VIII

Local solvability in locally integrable structures

Throughout this chapter we will work with a locally integrable structure $\mathcal{V}$ over a smooth manifold Ω. Our analysis will be for most of the chapter strictly local, and thus, we shall work in a neighborhood of a fixed point $p \in \Omega$. By Corollary I.10.2 there is a coordinate system $(x_1, \ldots, x_m, t_1, \ldots, t_n)$ centered at p and there are smooth, real-valued functions $\phi_1, \ldots, \phi_m$ defined in a neighborhood of the origin of $\mathbb{R}^{m+n}$ and satisfying

$$\phi_k(0,0) = 0, \quad \mathrm{d}_x\phi_k(0,0) = 0, \quad k = 1, \ldots, m, \tag{VIII.1}$$

such that the differentials of the functions

$$Z_k(x,t) = x_k + i\phi_k(x,t), \quad k = 1, \ldots, m \tag{VIII.2}$$

span T' near $p = (0,0)$.

We shall set $Z = (Z_1, \ldots, Z_m)$, $\phi = (\phi_1, \ldots, \phi_m)$. Thus we can write

$$Z(x,t) = x + i\phi(x,t),$$

which we assume is defined in an open neighborhood of the closure of $B_0 \times \Theta_0$, where $B_0 \subset \mathbb{R}^m$ and $\Theta_0 \subset \mathbb{R}^n$ are open balls centered at the corresponding origins. Thanks to (VIII.1) we can assume that

$$|\phi(x,t) - \phi(x',t)| \leq \frac{1}{2}|x - x'|, \ x, x' \in B_0, \ t \in \Theta_0. \tag{VIII.3}$$

Also recall that $\mathcal{V}$ is spanned, in an open set that contains the closure of $B_0 \times \Theta_0$, by the linearly independent, pairwise commuting vector fields (*cf.* (I.37))

$$L_j = \frac{\partial}{\partial t_j} - i\sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(x,t) M_k, \quad j = 1, \ldots, n, \tag{VIII.4}$$

where the vector fields

$$M_k = \sum_{\ell=1}^{m} \mu_{k\ell}(x,t)\,\frac{\partial}{\partial x_\ell}\,, \quad k = 1,\dots,m \tag{VIII.5}$$

are characterized by the relations $M_k Z_\ell = \delta_{k\ell}$ (*cf.* (I.35) and (I.36)).

VIII.1 Local solvability in essentially real structures

If $\mathcal{V}$ defines an essentially real structure over Ω of rank n then the functions ϕ_j can be taken identically zero (Theorem I.9.1). Hence $L_j = \partial/\partial t_j$, $j = 1,\dots,n$ and the operator d' equals the partial exterior derivative

$$\mathrm{d}_t f = \sum_{j=1}^{n} \sum_{|J|=q} \frac{\partial f_J}{\partial t_j}\,\mathrm{d}t_j \wedge \mathrm{d}t_J\,, \tag{VIII.6}$$

that is, the d'-complex is nothing other than the standard de Rham complex along the leaves of the foliation defined by $\mathcal{V}$. In particular, if we apply the Poincaré Lemma (Section VII.3) we conclude that *local solvability holds for an essentially real structure near any point and at any degree*.

VIII.2 Local solvability in the analytic category

Now we assume that the manifold Ω and the given locally integrable structure $\mathcal{V}$ are real-analytic. In this case Corollary I.11.1 asserts that the coordinates, functions, and vector fields described at the beginning of the chapter can all be taken real-analytic. We shall show:

PROPOSITION VIII.2.1. *Let* $\underline{f} \in \mathfrak{U}_q^{\mathcal{V}}(p)$ *have analytic coefficients and satisfy* $\mathrm{d}'\underline{f} = 0$. *If* $q \geq 1$ *then there is* $\mathbf{u} \in \mathfrak{U}_{q-1}^{\mathcal{V}}(p)$, *also with analytic coefficients, solving* $\mathrm{d}'\mathbf{u} = \underline{f}$.

PROOF. We let

$$f = \sum_{|I|=q} f_I(x,t)\,\mathrm{d}t_I$$

represent $\underline{f}$; the functions f_I are thus real-analytic in a neighborhood of the origin and

$$\mathrm{d}'f = \sum_{j=1}^{n} \sum_{|I|=q} L_j f_I(x,t)\,\mathrm{d}t_j \wedge \mathrm{d}t_I = 0. \tag{VIII.7}$$

Let $1 \le r \le n$ be an integer such that f only involves $\mathrm{d}t_1, \dots, \mathrm{d}t_r$. Hence we can write $f = f_1 + f_2$, where

$$f_1 = \sum_{|I|=q, I\subset\{1,\dots,r-1\}} f_I(x,t)\,\mathrm{d}t_I$$

and

$$f_2 = \sum_{|J|=q-1, J\subset\{1,\dots,r-1\}} (-1)^{q-1} f_{J\cup\{r\}} \mathrm{d}t_r \wedge \mathrm{d}t_J\,.$$

Notice in particular that (VIII.7) implies

$$L_s f_{J\cup\{r\}} = 0, \quad s > r. \tag{VIII.8}$$

We then apply the Cauchy–Kowalevsky theorem in order to solve, in a neighborhood of the origin, the problems

$$\begin{cases} L_r u_J &= (-1)^{q-1} f_{J\cup\{r\}} \\ u_J\big|_{t_r=0} &= 0. \end{cases} \tag{VIII.9}$$

Since the vector fields L_j are pairwise commuting, (VIII.8) implies

$$L_r L_s u_J = 0, \quad s > r.$$

Since moreover $L_s u_J = 0$ when $t_r = 0$ it follows from the uniqueness part in the Cauchy–Kowalevsky theorem that $L_s u_J = 0$ for all $s > r$ and all J. Consequently, if we set $u = \sum_J u_J\,\mathrm{d}t_J$ then

$$\begin{aligned} \mathrm{d}'u \;=\; & \sum_{|J|=q-1, J\subset\{1,\dots,r-1\}} \sum_{j=1}^{r-1} L_j u_J\,\mathrm{d}t_j \wedge \mathrm{d}t_J \;+ \\ & \sum_{|J|=q-1, J\subset\{1,\dots,r-1\}} L_r u_J\,\mathrm{d}t_r \wedge \mathrm{d}t_J \end{aligned}$$

and hence (VIII.9) implies that the d'-closed form $\mathrm{d}'u - f$ only involves $\mathrm{d}t_1, \dots, \mathrm{d}t_{r-1}$. The proof can then be concluded by an elementary inductive argument, whose details are left to the reader. □

VIII.3 Elliptic structures

When the structure $\mathcal{V}$ is elliptic the discussion presented at the end of Section I.12 shows that the differential complex associated with $\mathcal{V}$ can be locally realized as the standard elliptic complex in $\mathbb{C}^m \times \mathbb{R}^{n'}$, $n' = n - m$, which we now describe and study in some detail.

Let $m \in \mathbb{Z}_+$ and write the variables in $\mathbb{C}^m \times \mathbb{R}^{n'}$ as

$$(z, t) = (z_1, \dots, z_m, t_1, \dots, t_{n'}).$$

We shall also write $z_j = x_j + iy_j$, $j = 1, \dots, m$, and $n = m + n'$.

