DIFFERENTIABLE MONTGOMERY-SAMELSON FIBERINGS WITH FINITE SINGULAR SETS

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1. Introduction. In 1946 Montgomery and Samelson (**11**) introduced a generalization of the notion of a differentiable group action with one type of orbit besides fixed points. Such an object is essentially a locally trivial fibering except on a certain *singular set* over which fibres are pinched to points. In recent years there has been a fair amount of research on these MS-*fiberings* and similar singular fiberings. This paper is another effort in this direction. For a fairly complete bibliography of the literature, the reader should consult the references, and in particular, (**5**).

Let $f: M^n \to S^p$, with M^n a closed connected n-manifold and S^p the unit p-sphere with standard differentiable structure, be the projection map of a smooth MS-fibering with finite non-empty singular set. It is known that $(n, p) = (2m, m + 1)$, where $m = 2$, 4 or 8 and that the fibre must be a homotopy 1-sphere, 3-sphere or 7-sphere (**13**).

In (**2**) it was proved that the mth Betti number $b_m(M^{2m})$ determines the number of singular points. In the case where $b_m(M^{2m}) = 0$, it is known that f is topologically a suspension of a Hopf-type fibering $S^{2m-1} \to S^m$. Examples where the Betti number is non-vanishing may be obtained by smoothly plumbing (away from the singular set) suspensions of Hopf-type fiberings (**2**). The total spaces of these MS-fiberings all have the oriented homotopy type of a connected sum of $\frac{1}{2}b_m(M^{2m})$ copies of $S^m \times S^m$. The main result of this paper is that this is the *only* possible kind of total space, thus classifying all MS-fiberings over spheres with finite singular set. In the cases $m = 4$ and $m = 8$, our classification is actually up to homeomorphism, and it seems reasonable to think that this is true for $m = 2$. We believe that use of results in (**7**) may give the classification up to diffeomorphism, at least for $m = 4$ and $m = 8$.

Our results have direct application to transformation groups, as do the results of (**1**; **2**; **3**); an example is the following result.

THEOREM. *Let (M^n, G) be a smooth action of the compact connected Lie group G with orbit space S^p and one type of orbit other than isolated fixed points. We can then prove the following:*

(i) $(n, p) = (2m, m + 1)$ *with* $m = 2$ *and* $G = \mathrm{SO}(2)$ *or* $m = 4$ *and* $G = \mathrm{Sp}(1)$;

Received August 15, 1968. This research was partially supported by NSF Contract Number GP-5420.

(ii) *The number of fixed points is equal to the Euler characteristic of* M^{2m} *and is* $2(k + 1)$ *for some* $k \geqq 0$;

(iii) *The G-space* M^{2m} *has the oriented homotopy type of the connected sum*

$$[\sum^k \# (S^m \times S^m)] \# S^{2m},$$

and if $m = 4$, *then* M^{2m} *is homeomorphically this sum;*

(iv) *If* $k = 0$, *then* (M^n, G) *is topologically one or the other of the suspended actions* $(S^4, \mathrm{SO}(2))$ *or* $(S^8, \mathrm{Sp}(1))$.

2. Preliminaries. Let M^n and N^p be closed connected C^∞-manifolds, $n > p$. A map $f: M^n \to N^p$ is the projection map of a *smooth MS-fibering with finite singular set A* if f is a locally trivial C^∞-fibering on $M^n - A$ and f is a homeomorphism on the finite non-empty set A. In particular, the local product maps are to be diffeomorphisms. The usual example has been the suspension of a Hopf map $S^{2m-1} \to S^m$ for $m = 2, 4, 8$. The singular set in either of these cases is just a set of two points. The map f is of less than maximal rank on the singular set.

A C^∞-manifold M^n, possibly with boundary, is *m-parallelizable* if its tangent bundle is trivial on the m-skeleton of M^n provided with some piecewise linear structure. Clearly, every orientable manifold is 1-parallelizable. It was proved in (**8**) that if M^{2m} is *m-parallelizable and* $(m - 1)$*-connected, m even and* $m \geqq 4$ *with index* $I(M^{2m}) = 0$, then M^{2m} *is cobordant to an m-connected manifold.*

In (**9**) Milnor classified $2m$-manifolds, m even, according as to whether the usual quadratic form is of Type I or of Type II. It isproved that *an* $(m - 1)$*-connected* $2m$*-manifold* M^{2m} *is of Type* II *if and only if the* m*th Stiefel-Whitney class* $W_m(M^{2m})$ *is zero.* It follows that *any simply connected* 4*-manifold of Type* II *is a spin manifold* (**10**).

