows from (3.7)–(3.10) that

$$(\tfrac{1}{2}\lambda\alpha^2 - \tfrac{1}{2})!\left(\frac{2}{\pi}\right)^{1/2} A = A_1 \cos\pi(\tfrac{1}{4} - \tfrac{1}{4}\lambda\alpha^2) + \mathrm{i}A_2 \sin\pi(\tfrac{1}{4} - \tfrac{1}{4}\lambda\alpha^2), \tag{3.21}$$

$$(\tfrac{1}{2}\lambda\alpha^2 - \tfrac{1}{2})!\left(\frac{2}{\pi}\right)^{1/2} B = A_1 \sin\pi(\tfrac{1}{4} - \tfrac{1}{4}\lambda\alpha^2) - \mathrm{i}A_2 \cos\pi(\tfrac{1}{4} - \tfrac{1}{4}\lambda\alpha^2), \tag{3.22}$$

where

$$A_1 = \left(\frac{1-\sigma^2}{\alpha^2}\right)^{1/4} \frac{\lambda!2^{1/4-\lambda/2-\mu/2}}{(\frac{1}{2}\lambda + \frac{1}{2}\mu - \frac{1}{4})!} \tag{3.23}$$

and

$$A_2 = \left(\frac{\alpha^2}{1-\sigma^2}\right)^{1/4} \frac{\lambda!2^{1/4-\lambda/2-\mu/2}}{(\frac{1}{2}\lambda + \frac{1}{2}\mu - \frac{3}{4})!\lambda^{1/2}}. \tag{3.24}$$

Thus the approximation for $q(\zeta)$ which results from neglecting ψ in (2.10) is

$$\begin{aligned} q(\zeta) = {} & \frac{A_1 + A_2}{(\frac{1}{2}\lambda\alpha^2 - \frac{1}{2})!2}(\tfrac{1}{2}\pi)^{1/2}\mathrm{e}^{\mathrm{i}\pi(1/4-\lambda\alpha^2/4)} \\ & \qquad \times \{U(-\tfrac{1}{2}\lambda\alpha^2, (2\lambda)^{1/2}\zeta) - \mathrm{i}V(-\tfrac{1}{2}\lambda\alpha^2, (2\lambda)^{1/2}\zeta)\} \\ & + \frac{A_1 - A_2}{(\frac{1}{2}\lambda\alpha^2 - \frac{1}{2})!2}(\tfrac{1}{2}\pi)^{1/2}\mathrm{e}^{-\mathrm{i}\pi(1/4-\lambda\alpha^2/4)} \\ & \qquad \times \{U(-\tfrac{1}{2}\lambda\alpha^2, (2\lambda)^{1/2}\zeta) + \mathrm{i}V(-\tfrac{1}{2}\lambda\alpha^2, (2\lambda)^{1/2}\zeta)\}. \end{aligned} \tag{3.25}$$

The expression in (3.25) can be rewritten by calling on the connection formulae

$$U(a,z) \pm \mathrm{i}V(a,z) = (-\tfrac{1}{2} - a)!\left(\frac{2}{\pi}\right)^{1/2} \mathrm{e}^{\pm\mathrm{i}\pi(a/2+1/4)}U(-a, \mp\mathrm{i}z). \tag{3.26}$$

Consequently,

$$q(\zeta) = \tfrac{1}{2}(A_1 + A_2)U(\tfrac{1}{2}\lambda\alpha^2, \mathrm{i}(2\lambda)^{1/2}\zeta) + \tfrac{1}{2}(A_1 - A_2)U(\tfrac{1}{2}\lambda\alpha^2, -\mathrm{i}(2\lambda)^{1/2}\zeta). \tag{3.27}$$

It may be remarked that, since $\lambda + \mu$ is large,

$$A_2/A_1 \sim 1 + O(1/(\lambda+\mu)^2) \tag{3.28}$$

from (3.23) and (3.24). This suggests that, if $O(1/(\lambda+\mu)^2)$ is neglected, the term involving $A_1 - A_2$ in (3.27) can be dropped and $A_1 + A_2$ can be replaced by $2A_1$, provided that ζ does not have a value that makes the second U dominant. In any case, there is little point in retaining the order term in (3.28) until it has been ascertained whether the presence of ψ in (2.10) produces a contribution of like magnitude. The effect of ψ is considered in the next section.