The elliptic complex on $\mathbb{C}^m \times \mathbb{R}^{n'}$ is defined as follows: given $\Omega \subset \mathbb{C}^m \times \mathbb{R}^{n'}$ open and $0 \le q \le n$ we set $C^\infty(\Omega, \underline{\Lambda}^q)$ as being the space of all smooth differential forms of the kind

$$f = \sum_{|J|+|K|=q} f_{JK}\, \mathrm{d}\bar{z}_J \wedge \mathrm{d}t_K, \qquad f_{JK} \in C^\infty(\Omega). \tag{VIII.10}$$

We define the differential operator

$$\mathrm{D}_q : C^\infty(\Omega, \underline{\Lambda}^q) \longrightarrow C^\infty(\Omega, \underline{\Lambda}^{q+1}) \tag{VIII.11}$$

by the formula

$$\mathrm{D}_0 u = \sum_{j=1}^{m} \frac{\partial u}{\partial \bar{z}_j}\, \mathrm{d}\bar{z}_j + \sum_{k=1}^{n'} \frac{\partial u}{\partial t_k}\, \mathrm{d}t_k \tag{VIII.12}$$

if $u \in C^\infty(\Omega) = C^\infty(\Omega, \underline{\Lambda}^0)$, and by

$$\mathrm{D}_q f = \sum_{|J|+|K|=q} \mathrm{D}_0 f_{JK} \wedge \mathrm{d}\bar{z}_J \wedge \mathrm{d}t_K \tag{VIII.13}$$

if f is as in (VIII.10). In particular, when $m = 0$, we have $\mathrm{D}_q = d_q$, the exterior derivative acting on q-forms.

It is clear that $\mathrm{D}_{q+1} \circ \mathrm{D}_q = 0$ and consequently (VIII.11) defines a complex D of differential operators, whose cohomology will be denoted by $\{H^q_{\mathrm{D}}(\Omega) : q = 0, \dots, n\}$. In particular, $H^0_{\mathrm{D}}(\Omega)$ is the space of all solutions $u \in C^\infty(\Omega)$ of the system $\mathrm{D}_0(u) = 0$, that is, the space of all smooth functions on Ω that are holomorphic in z and locally constant in t. Furthermore, when $m = 0$, there are isomorphisms between $H^q_{\mathrm{D}}(\Omega)$ and $H^q(\Omega, \mathbb{C})$, the cohomology groups of Ω with complex coefficients (de Rham's theorem).

THEOREM VIII.3.1. *Let $U \subset \mathbb{C}^m$ be open and pseudo-convex and let $\Theta \subset \mathbb{R}^{n'}$ be open and convex. Then* D *is solvable in $U \times \Theta$ in degree q, for every $q \ge 1$.*

PROOF. For the proof it is convenient to introduce the natural decomposition

$$C^\infty(U \times \Theta, \underline{\Lambda}^q) = \bigoplus_{r+s=q} C^\infty(U \times \Theta, \underline{\Lambda}^{r,s}),$$

where $C^\infty(U \times \Theta, \underline{\Lambda}^{r,s})$ is the space of forms of the kind

$$f = \sum_{|J|=r,\, |K|=s} f_{JK}\, \mathrm{d}\bar{z}_J \wedge \mathrm{d}t_K.$$

Notice that $C^\infty(U\times\Theta, \underline{\Lambda}^{r,s}) = 0$ if either $r > m$ or $s > n'$. We also observe that we have homomorphisms

$$\overline{\partial}_z : C^\infty(U\times\Theta, \underline{\Lambda}^{r,s}) \longrightarrow C^\infty(U\times\Theta, \underline{\Lambda}^{r+1,s}),$$

$$\mathrm{d}_t : C^\infty(U\times\Theta, \underline{\Lambda}^{r,s}) \longrightarrow C^\infty(U\times\Theta, \underline{\Lambda}^{r,s+1})$$

such that $\mathrm{D} = \overline{\partial}_z + \mathrm{d}_t$.

Let $f \in C^\infty(U\times\Theta, \underline{\Lambda}^{q})$ satisfy $\mathrm{D}_q f = 0$ and decompose $f = \sum_{r,s} f_{r,s}$, where $f_{r,s} \in C^\infty(U\times\Theta, \underline{\Lambda}^{r,s})$ and the sum runs over the pairs (r, s) such that $r+s = q$, $r \le m$, $s \le n'$. Consider, in this decomposition, the term $f_{r,s}$ whose value of s is maximum. From the fact that $\mathrm{D}f = 0$ it follows that $\mathrm{d}_t f_{r,s} = 0$ and consequently we can apply the Poincaré Lemma (Section VII.3) in order to find $h \in C^\infty(U\times\Theta, \underline{\Lambda}^{r,s-1})$ such that $\mathrm{d}_t h = f_{r,s}$. If we set $f^\bullet = f - \mathrm{D}_{q-1} h$ it follows that in the analogous decomposition $f^\bullet = \sum_{r,s} f^\bullet_{r,s}$ the maximum value of s that occurs has dropped by one and $\mathrm{D}_q f^\bullet = 0$.

If we iterate the argument we will, after a finite number of steps, either solve the equation $\mathrm{D}_{q-1} u = f$ or at least find $v \in C^\infty(U\times\Theta, \underline{\Lambda}^{q-1})$ such that $g \doteq f - \mathrm{D}_{q-1} v$ does not involve $\mathrm{d}t_1, \ldots, \mathrm{d}t_{n'}$. If this is the case we can write

$$g = \sum_{|J|=q} g_J(z, t)\mathrm{d}\overline{z}_J$$

and the fact that $\mathrm{D}_q g = 0$ gives in particular that $\mathrm{d}_t g_J = 0$ for all J, that is, g_J are independent of t. Hence g defines a Dolbeault class in U and by the standard complex analysis theory we can determine $w = \sum_{|L|=q-1} w_L(z)\mathrm{d}\overline{z}_L$ solving $\overline{\partial}_z w = g$. If we set $u \doteq v + w$ we obtain $\mathrm{D}_{q-1} u = f$, which completes the proof. □

Likewise we can introduce the spaces $\mathcal{D}'(\Omega, \underline{\Lambda}^q)$, which are the spaces of all currents of the form (VIII.11) where now the coefficients are allowed to be elements of $\mathcal{D}'(\Omega)$. By the same expressions (VIII.12) and (VIII.13) we obtain new differential complexes

$$\mathrm{D}_q : \mathcal{D}'(\Omega, \underline{\Lambda}^q) \longrightarrow \mathcal{D}'(\Omega, \underline{\Lambda}^{q+1}) \tag{VIII.14}$$

whose cohomology will be denoted by $\{H^q_{\mathrm{D}}(\Omega, \mathcal{D}') : q = 0, \ldots, n\}$.