THEOREM (Milnor). *A simply connected* 4*-manifold* M^4 *of Type* II *and index* $I(M^4) = 0$ *has the oriented homotopy type of the connected sum of* $\frac{1}{2}b_2(M^4)$ *copies of* $S^2 \times S^2$.

A similar result is proved for $(m - 1)$-connected $2m$-manifolds, $m \geqq 2$, by Wall (**14**). However, it is necessary to bring into play the additional invariant $X^2(M^{4m})$, where $X(M^{4m})$ is the obstruction to the triviality of the stable tangent bundle of M^{4m}. In (**6**) Kervaire proved that

$$P_m(M^{4m}) = \pm a_m(2m - 1)!X(M^{4m}),$$

where the left side is the mth Pontryagin class of M^{4m} and $a_m = 1$ for m even, $a_m = 2$ for m odd.

THEOREM (Wall). *If* M^{4m} *is of Type* II *and* $(2m - 1)$*-connected with* $I(M^{4m}) = X^2(M^{4m}) = 0$ *for* $m = 2, 4$, *then* M^{4m} *is homeomorphic to the connected sum of* $\frac{1}{2}b_{2m}(M^{4m})$ *copies of* $S^{2m} \times S^{2m}$.

3. Statement of the Main Theorem. Let $f\colon M^{2m} \to S^{m+1}$, $m = 2, 4, 8$, be a smooth MS-fibering with finite non-empty singular set. We have seen that this is no restriction on the dimension of the base or total space.

MAIN THEOREM. *The total space M^{2m} has the oriented homotopy type of the connected sum $[\sum^k \# (S^m \times S^m)] \# S^{2m}$, where the Euler characteristic $e(M^{2m}) = 2(k+1)$, $k \geqq 0$. If $m = 4$ or 8, M^{2m} is homeomorphic to this space.*

To prove this theorem we first show that M^{2m} must be $(m-1)$-connected. Then the following two propositions hold.

PROPOSITION 1. *M^{2m} is of Type* II *with $I(M^{2m}) = 0$ and $b_m(M^{2m}) = 2k$ for some $k \geqq 0$.*

PROPOSITION 2. *The tangent bundle $\tau(M^{2m})$ is m-parallelizable.*

As an immediate consequence of Proposition 1, the fact that $e(M^{2m}) = 2 + b_m(M^{2m})$ (see **2**), and Milnor's classification theorem stated above, we have the following result.

COROLLARY 3. *The total space M^4 has the oriented homotopy type of $[\sum^k \# (S^2 \times S^2)] \# S^4$, where $e(M^4) = 2(k+1)$.*

In addition, M^4 must be a spin manifold. Indeed, any spin manifold M^4 of index zero occurs as the total space of an MS-fibering over a sphere up to homotopy type, as can be seen by plumbing techniques.

In the case $m = 4$ or $m = 8$, m-parallelizability implies that M^{2m} is cobordant to S^{2m}, and hence that $X^2 = 0$ by Kervaire's equation, stated above. The classification of Wall then yields the following result.

COROLLARY 4. *The total space M^{2m}, $m = 4$ or $m = 8$, is homeomorphic to $[\sum^k \# (S^m \times S^m)] \# S^{2m}$, where $e(M^{2m}) = 2(k+1)$.*

Clearly, Corollaries 3 and 4 yield the desired theorem.

4. Proof that M^{2m} is $(m-1)$-connected. We first consider the case $m = 2$ by showing that M^{2m} is simply connected. The gist of the argument in this case was suggested to us by J. G. Timourian (oral communication).

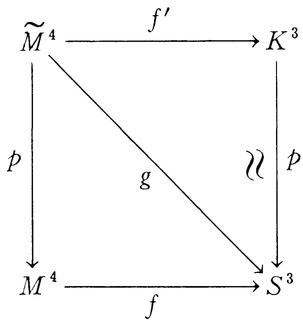
It follows from the local structure theory (**13**) that the fibre of $f\colon M^4 \to S^3$ is diffeomorphically a circle. Thus the fibre homotopy sequence yields

$$\begin{array}{ccccc} \to Z \to & \pi_1(M^4 - A) & \to & \pi_1(S^3 - f(A)) & \to \\ & \wr\wr & & \wr\wr & \\ & \pi_1(M^4) & \to & \pi_1(S^3) \approx 0 & \end{array}$$

where the vertical isomorphisms follow from dimension considerations. We therefore see that the fundamental group of M^4 is cyclic.