4. Higher approximations

To allow for the influence of ψ in (2.10) as $\lambda \to \infty$, take as a possible solution

$$q(\zeta) = U(-\tfrac{1}{2}\lambda\alpha^2, (2\lambda)^{1/2}\zeta) \sum_{m=0} \frac{r_m(\zeta)}{\lambda^m} + \frac{(2\lambda)^{1/2}}{\lambda^2} U'(-\tfrac{1}{2}\lambda\alpha^2, (2\lambda)^{1/2}\zeta) \sum_{m=0} \frac{s_m(\zeta)}{\lambda^m}. \quad (4.1)$$

Then

$$\frac{\mathrm{d}q}{\mathrm{d}\zeta} = U \sum_{m=0} \left\{ \frac{r'_m}{\lambda^m} + (\zeta^2 - \alpha^2)\frac{s_m}{\lambda^m} \right\} + (2\lambda)^{1/2} U' \sum_{m=0} \left\{ \frac{r_m}{\lambda^m} + \frac{s'_m}{\lambda^{m+2}} \right\} \quad (4.2)$$

and

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\zeta^2} = \lambda^2(\zeta^2 - \alpha^2)q + U \sum_{m=0} \{r''_m + 2\zeta s_m + 2(\zeta^2 - \alpha^2)s'_m\} \frac{1}{\lambda^m} + (2\lambda)^{1/2} U' \sum_{m=0} \left(2r'_m + \frac{s''_m}{\lambda^2} \right) \frac{1}{\lambda^m}.$$

Therefore, (2.10) can be satisfied by requiring that $r'_0 = 0$, $r'_1 = 0$ and

$$r'_{m+2} = \tfrac{1}{2}(\psi s_m - s''_m), \quad (4.3)$$

$$(\zeta^2 - \alpha^2)s'_m + \zeta s_m = \tfrac{1}{2}(\psi r_m - r''_m) \quad (4.4)$$

for $m \geqslant 0$.

With r_m known, s_m can be found from (4.4) and then r_{m+2} can be determined from (4.3). The iterative process is started with $r_0 = 1$ and $r_1 = 0$. The constants of integration arising in (4.3) and (4.4) are fixed by requiring that s_m $(m \geqslant 0)$ and r_m $(m \geqslant 2)$ vanish at the origin. Then

$$r_{m+2} = \frac{1}{2} \int_0^\zeta (\psi s_m - s''_m)\,\mathrm{d}\zeta \quad (4.5)$$

and

$$s_m = \frac{1}{2(\zeta^2 - \alpha^2)^{1/2}} \int_0^\zeta \frac{\psi r_m - r''_m}{(\zeta^2 - \alpha^2)^{1/2}}\,\mathrm{d}\zeta. \quad (4.6)$$

Clearly, both s_m and r_m are identically zero when m is odd.

Observe that the choice $r_0 = 0$, $r_1 = 1$ generates the same series as in (4.1) multiplied by $1/\lambda$. Hence there is no loss of generality in the selection that has been made.

Since $\psi = O(1/\zeta^2)$ as $\psi \to \infty$, it is evident that r_m is bounded at infinity, whereas $s_m = O(1/\zeta)$. Furthermore, ψ is an even function of ζ so that s_m is an odd function and r_m an even function of ζ. This is verified by the explicit formula for s_0, namely

$$s_0 = \frac{\zeta^3 - 6\alpha^2\zeta}{24\alpha^2(\zeta^2-\alpha^2)^2} - \frac{v}{8(\zeta^2-\alpha^2)^{1/2}(1-\sigma^2)}\left[\frac{\sigma^2-4}{\{\sigma^2(1+v^2)-1\}^{1/2}} + \frac{5}{3}\frac{2\sigma^2v^2-3(1-\sigma^2)}{\{\sigma^2(1+v^2)-1\}^{3/2}}\right]. \quad (4.7)$$

Another solution of (2.10) is obtained by replacing U with V in (4.1). Two further solutions are given by

$$q_{\pm}(\zeta) = U(\tfrac{1}{2}\lambda\alpha^2, \pm\mathrm{i}(2\lambda)^{1/2}\zeta)\sum_{m=0}\frac{r_m(\zeta)}{\lambda^m} \pm \frac{\mathrm{i}(2\lambda)^{1/2}}{\lambda^2}U'(\tfrac{1}{2}\lambda\alpha^2, \pm\mathrm{i}(2\lambda)^{1/2}\zeta)\sum_{m=0}\frac{s_m(\zeta)}{\lambda^m}, \quad (4.8)$$

the upper and lower signs being taken together.