The natural injections $C^\infty(\Omega, \underline{\Lambda}^q) \hookrightarrow \mathcal{D}'(\Omega, \underline{\Lambda}^q)$ commute with the operator D and then induce homomorphisms

$$H^q_{\mathrm{D}}(\Omega) \longrightarrow H^q_{\mathrm{D}}(\Omega, \mathcal{D}'). \tag{VIII.15}$$

Finally we shall also consider the spaces

$$C^\infty_c(\Omega, \underline{\Lambda}^q) = \{f \in C^\infty(\Omega, \underline{\Lambda}^q) : \operatorname{supp} f \subset\subset \Omega\}; \tag{VIII.16}$$

$$\mathcal{E}'(\Omega, \underline{\Lambda}^q) = \{f \in \mathcal{D}'(\Omega, \underline{\Lambda}^q) : \operatorname{supp} f \subset\subset \Omega\}. \tag{VIII.17}$$

The natural pairing

$$C^\infty(\Omega, \underline{\Lambda}^q) \times C_c^\infty(\Omega, \underline{\Lambda}^{n-q}) \longrightarrow \mathbb{C}$$

defined by

$$(f, \psi) \longrightarrow \int f \wedge \mathrm{d}z \wedge \psi,$$

where $\mathrm{d}z = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m$, extends to a bilinear form

$$\mathcal{D}'(\Omega, \underline{\Lambda}^q) \times C_c^\infty(\Omega, \underline{\Lambda}^{n-q}) \longrightarrow \mathbb{C}$$

which allows us to identify $\mathcal{D}'(\Omega, \underline{\Lambda}^q)$ with the topological dual of $C_c^\infty(\Omega, \underline{\Lambda}^{n-q})$, when the latter carries its natural structure of an inductive limit of Fréchet spaces. We shall use the standard notation of the de Rham theory: if $T \in \mathcal{D}'(\Omega, \underline{\Lambda}^q)$ and $\psi \in C_c^\infty(\Omega, \underline{\Lambda}^{n-q})$ we shall set

$$T[\psi] = (T \wedge \psi)[1] = \int T \wedge \mathrm{d}z \wedge \psi.$$

Likewise we have a natural identification between $\mathcal{E}'(\Omega, \underline{\Lambda}^q)$ and the topological dual of $C^\infty(\Omega, \underline{\Lambda}^{n-q})$, where now the latter carries its natural topology of a Fréchet space.

We shall always consider the weak topology in the spaces $\mathcal{D}'(\Omega, \underline{\Lambda}^q)$ and $\mathcal{E}'(\Omega, \underline{\Lambda}^q)$.

Lemma VIII.3.2. *If $T \in \mathcal{D}'(\Omega, \underline{\Lambda}^q)$, $\psi \in C^\infty(\Omega, \underline{\Lambda}^{n-q-1})$ and one of them has compact support then*

$$\int \mathrm{D}_q T \wedge \mathrm{d}z \wedge \psi = (-1)^{q+m-1} \int T \wedge \mathrm{d}z \wedge \mathrm{D}_{n-q-1} \psi.$$

Proof. Using the fact that $C^\infty(\Omega, \underline{\Lambda}^q) \subset \mathcal{D}'(\Omega, \underline{\Lambda}^q)$ as well as $C_c^\infty(\Omega, \underline{\Lambda}^q) \subset \mathcal{E}'(\Omega, \underline{\Lambda}^q)$ are dense inclusions we can assume that $T = f$ is smooth. We have

$$\begin{aligned}
\mathrm{d}_{2m+n'-1}(f \wedge \mathrm{d}z \wedge \psi) &= \mathrm{d}_q f \wedge \mathrm{d}z \wedge \psi + (-1)^q f \wedge \mathrm{d}_{m+n-q-1}(\mathrm{d}z \wedge \psi) \\
&= \mathrm{D}_q f \wedge \mathrm{d}z \wedge \psi + (-1)^{q+m} f \wedge \mathrm{d}z \wedge \mathrm{d}_{n-q-1} \psi \\
&= \mathrm{D}_q f \wedge \mathrm{d}z \wedge \psi + (-1)^{q+m} f \wedge \mathrm{d}z \wedge \mathrm{D}_{n-q-1} \psi.
\end{aligned}$$

Since

$$\int \mathrm{d}_{2m+n'-1}(f \wedge \mathrm{d}z \wedge \psi) = 0$$

we obtain the desired conclusion. □

VIII.4 The Box operator associated with D

If $f, g \in C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^q)$ and one of them has compact support we set

$$(f, g)_q \doteq \sum_{|J|+|K|=q} \int_{\mathbb{C}^m \times \mathbb{R}^{n'}} f_{JK}\, \overline{g_{JK}}\, \mathrm{d}x\mathrm{d}y\mathrm{d}t. \tag{VIII.18}$$

The *formal adjoint* of the operator (VIII.13) is the differential operator

$$\mathrm{D}_q^* : C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^{q+1}) \longrightarrow C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^q) \tag{VIII.19}$$

defined by the expression

$$\left(\mathrm{D}_q f, u\right)_{q+1} = \left(f, \mathrm{D}_q^* u\right)_q, \tag{VIII.20}$$

where $u \in C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^{q+1})$ and $f \in C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^q)$, the latter with compact support.
We then set $\mathrm{D}_{-1} = \mathrm{D}_{n+1} = 0$ and define

$$\Box_q = \mathrm{D}_{q-1}\mathrm{D}_{q-1}^* + \mathrm{D}_q^*\mathrm{D}_q. \tag{VIII.21}$$

Notice that $\Box_q$ is a second-order differential operator acting on the space $C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^q)$. Actually an elementary but long computation shows that

$$\Box_q f = \sum_{|J|+|K|=q} (\mathrm{P} f_{JK})\, \mathrm{d}\overline{z}_J \wedge \mathrm{d}t_K, \tag{VIII.22}$$

where f is as in (VIII.10) and

$$\mathrm{P} = -\sum_{j=1}^{m} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} - \sum_{k=1}^{n'} \frac{\partial^2}{\partial t_k^2}. \tag{VIII.23}$$

The following crucial properties of the operators $\Box_q$, $q = 0, 1, \ldots, n$, will be used in the sequel:

$$\mathrm{D}_q \Box_q = \Box_{q+1}\mathrm{D}_q = \mathrm{D}_q\mathrm{D}_q^*\mathrm{D}_q; \tag{VIII.24}$$

$$\mathrm{D}_q^* \Box_{q+1} = \Box_q \mathrm{D}_q^* = \mathrm{D}_q^*\mathrm{D}_q\mathrm{D}_q^*; \tag{VIII.25}$$

$$\Box_0 \text{ is hypoelliptic in } \mathbb{C}^m \times \mathbb{R}^{n'}. \tag{VIII.26}$$

Moreover, since any open subset of $\mathbb{C}^m \times \mathbb{R}^{n'}$ is P-convex for singular supports ([**H3**]), we also have

$$\text{Given any open set } \Omega \subset \mathbb{C}^m \times \mathbb{R}^{n'} \text{ the maps} \tag{VIII.27}$$

$$\Box_q : \mathcal{D}'(\Omega, \underline{\Lambda}^q) \longrightarrow \mathcal{D}'(\Omega, \underline{\Lambda}^q) \text{ are surjective.}$$

PROPOSITION VIII.4.1. *For any open set $\Omega \subset \mathbb{C}^m \times \mathbb{R}^{n'}$ the maps (VIII.15) are isomorphims. More precisely:*

(i) *If $u \in \mathcal{D}'(\Omega)$ satisfies $\mathrm{D}_0 u = 0$ then $u \in C^\infty(\Omega)$.*
(ii) *If $q \geq 1$ and if $u \in \mathcal{D}'(\Omega, \underline{\Lambda}^{q-1})$ is such that $\mathrm{D}_{q-1}u \in C^\infty(\Omega, \underline{\Lambda}^q)$ then there is $v \in C^\infty(\Omega, \underline{\Lambda}^{q-1})$ such that $\mathrm{D}_{q-1}u = \mathrm{D}_{q-1}v$.*
(iii) *If $q \geq 1$ and if $f \in \mathcal{D}'(\Omega, \underline{\Lambda}^q)$ satisfies $\mathrm{D}_q f = 0$ then there are $g \in C^\infty(\Omega, \underline{\Lambda}^q)$ and $u \in \mathcal{D}'(\Omega, \underline{\Lambda}^{q-1})$ such that $f - g = \mathrm{D}_{q-1}u$.*