Suppose that $\pi_1(M^4)$ has torsion. Then there is a q-to-1 cover $\tilde{M}^4$ of M^4 for some prime q. Let q denote this covering map and g the composition $f \circ p$.

It is easy to see that g is a proper singular fibering in the sense of Timourian (**13**). As such, g may be factored into a smooth MS-fibering f' followed by a covering map p' also of index q. Clearly,



K^3 must be homeomorphic to S^3 and thus it follows that p is a homeomorphism, a contradiction. Therefore, $\pi_1(\tilde{M}^4)$ is torsion-free and is a free abelian group of rank at most one.

Suppose that $\pi_1(M^4) \neq 0$. Then we can always find a 2-to-1 cover of M^4. Applying the argument above we again obtain a contradiction. Thus the proof of simple connectivity in the case $m = 2$ is complete.

The cases $m = 4$ and $m = 8$ are similar, thus we will give details only for the case $m = 4$.

By Timourian's local structure theory (**13**), the fibre is a homotopy 3-sphere. Then the fibre-homotopy sequence implies that M^8 is 2-connected, while the Hurewicz isomorphism theorem implies that $\pi_3(M^8) = H_3(M^8; Z)$. We complete the proof by showing that $H_3(M^8; Z)$ is torsion-free and $b_3(M^8) = 0$.

By torsion duality, there is torsion in dimension 3 if and only if there is torsion in dimension 4. Letting F denote a field, it follows, as in (**2**), that

$$e(M^8; F) = b_4(M^8; F) + 2 = \#(A), \tag{1}$$

where $\#(A)$ is the number of singular points. It follows from the universal coefficient theorem that if there is torsion in dimension 4, there is a prime p such that $b_4(M^8; Z_p) \neq 0$. Then, from the above equation and the fact that $b_4(M^8) \leqq b_4(M^8; Z_p)$ for any prime p, we obtain a contradiction. Thus $H_3(M^8; Z)$ is torsion-free.

It remains to show that $b_3(M^8) = 0$. However, from (**2**, pp. 182–183), we have

$$b_3(M^8) = b_2(M^8), \tag{2}$$

which yields the result since M^8 is already 2-connected.

Note that in the case $m = 8$ we would use

$$b_7(M^{16}) = b_2(M^{16}) \tag{3}$$

from (**2**, p. 183), instead of (2) above.

5. Proof of Proposition 1. We will need the following result.

LEMMA 5. *Let* $f\colon M^{2m} \to S^{m+1}$, $m = 2, 4, 8$, *be a smooth MS-fibering with finite non-empty singular set* A *and* $i\colon M^{2m} - A \to M^{2m}$ *the inclusion map. Then* $i^*\colon H^q(M^{2m}; Z_2) \to H^q(M^{2m} - A; Z_2)$ *is an isomorphism for* $q \leqq 2m - 2$.

Proof. It is easy to see that $i_*\colon \pi_q(M^{2m} - A) \to \pi_q(M^{2m})$ is an isomorphism for $q \leqq 2m - 2$. By Whitehead's theorem, $i_*\colon H_q(M^{2m} - A) \to H_q(M^{2m})$ is an isomorphism for $q \leqq 2m - 2$. From the diagram

$$\begin{array}{ccccccc} 0 \to \mathrm{Ext}(H_{q-1}(M^{2m}), Z_2) & \to & H^q(M^{2m}; Z_2) & \to & \mathrm{Hom}(H_q(M^{2m}), Z_2) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to \mathrm{Ext}(H_{q-1}(M^{2m} - A), Z_2) & \to & H^q(M^{2m} - A; Z_2) & \to & \mathrm{Hom}(H_q(M^{2m} - A), Z_2) & \to & 0 \end{array}$$

the desired conclusion follows via the 5-lemma.