If the analysis of § 3 is repeated, it is found that (3.27) becomes

$$q(\zeta) = \tfrac{1}{2}(A_1+A_3)q_+(\zeta) + \tfrac{1}{2}(A_1-A_3)q_-(\zeta), \quad (4.9)$$

where

$$A_3 = A_2 \Big/ \left\{1 + \sum_{m=0}\frac{s'_m(0)}{\lambda^{m+2}}\right\}. \quad (4.10)$$

The formula (4.9) can be simplified. As v approaches $-\mathrm{i}$ from the origin along the imaginary axis, $\zeta \to -\mathrm{i}\infty$. Indeed, if $\zeta = -\mathrm{i}\eta$ and $\eta > 0$,

$$\eta^2 + \alpha^2 \ln\eta \sim -\ln(1-\mathrm{i}v).$$

Hence, when $v \to -\mathrm{i}$, it is evident from (3.12) that $q_+(\zeta)$ tends to zero like $(1-\mathrm{i}v)^{\lambda/2}$ when the behaviour of r_m and s_m at infinity is borne in mind. On the other hand, it is transparent from (3.13) that $q_-(\zeta)$ becomes infinite like $(1-\mathrm{i}v)^{-\lambda/2}$. But, as $v \to -\mathrm{i}$,

$$F(\tfrac{1}{2}-\mu, \mu+\tfrac{1}{2}; \lambda+1; \tfrac{1}{2}(1-\mathrm{i}v)) \to 1,$$

which is inconsistent with (4.9) and (2.12) unless the term in $q_-(\zeta)$ is absent. In other words, consistency requires that $A_1 = A_3$ or

$$1 + \sum_{m=0}\frac{s'_m(0)}{\lambda^{m+2}} = \frac{A_2}{A_1} = \left(\frac{\alpha^2}{1-\sigma^2}\right)^{1/2}\frac{(\frac{1}{2}\lambda+\frac{1}{2}\mu-\frac{1}{4})!}{(\frac{1}{2}\lambda+\frac{1}{2}\mu-\frac{3}{4})!\lambda^{1/2}} \quad (4.11)$$

from (4.10), (3.23) and (3.24).

The expression for $q(\zeta)$ now reduces to

$$q(\zeta) = A_1 q_+(\zeta) \quad (4.12)$$

and

$$F(\tfrac{1}{2}-\mu,\mu+\tfrac{1}{2};\lambda+1;\tfrac{1}{2}(1-\mathrm{i}v)) \sim (1-\mathrm{i}v)^{-\lambda/2}(1+\mathrm{i}v)^{\lambda/2}\left(\frac{\zeta^2-\alpha^2}{\sigma^2(1+v^2)-1}\right)^{1/4} A_1 q_+(\zeta). \tag{4.13}$$

An additional check on (4.13) is provided by allowing v and ζ to tend to infinity along the positive real axis, taking into account the largeness of the parameters.

Note that it has now been verified that putting $A_1 = A_2$ in (3.27) does offer a tolerable first approximation.

It follows from (1.9) and (4.13) that, as $\nu+\mu \to \infty$,

$$I\left(\nu,\mu-\tfrac{1}{2},\mu-\tfrac{1}{2},\frac{\mathrm{i}v-1}{\mathrm{i}v+1}\right) \sim \frac{\nu!(\mu-\frac{1}{2})!(1+\mathrm{i}v)^{\lambda/2-\mu+1/2}}{(\frac{1}{2}\lambda+\frac{1}{2}\mu-\frac{1}{4})!(1-\mathrm{i}v)^{\lambda/2}}\left(\frac{1-\sigma^2}{\alpha^2}\right)^{1/4} 2^{\mu/2-\lambda/2-1/4}\left\{\frac{\zeta^2-\alpha^2}{\sigma^2(1+v^2)-1}\right\}^{1/4} q_+(\zeta). \tag{4.14}$$

When μ is an odd half integer, F is a polynomial in $1-\mathrm{i}v$. For these values of μ, the polynomial will supply more convenient values for I than (4.14) so long as μ is not too large.

Asymptotic formulae for other hypergeometric functions can be deduced from (4.13) by means of the well-known relations between hypergeometric functions but details are omitted.

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