PROOF. (i) is a consequence of (VIII.26), since $\square_0 = \mathrm{D}_0^*\mathrm{D}_0$. Next take u as in (ii) and apply (VIII.27). We can solve

$$\square_{q-1} w = u$$

for some $w \in \mathcal{D}'(\Omega, \underline{\Lambda}^{q-1})$. Then, by (VIII.24),

$$\mathrm{D}_{q-1}u = \mathrm{D}_{q-1}\square_{q-1}w = \square_q \mathrm{D}_{q-1}w.$$

If we apply (VIII.26) we conclude that $\mathrm{D}_{q-1}w \in C^\infty(\Omega, \underline{\Lambda}^q)$ and consequently $v \doteq \mathrm{D}_{q-1}^*\mathrm{D}_{q-1}w \in C^\infty(\Omega, \underline{\Lambda}^{q-1})$. Since using (VIII.24) we also have

$$\mathrm{D}_{q-1}u = \mathrm{D}_{q-1}\square_{q-1}w = \mathrm{D}_{q-1}\mathrm{D}_{q-1}^*\mathrm{D}_{q-1}w = \mathrm{D}_{q-1}v$$

(ii) is proved.

Finally let f be as in (iii) and solve

$$\square_q U = \mathrm{D}_{q-1}\mathrm{D}_{q-1}^* U + \mathrm{D}_q^*\mathrm{D}_q U = f,$$

for some $U \in \mathcal{D}'(\Omega, \underline{\Lambda}^{q-1})$. We set $u \doteq \mathrm{D}_{q-1}^* U$ and $g \doteq \mathrm{D}_q^*\mathrm{D}_q U$. In order to conclude the proof it remains to show that g is smooth. But (VIII.24) and (VIII.25) imply

$$\square_q g = \square_q \mathrm{D}_q^*\mathrm{D}_q U = \mathrm{D}_q^*\square_{q+1}\mathrm{D}_q U = \mathrm{D}_q^*\mathrm{D}_q\square_q U = \mathrm{D}_q^*\mathrm{D}_q f = 0.$$

By (VIII.26) g is smooth and we are done. □

REMARK VIII.4.2. The preceding argument gives indeed the proof of a stronger statement than (iii): every cohomology class in $H^q_{\mathrm{D}}(\Omega, \mathcal{D}')$ contains a representative which is in the kernel of $\square_q$ (and consequently it is real-analytic).

By a similar argument we have:

PROPOSITION VIII.4.3. *If Ω is any open set on $\mathbb{C}^m \times \mathbb{R}^{n'}$ then*

$$\mathrm{H}^n_{\mathrm{D}}(\Omega) = 0.$$

PROOF. Given $f \in C^\infty(\Omega, \underline{\Lambda}^n)$ we apply (VIII.26) and (VIII.27) in order to find $v \in C^\infty(\Omega, \underline{\Lambda}^n)$ solving

$$\Box_n v = f \tag{VIII.28}$$

in Ω. Since moreover $\Box_n = \mathrm{D}_{n-1}\mathrm{D}^*_{n-1}$ we then have $\mathrm{D}_{n-1}u = f$, where $u = \mathrm{D}^*_{n-1}v \in C^\infty(\Omega, \underline{\Lambda}^{n-1})$, thanks to (VIII.28). $\Box$

Consider the function $E \in L^1_{\mathrm{loc}}(\mathbb{C}^m \times \mathbb{R}^{n'})$ defined by

$$E(z,t) = \begin{cases} \omega^{-1}_{m,n}\left\{|z|^2 + |t/2|^2\right\}^{-m-n'/2+1} & \text{if } m \geq 1, \\ -t/(2|t|) & \text{if } m = 0,\ n' = 1, \\ -(\log|t|)/2\pi & \text{if } m = 0,\ n' = 2, \\ \omega^{-1}_{m,n}\left\{|z|^2 + |t/2|^2\right\}^{-m-n'/2+1} & \text{if } m = 0 \text{ and } n' \geq 3, \end{cases}$$

where $\omega_{m,n} = 2^{n'-2}(2m+n'-2)|S^{2m+n'-1}|$. It is a well-known fact that E is a fundamental solution of P. If we then set, for $U \in \mathcal{E}'(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^q)$,

$$E \star U \doteq \sum_{|J|+|K|=q} (E \star U_{JK})\, \mathrm{d}\bar{z}_J \wedge \mathrm{d}t_K\,, \tag{VIII.29}$$

we obtain

$$\Box_q(E \star U) = \Box_q E \star U = U, \tag{VIII.30}$$

$$\mathrm{D}_{q-1}\left[\mathrm{D}^*_{q-1}(E \star U)\right] + \mathrm{D}^*_q\left[E \star \mathrm{D}_q U\right] = U. \tag{VIII.31}$$

We push the argument further. Let Ω be a regular, bounded open subset of $\mathbb{C}^m \times \mathbb{R}^{n'}$. If $f \in C^\infty(\overline{\Omega}, \underline{\Lambda}^q)$ we consider $f\mathcal{X}_\Omega \in \mathcal{E}'(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^q)$, where $\mathcal{X}_\Omega$ denotes the characteristic function of Ω. We obtain, from (VIII.31),

$$f\mathcal{X}_\Omega = \mathrm{D}_{q-1}\left[\mathrm{D}^*_{q-1}(E \star f\mathcal{X}_\Omega)\right] + \mathrm{D}^*_q\left[E \star (\mathrm{D}_q f)\mathcal{X}_\Omega\right] + \mathrm{D}^*_q\left[E \star (\mathrm{D}_0\mathcal{X}_\Omega \wedge f)\right]. \tag{VIII.32}$$

If we now introduce the operators

$$\mathfrak{G}_q : C^\infty(\overline{\Omega}, \underline{\Lambda}^q) \longrightarrow C^\infty(\Omega, \underline{\Lambda}^{q-1}), \quad \mathfrak{H}_q : C^\infty(\overline{\Omega}, \underline{\Lambda}^q) \longrightarrow C^\infty(\Omega, \underline{\Lambda}^q)$$

defined by the expressions

$$\mathfrak{G}_q(f) = \mathrm{D}^*_{q-1}(E \star f\mathcal{X}_\Omega)|_\Omega\,, \quad \mathfrak{H}_q(f) = \mathrm{D}^*_q[E \star (\mathrm{D}_0\mathcal{X}_\Omega \wedge f)]|_\Omega, \tag{VIII.33}$$

formula (VIII.32) then gives a natural extension of the so-called *Bochner–Martinelli formula*:

THEOREM VIII.4.4. *If Ω is a regular, bounded open subset of $\mathbb{C}^m \times \mathbb{R}^{n'}$ with a smooth boundary and if $f \in C^\infty(\overline{\Omega}, \underline{\Lambda}^q)$ then*

$$\mathrm{D}_{q-1}\mathfrak{G}_q(f) + \mathfrak{G}_{q+1}(\mathrm{D}_q f) + \mathfrak{H}_q(f) = f. \tag{VIII.34}$$

$\Box$

Observe that E is real-analytic in the complement of the origin and that $\operatorname{supp} \mathrm{D}_0 \mathcal{X}_\Omega \subset \partial\Omega$ and so there exists a neighborhood $\Omega^\bullet$ of Ω in the complexification of $\mathbb{C}^m \times \mathbb{R}^{n'}$ such that the following is true: for every $f \in \mathcal{C}^\infty(\overline{\Omega}, \underline{\Lambda}_q)$ the coefficients of $\mathfrak{H}_q(f)$ extend as holomorphic functions to $\Omega^\bullet$. This fact, in conjunction with Proposition VIII.2.1, provides another proof for the local solvability of the complex D.