In order to show that M^{2m} is of Type II, it is enough to show that $W_m(M^{2m}) = 0$ since M^{2m} is $(m - 1)$-connected. Since $S^{m+1} - f(A) \subseteq R^{m+1}$, we have the tangent bundle isomorphism $\tau(S^{m+1} - f(A)) \simeq \theta^{m+1}$, where θ^{m+1} denotes the trivial $(m + 1)$-plane bundle. However, $f\colon M^{2m} - A \to S^{m+1} - f(A)$, the restriction of f, is of maximal rank so that the tangent bundle $\tau(M^{2m} - A)$ splits off a trivial $(m + 1)$-plane bundle, i.e., $\tau(M^{2m} - A) \simeq \xi^{m-1} \oplus \theta^{m+1}$ for some bundle ξ^{m-1}. Thus $W_m(\tau(M^{2m} - A)) = W_m(\xi^{m-1}) = 0$. By the naturality of Stiefel-Whitney classes, we therefore have

$$W_m(\tau(M^{2m} - A)) = W_m(i^*\tau(M^{2m})) = i^*(W_m(M^{2m})) = 0.$$

Now Lemma 5 yields $W_m(M^{2m}) = 0$ since $m \leqq 2m - 2$ for $m = 2, 4, 8$.

It follows that $I(M^{2m}) \equiv 0 \pmod 8$ and that $I(M^{2m}) \equiv b_m(M^{2m}) \pmod 2$ since M^{2m} is of Type II (**9**). Therefore, $b_m(M^{2m}) = 2k$ for some $k \geqq 0$.

It remains to show that M^{4m} has vanishing index. In order to prove this, we will need the following well-known result, which we state without proof.

LEMMA 6. *Let* W^{4m} *be a compact connected* $(2m - 1)$*-connected* $4m$*-manifold with boundary the union of homotopy* $(4m - 1)$*-spheres* ∂D_i, $i = 1, 2, \ldots, n$, *which bound smooth manifolds* D_i *homeomorphic to* $4m$*-disks. Let the diffeomorphisms* $f_i\colon \partial D_i \to W^{4m}$ *be the attaching maps and* W_*^{4m} *the identification space. Then*

$$I(W^{4m}) = I(W_*^{4m}). \tag{4}$$

The local structure theory (**13**) tells us that the MS-fibering $f\colon M^{2m} \to S^{m+1}$ is topologically the cone of a Hopf-type map $S^{2m-1} \to S^m$, $m = 2, 4, 8$. Letting $\#(A) = n$, we obtain the smooth fibering

$$f\colon W^{2m}\left(= M^{2m} - \bigcup_{i=1}^{n} \operatorname{int} D_i^{2m}\right) \to U^{m+1}\left(= S^{m+1} - \bigcup_{i=1}^{n} \operatorname{int} D_i^{m+1}\right)$$

https://doi.org/10.4153/CJM-1969-163-9 Published online by Cambridge University Press

given by restriction of f, where D_i^{2m} fibers over D_i^{m+1} as a cone of a Hopf-type map. Since the fibre of f is a homotopy 1-sphere, 3-sphere, or 7-sphere T^{m-1}, the equation

$$I(W^{2m}) = I(U^{m+1}) \cdot I(T^{m-1})$$

yields $I(W^{2m}) = 0$. Lemma 6 now implies that $I(W^{2m}) = I(M^{2m})$, and the proof is complete.

Remark. We know of no reference for the above index formula as such. However, the case of a fibering of manifolds without boundary is a well-known result of Chern-Hirzebruch-Serre (**4**). The proof in the boundary-free case may be modified easily to obtain the above formula.

6. Proof of Proposition 2. It is well known that smooth fiberings may be smoothly triangulated; see for instance (**12**). Thus we may suppose that the fibering f: $W^{2m} \to U^{m+1}$ of § 5 is simplicial. Moreover, the triangulations may be chosen so that the m-skeleton K^m fibers over the 1-skeleton L^1 of U^{m+1}. Since the tangent bundle restricted to L^1 is trivial, it follows that its restriction to K^m is $\xi^m \oplus \theta^m$, by the maximal rank of f. For $q < m$, the isomorphism $H^q(K^m; G) \approx H^q(M^{2m}; G)$ implies that the only existing obstruction to the triviality of $\xi^m \oplus \theta^m$ lies in $H^m(K^m; \pi_{m-1}(\mathrm{SO}(2m)))$. It follows from (**6**, p. 773) that this obstruction will vanish for $m = 4$, $m = 8$ if $P_{m/2}(\xi^m \oplus \theta^m)$ is zero. However, the Pontryagin classes are stable invariants and $P_{m/2}(\xi^m)$ is the square of the Euler class of ξ^m. The maximality of the rank of f implies that ξ^m splits off a trivial line bundle, and therefore the Euler class vanishes. For the case $m = 2$, we apply the above maximality argument and note that the pertinent obstruction is the Euler class of ξ^2. This completes the proof of Proposition 2.

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