VIII.5 The intersection number

We fix a pair (Ω, Ω') of open subsets of $\mathbb{C}^m \times \mathbb{R}^{n'}$, with $\Omega' \subset \Omega$, and an integer $q \geq 1$. The intersection number for the pair (Ω, Ω') in degree q is the $\mathbb{C}$-bilinear form defined on the product

$$\{f \in C^\infty(\Omega, \underline{\Lambda}^q) : \mathrm{D}_q f = 0\} \times \{\Theta \in C_c^\infty(\Omega', \underline{\Lambda}^{n-q}) : \mathrm{D}_{n-q}\Theta = 0\}$$

defined by

$$\mathfrak{I}^q_{(\Omega,\Omega')}[f, \Theta] = \int f \wedge \mathrm{d}z \wedge \Theta.$$

The intersection number for the pair (Ω, Ω') in degree 0 is the $\mathbb{C}$-bilinear form defined on the product

$$\{f \in C^\infty(\Omega) : \mathrm{D}_0 f = 0\} \times \{\Theta \in C_c^\infty(\Omega', \underline{\Lambda}^n) : \int F \mathrm{d}z \wedge \Theta = 0, \\ \forall F \in C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}),\ \mathrm{D}_0 F = 0\}$$

defined by

$$\mathfrak{I}^0_{(\Omega,\Omega')}[f, \Theta] = \int f \mathrm{d}z \wedge \Theta.$$

We have the following result:

Proposition VIII.5.1. *Let $q \geq 1$. The intersection number $\mathfrak{I}^q_{(\Omega,\Omega')}$ vanishes identically if and only if for every $f \in C^\infty(\Omega, \underline{\Lambda}^q)$ satisfying $\mathrm{D}_q f = 0$ its restriction to Ω' belongs to the closure of the image of the map*

$$\mathrm{D}_{q-1} : C^\infty(\Omega', \underline{\Lambda}^{q-1}) \longrightarrow C^\infty(\Omega', \underline{\Lambda}^q). \tag{VIII.35}$$

Proof. Let $f \in C^\infty(\Omega, \underline{\Lambda}^q)$ satisfy $\mathrm{D}_q f = 0$. If

$$f|_{\Omega'} = \lim_{\nu \to \infty} \mathrm{D}_{q-1} u_\nu \quad \text{in } C^\infty(\Omega', \underline{\Lambda}^q)$$

for some sequence (u_ν) in $C^\infty(\Omega', \underline{\Lambda}^{q-1})$, and if $\Theta \in C_c^\infty(\Omega', \underline{\Lambda}^{n-q})$ satisfies $\mathrm{D}_{n-q}\Theta = 0$, we have

$$\mathfrak{I}^q_{(\Omega,\Omega')}[f, \Theta] = \lim_{\nu\to\infty} \int \mathrm{D}_{q-1}u_\nu \wedge \mathrm{d}z \wedge \Theta = \lim_{\nu\to\infty} (-1)^{q+m} \int u_\nu \wedge \mathrm{d}z \wedge \mathrm{D}_{n-q}\Theta = 0$$

thanks to Lemma VIII.3.2.

For the converse we reason by contradiction and apply the Hahn–Banach theorem. Thus we assume that there are $f_0 \in C^\infty(\Omega, \underline{\Lambda}^q)$ satisfying $\mathrm{D}_q f_0 = 0$ and $T \in \mathcal{E}'(\Omega', \underline{\Lambda}^{n-q})$ such that

$$T[f_0] = 1, \qquad T[\mathrm{D}_{q-1}u] = 0, \quad \forall u \in C^\infty(\Omega', \underline{\Lambda}^{q-1}).$$

In particular we have $\mathrm{D}_{n-q}T = 0$.

Let now $\rho \in C_c^\infty(\mathbb{C}^m \times \mathbb{R}^{n'})$ be such that $\int \rho = 1$ and set, for $\epsilon > 0$,

$$\rho_\epsilon(z, t) = \frac{1}{\epsilon^{2m+n'}} \rho\left(\frac{z}{\epsilon}, \frac{t}{\epsilon}\right).$$

If we introduce the regularizations

$$\Theta_\epsilon = \rho_\epsilon \star T \in C_c^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^{n-q})$$

then there is $\epsilon_0 > 0$ such that $\Theta_\epsilon \in C_c^\infty(\Omega', \underline{\Lambda}^{n-q})$ if $0 < \epsilon \le \epsilon_0$. Moreover, $\Theta_\epsilon \to T$ in $\mathcal{E}'(\Omega', \underline{\Lambda}^{n-q})$ and $\mathrm{D}_{n-q}\Theta_\epsilon = \rho_\epsilon \star \mathrm{D}_{n-q}T = 0$ for every $\epsilon > 0$. Now

$$\int f_0 \wedge \mathrm{d}z \wedge \Theta_\epsilon = \Theta_\epsilon[f_0] \longrightarrow T[f_0] = 1$$

and consequently there is $0 < \epsilon_1 \le \epsilon_0$ such that

$$\mathfrak{I}^q_{(\Omega,\Omega')}\left[f_0, \Theta_{\epsilon_1}\right] = \int f \wedge \mathrm{d}z \wedge \Theta_{\epsilon_1} \ne 0. \qquad \square$$

Next we turn to the case $q = 0$:

PROPOSITION VIII.5.2. *The intersection number $\mathfrak{I}^0_{(\Omega,\Omega')}$ vanishes identically if and only if the following holds:*

$$\text{For every } f \in C^\infty(\Omega) \text{ satisfying } \mathrm{D}_0 f = 0 \text{ there is} \qquad \text{(VIII.36)}$$

$$\{F_\nu\} \subset C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}) \text{ satisfying } \mathrm{D}_0 F_\nu = 0 \text{ such that}$$

$$F_\nu|_{\Omega'} \longrightarrow f|_{\Omega'} \text{ in } C^\infty(\Omega').$$

PROOF. The proof that (VIII.36) implies the vanishing of $\mathfrak{I}^0_{(\Omega,\Omega')}$ is immediate. We then prove the converse and for this we argue by contradiction as in the proof of Proposition VIII.5.1. Thus we assume that there is $f_0 \in C^\infty(\Omega)$ satisfying $\mathrm{D}_0 f = 0$ for which no sequence $\{F_\nu\} \subset C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'})$ as stated exists and apply once more the Hahn–Banach theorem: there is $T \in \mathcal{E}'(\omega', \underline{\Lambda}^n)$ such that

$$T[f_0] = 1, \tag{VIII.37}$$

$$T[F] = 0, \quad \forall F \in C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}), \ \mathrm{D}_0 F = 0. \tag{VIII.38}$$

We next observe that the vanishing of $H^1_{\mathrm{D}}(\mathbb{C}^m \times \mathbb{R}^{n'})$ (Theorem VIII.3.1) implies, in particular, that the homomorphism of Fréchet spaces

$$\mathrm{D}_0 : C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}) \longrightarrow C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^1)$$

has closed image. Consequently its transpose, which is the map

$$\mathrm{D}_n : \mathcal{E}'(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^{n-1}) \longrightarrow \mathcal{E}'(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^n),$$

has a weakly closed image, that is, its image is precisely the orthogonal of the kernel of $\mathrm{D}_0 : C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}) \to C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^1)$ in $\mathcal{E}'(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^n)$.

Returning to our argument we conclude from (VIII.38) that there exists $S \in \mathcal{E}'(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^{n-1})$ such that $\mathrm{D}_{n-1} S = T$.

As in the proof of Proposition VIII.5.1 we introduce once more the regularizations

$$\Theta_\epsilon = \rho_\epsilon \star T \in C^\infty_c(\mathbb{C}^m \times \mathbb{R}^{n'}, \underline{\Lambda}^n).$$

There is $\epsilon_0 > 0$ such that $\Theta_\epsilon \in C^\infty_c(\Omega', \underline{\Lambda}^n)$ if $0 < \epsilon \leq \epsilon_0$. Furthermore, if $F \in C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'})$ satisfies $\mathrm{D}_0 F = 0$ then

$$\int F \mathrm{d}z \wedge \Theta_\epsilon = \int F \mathrm{d}z \wedge (\rho_\epsilon \star \mathrm{D}_{n-1} S) = \int F \mathrm{d}z \wedge \mathrm{D}_{n-1}(\rho_\epsilon \star S)$$
$$= (-1)^{m-1} \int \mathrm{D}_0 F \wedge \mathrm{d}z \wedge (\rho_\epsilon \star S) = 0$$

for $0 < \epsilon \leq \epsilon_0$ and also

$$\mathfrak{I}^0_{(\Omega,\Omega')}[f_0, \Theta_\epsilon] = \int f_0 \wedge \mathrm{d}z \wedge \Theta_\epsilon \xrightarrow{\epsilon \to 0} 1,$$

thanks to (VIII.37), which leads to the desired contradiction. □

REMARK VIII.5.3. It follows from the argument in the proof of Proposition VIII.5.2 that the space $\mathbb{C}^m \times \mathbb{R}^{n'}$ can be replaced in (VIII.36) by any open set containing Ω and of the form $U \times \Theta$, where U and Θ are as in Theorem VIII.3.1.

COROLLARY VIII.5.4. *Assume that $m = 0$ and let $\Omega' \subset \Omega$ be open subsets of $\mathbb{R}^{n'}$. Then, if $q \geq 1$, the vanishing of $\mathfrak{I}^q_{(\Omega,\Omega')}$ is equivalent to the vanishing of the natural map induced by restriction $H^q(\Omega, \mathbb{C}) \to H^q(\Omega', \mathbb{C})$. Also, the* *vanishing of $\mathfrak{I}^0_{(\Omega,\Omega')}$ is equivalent to the property that Ω' is contained in a single connected component of Ω.*

PROOF. Thanks to de Rham's theorem we can assert:

(a) The exterior derivative defines a map with closed image when defined on an arbitary smooth manifold.
(b) The d-cohomology is isomorphic to the standard singular cohomology with complex coefficients for any smooth manifold.

These two properties in conjunction with Proposition VIII.5.1 prove the assertion for $q \geq 1$. Furthermore, since a scalar function is d-closed if and only if it is locally constant, the assertion for $q = 0$ is an immediate consequence of Proposition VIII.5.2. □

We shall now draw an important corollary of Propositions VIII.5.1 and VIII.5.2. Let Ω be a regular, bounded open subset of $\mathbb{C}^m \times \mathbb{R}^{n'}$. Since we are dealing with an elliptic structure on $\mathbb{C}^m \times \mathbb{R}^{n'}$ it follows that $\partial\Omega$ is noncharacteristic and consequently we can apply the one-sided approximate Poincaré Lemma (Theorem VII.8.4) and obtain:

COROLLARY VIII.5.5. *Let $p \in \partial\Omega$. Given any neighborhood W of p in $\mathbb{C}^m \times \mathbb{R}^{n'}$ there is another such neighborhood $W' \subset\subset W$ such that $\mathfrak{I}^q_{(W\cap\Omega,W'\cap\Omega)} = 0$ for all $q = 0, \ldots, m + n'$.* □

VIII.6 The intersection number under certain geometrical assumptions

In this section we shall give a special meaning to one of the complex variables. Thus we shall assume $m \geq 1$ and write the complex variables as $(z_1, \ldots, z_\nu, w)$, where now $m = \nu + 1$. If Ω is an open subset of $\mathbb{C}^m \times \mathbb{R}^{n'}$ we shall denote by $\mathcal{R}(\Omega)$ the space of all $u \in C^\infty(\Omega)$ which satisfy $\partial u/\partial \overline{w} = 0$.

If Ω is an open subset of $\mathbb{C}^m \times \mathbb{R}^{n'}$ and if $w_0 \in \mathbb{C}$ we shall write

$$\Omega(w_0) = \{(z, w, t) \in \Omega : w = w_0\}.$$

In the sequel, when dealing with functions defined on $\Omega(w_0)$, we shall identify the latter with the open subset of $\mathbb{C}^\nu \times \mathbb{R}^{n'}$ given by $\{(z, t) \in \mathbb{C}^\nu \times \mathbb{R}^{n'} : (z, w_0, t) \in \Omega(w_0)\}$. We start with an important result:

PROPOSITION VIII.6.1. *Let $\Omega \subset \mathbb{C}^m \times \mathbb{R}^{n'}$ be open and $\partial/\partial\overline{w}$-convex, that is:*

$$\text{The homomorphism } \partial/\partial\overline{w} : C^\infty(\Omega) \to C^\infty(\Omega) \text{ is surjective.} \quad \text{(VIII.39)}$$

Then given $w_0 \in \mathbb{C}$ the restriction map $\mathcal{R}(\Omega) \to C^\infty(\Omega(w_0))$ is surjective.

PROOF. There is a continuous function $\delta : \Omega(w_0) \to]0, \infty[$ such that the open set

$$\mathcal{U}_\delta = \{(z, w, t) : (z, w_0, t) \in \Omega(w_0),\ |w - w_0| < \delta(z, t)\}$$

is contained in Ω. Let $f = f(z, t) \in C^\infty(\Omega(w_0))$ and select $f^\bullet \in C^\infty(\Omega)$ such that $f^\bullet(z, w, t) = f(z, t)$ if $(z, w, t) \in \mathcal{U}_{\delta/2}$. In particular, $f^\bullet|_{\Omega(w_0)} = f$ and

$$\frac{\partial f^\bullet}{\partial \overline{w}} = 0 \qquad \text{in} \quad \mathcal{U}_{\delta/2}. \tag{VIII.40}$$

We must find $g \in C^\infty(\Omega)$ such that

$$F(z, w, t) = f^\bullet(z, w, t) + (w - w_0) g(z, w, t)$$

belongs to $\mathcal{R}(\Omega)$. For this we must have

$$\frac{\partial f^\bullet}{\partial \overline{w}} + (w - w_0)\frac{\partial g^\bullet}{\partial \overline{w}} = 0,$$

which is possible to achieve, since by hypothesis and by (VIII.40) we can solve the equation

$$\frac{\partial g^\bullet}{\partial \overline{w}} = -(w - w_0)^{-1}\frac{\partial f^\bullet}{\partial \overline{w}}$$

in order to determine the desired g. □

Denote by $C^\infty(\Omega, \underline{\underline{\Lambda}}^q)$, $q = 0, \ldots, n-1$, the space of all forms in $C^\infty(\Omega, \underline{\Lambda}^q)$ which do not involve $\mathrm{d}\overline{w}$, that is, the space of all forms of the kind

$$f = \sum_{|J|+|K|=q} f_{JK}\, \mathrm{d}\overline{z}_J \wedge \mathrm{d}t_K\,, \tag{VIII.41}$$

with $f_{JK} \in C^\infty(\Omega)$. It is important to observe that if f is as in (VIII.41) and satisfies $\mathrm{D}_q f = 0$ then *a fortiori* we have $f_{JK} \in \mathcal{R}(\Omega)$ for every J and K. Notice also that the pullback of an element in $C^\infty(\Omega, \underline{\underline{\Lambda}}^q)$ to any slice $\Omega(w_0)$ is simply obtained by setting $w = w_0$ in its coefficients.

PROPOSITION VIII.6.2. *Let $\Omega' \subset \Omega$ be open subsets of $\mathbb{C}^m \times \mathbb{R}^{n'}$, both satisfying (VIII.39). If for some $q \geq 1$ the homomorphism $H^q_{\mathrm{D}}(\Omega) \to H^q_{\mathrm{D}}(\Omega')$ is trivial then for every $w_0 \in \mathbb{C}$ and every $f \in C^\infty(\Omega(w_0), \underline{\Lambda}^{q-1})$ satisfying $\mathrm{D}_{q-1} f = 0$ there is $F \in C^\infty(\Omega', \underline{\underline{\Lambda}}^{q-1})$ satisfying $\mathrm{D}_{q-1} F = 0$ and $F(z, w_0, t) = f(z, t)$ on $\Omega'(w_0)$.*

PROOF. Let $f = f(z,t) \in C^\infty(\Omega(w_0), \underline{\Lambda}^{q-1})$ satisfy $\mathrm{D}_{q-1} f = 0$. We apply Proposition VIII.6.1 in order to get $f^\bullet(z,w,t) \in C^\infty(\Omega, \underline{\underline{\Lambda}}^{q-1})$, with coefficients in $\mathcal{R}(\Omega)$, such that $f^\bullet(z, w_0, t) = f(z,t)$. We have

$$(\mathrm{D}_{q-1} f^\bullet)(z, w_0, t) = (\mathrm{D}_{q-1} f)(z,t) = 0$$

and consequently we can write $\mathrm{D}_{q-1} f^\bullet = (w - w_0) G$ for some $G \in C^\infty(\Omega, \underline{\underline{\Lambda}}^q)$, also with coefficients in $\mathcal{R}(\Omega)$. It is clear that $\mathrm{D}_q G = 0$ and thus by hypothesis there is $u \in C^\infty(\Omega', \underline{\Lambda}^{q-1})$ satisfying $\mathrm{D}_{q-1} u = G$ in Ω'. Write

$$u = u_0 + u_1 \wedge \mathrm{d}\overline{w},$$

with $u_j \in C^\infty(\Omega', \underline{\underline{\Lambda}}^{q-j-1})$, $j = 0, 1$. We now use the fact that Ω' also satisfies (VIII.39) in order to solve

$$\frac{\partial v}{\partial \overline{w}} = (-1)^q u_1,$$

with $v \in C^\infty(\Omega', \underline{\underline{\Lambda}}^{q-2})$ (we set $v = u_1 = 0$ if $q = 1$). A simple computation shows that $u - \mathrm{D}_{q-2} v \in C^\infty(\Omega', \underline{\underline{\Lambda}}^q)$ and consequently if we set

$$F \doteq f^\bullet - (w - w_0)\left(u - \mathrm{D}_{q-2} v\right)$$

we obtain $F \in C^\infty(\Omega', \underline{\underline{\Lambda}}^q)$, $F(z, w_0, t) = f(z,t)$, and

$$\mathrm{D}_{q-1} F = \mathrm{D}_{q-1} f^\bullet - (w - w_0)\mathrm{D}_{q-1} u = (w - w_0)(G - \mathrm{D}_{q-1} u) = 0. \qquad \square$$

We can now prove:

THEOREM VIII.6.3. *Let $\Omega'' \subset \Omega' \subset \Omega$ be open subsets of $\mathbb{C}^m \times \mathbb{R}^{n'}$, all of them satisfying (VIII.39). Let $q \geq 1$ and assume that $\mathfrak{I}^{q-1}_{(\Omega',\Omega'')} = 0$ and that $H^q_{\mathrm{D}}(\Omega) \to H^q_{\mathrm{D}}(\Omega')$ is the trivial map. Then $\mathfrak{I}^{q-1}_{(\Omega(w_0),\Omega''(w_0))} = 0$ for every $w_0 \in \mathbb{C}$.*

PROOF. First we observe that, after taking regularizations, the vanishing of $\mathfrak{I}^{q-1}_{(\Omega',\Omega'')} = 0$ allows one to assert that

$$\int F \wedge \mathrm{d}z \wedge \mathrm{d}w \wedge T = 0, \qquad \text{(VIII.42)}$$

for every $F \in C^\infty(\Omega', \underline{\Lambda}^{q-1})$ satisfying $\mathrm{D}_{q-1} F = 0$ and every $T \in \mathcal{E}'(\Omega'', \underline{\Lambda}^{n-q+1})$ satisfying $\mathrm{D}_{n-q+1} T = 0$ (when $q = 1$ this condition must be replaced by $T[G] = 0$ for every $G \in C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'})$ satisfying $\mathrm{D}_0 G = 0$).

Fix $w_0 \in \mathbb{C}$ and let $f \in C^\infty(\Omega(w_0), \underline{\underline{\Lambda}}^{q-1})$, $\Theta \in C^\infty_c(\Omega''(w_0), \underline{\Lambda}^{n-q})$ be both D-closed (we assume $\Theta \in \{g \in C^\infty(\mathbb{C}^\nu \times \mathbb{R}^{n'}) : \mathrm{D}_0 g = 0\}^\perp$ when $q = 1$). Thanks to our hypotheses we can apply Proposition VIII.6.2 in order to obtain $F \in C^\infty(\Omega', \underline{\underline{\Lambda}}^{q-1})$ satisfying $\mathrm{D}_{q-1} F = 0$ and $F|_{w=w_0} = f$.

On the other hand, if we write

$$\Theta = \sum_{|J|+|K|=n-q} \Theta_{JK}(z,t)\, \mathrm{d}\bar{z}_J \wedge \mathrm{d}t_K$$

and define $T_\Theta \in \mathcal{E}'(\Omega'', \underline{\Lambda}^{n-q+1})$ by the formula

$$T_\Theta \doteq \sum_{|J|+|K|=n-q} \Theta_{JK}(z,t) \otimes \delta(w-w_0)\, \mathrm{d}\bar{z}_J \wedge \mathrm{d}\overline{w} \wedge \mathrm{d}t_K$$

we have $\mathrm{D}_{n-q+1} T_\Theta = 0$ and also $T_\Theta \in \{G \in C^\infty(\mathbb{C}^m \times \mathbb{R}^{n'}) : \mathrm{D}_0 G = 0\}^\perp$ when $q = 1$. Then (VIII.42) gives

$$\mathfrak{I}^{q-1}_{(\Omega(w_0),\Omega''(w_0))}(f,\Theta) = \int \left(F|_{w=w_0}\right) \wedge \mathrm{d}z \wedge \Theta = \pm \int F \wedge \mathrm{d}z \wedge \mathrm{d}w \wedge T_\Theta = 0,$$

which concludes the proof. □

VIII.7 A necessary condition for one-sided solvability

We keep the notation established in Section VIII.6 and consider now a regular open subset Ω of $\mathbb{C}^m \times \mathbb{R}^{n'}$. We fix a defining function ρ for $\partial\Omega$: thus ρ is a smooth, real-valued function such that $\partial\Omega$ is defined by the equation $\rho = 0$, with $\mathrm{d}\rho \neq 0$ on $\partial\Omega$.

Theorem VIII.7.1. *Let $p \in \partial\Omega$ be such that*

$$\frac{\partial \rho}{\partial w}(p) \neq 0. \tag{VIII.43}$$

Then if for some $q \geq 1$ D is solvable near p in degree q with respect to Ω it follows that the following property holds: given any open neighborhood U of p in $\mathbb{C}^m \times \mathbb{R}^{n'}$ there is another such neighborhood $V \subset U$ such that, for every $w_0 \in \mathbb{C}$, the intersection number $\mathfrak{I}^{q-1}_{(\Omega(w_0)\cap U, \Omega(w_0)\cap V)}$ vanishes identically.

This result is a direct consequence of Theorem VIII.6.3 in conjunction with Corollary VIII.5.5 and the following proposition:

Proposition VIII.7.2. *Suppose that (VIII.43) is satisfied. Then there is an open neighborhood W of p in $\mathbb{C}^m \times \mathbb{R}^{n'}$ such that given any open convex set $D \subset W$ the set $D \cap \Omega$ is $\partial/\partial\overline{w}$-convex.*

Proof. It suffices to prove the analogous statement for the operator

$$\Delta_w = 4 \frac{\partial^2}{\partial w \partial \overline{w}},$$

since every open set which is Δ_w-convex is *a fortiori* $\partial/\partial\overline{w}$-convex.

We write $w = s + ir$, $p = (z_0, s_0 + ir_0, t_0)$, and assume, say, that $\partial\rho/\partial r \neq 0$ at p. By the implicit function theorem, there are an open neighborhood W of p and a smooth function $\psi: \mathbb{C}^\nu \times \mathbb{R} \times \mathbb{R}^{n'} \to \mathbb{R}$ such that $\psi(z_0, s_0, t_0) = r_0$ and

$$W \cap \Omega = \{(z, w, t) \in W : r < \psi(z, s, t)\}.$$

Now the set $\mathcal{U} = \{(z, w, t) \in \mathbb{C}^m \times \mathbb{R}^{n'} : r < \psi(z, s, t)\}$ is Δ_w-convex since Δ_w is real and any normal to $\partial\mathcal{U}$ is not a characteristic vector for Δ_w ([**H1**], theorem 3.7.4). Consequently, given any open convex set $D \subset \mathbb{C}^m \times \mathbb{R}^{n'}$, the set $D \cap \mathcal{U}$, being the intersection of Δ_w-convex open sets, is also Δ_w-convex. If we finally observe that if $D \subset W$ then $D \cap \Omega = D \cap \mathcal{U}$, the result follows at once. □

Remark VIII.7.3. As in Section VII.12, we introduce the spaces of germs:

$$C^\infty_\Omega(p, \underline{\Lambda}^q) = \lim_{U \to \{p\}} C^\infty(U \cap \Omega, \underline{\Lambda}^q);$$

$$C^\infty_{\overline{\Omega}}(p, \underline{\Lambda}^q) = \lim_{U \to \{p\}} C^\infty(U \cap \overline{\Omega}, \underline{\Lambda}^q).$$

It can be proved, via methods of hyperfunction theory, that if solvability for D near p in degree q with respect to Ω does not occur then there is $\underline{f} \in C^\infty_{\overline{\Omega}}(p, \underline{\Lambda}^q)$ for which no $\mathbf{u} \in C^\infty_\Omega(p, \underline{\Lambda}^{q-1})$ satisfies $\mathrm{D}\mathbf{u} = \underline{f}$. In particular, Corollary VIII.5.5 also gives a necessary condition for solvability for D near p in degree q with respect to $\overline{\Omega}$.

In the particular case when $m = 1$, Corollary VIII.5.4 allows us to state the necessary condition in terms of the de Rham cohomology. We give first a definition.

Definition VIII.7.4. *Assume that $m = 1$ and let $p \in \Omega$. We shall say condition $(\star)_q$ $(q \geq 1)$ holds at p with respect to Ω if given any open neighborhood U of p in $\mathbb{C} \times \mathbb{R}^{n'}$ there is another such neighborhood $V \subset U$ such that, for all $w_0 \in \mathbb{C}$, the natural homomorphism $H^q(\Omega(w_0) \cap U, \mathbb{C}) \to H^q(\Omega(w_0) \cap V, \mathbb{C})$ is trivial. We further say that condition $(\star)_0$ holds at p with respect to Ω if given any open neighborhood U of p in $\mathbb{C} \times \mathbb{R}^{n'}$ there is another such neighborhood $V \subset U$ such that, for all $w_0 \in \mathbb{C}$, $\Omega(w_0) \cap V$ is contained in one of the connected components of $\Omega(w_0) \cap U$.*

Corollary VIII.7.5. *Suppose that $m = 1$ and that (VIII.43) is satisfied. Then if for some $q \geq 1$, D is solvable near $p \in \partial\Omega$ in degree q with respect to Ω it follows that condition $(\star)_{q-1}$ holds at p with respect to Ω.*

VIII.8 The sufficiency of condition $(\star)_0$

We shall now show the sufficiency, in a weak form, of condition $(\star)_0$ for solvability near $p \in \partial\Omega$ in degree 0 with respect to $\overline{\Omega}$ under the stronger assumption that

$$\textit{The boundary of } \Omega \textit{ is real-analytic.} \tag{VIII.44}$$

In other words, we shall assume that Ω is defined by $\rho > 0$, where ρ is real-valued, real-analytic and such that $\partial\Omega$ is defined by $\rho = 0$, with $\partial\rho/\partial z \neq 0$ near p.

The next result is the key tool for the proof of the result. In all the arguments that follow we shall denote by $\pi : \mathbb{C} \times \mathbb{R}^{n'} \to \mathbb{C}$ the projection $\pi(z,t) = z$.

We assume that the central point in the analysis is $p = (z_0, t_0) \in \partial\Omega$ in $\mathbb{C} \times \mathbb{R}^{n'}$. By applying the implicit function theorem we can assume that

$$\rho(z,t) = y - \Phi(x,t), \quad z = x + iy, \tag{VIII.45}$$

where Φ is real-analytic and $\Phi(x_0, t_0) = y_0$.

We shall also denote by $\mathfrak{V}(p)$ the set of all open sets D of the form $R \times \Theta$, where R (resp. Θ) is an open square in $\mathbb{C}$ with sides parallel to the coordinate axes (resp. open ball in $\mathbb{R}^{n'}$) centered at $z_0 \in \mathbb{C}$ (resp. $t_0 \in \mathbb{R}^{n'}$).

Proposition VIII.8.1. *Assume that both (VIII.44) and condition $(\star)_0$ hold. Then given any $D \in \mathfrak{V}(p)$ there is $D_\bullet \subset\subset D$ also belonging to $\mathfrak{V}(p)$ and a constant $M > 0$ such that, for any $z \in \mathbb{C}$, any two points in $\Omega(z) \cap D_\bullet$ can be joined by a piecewise real-analytic curve contained in $\overline{\Omega}(z) \cap D$ and with length $\leq M$.*

Proof. Given D as in the statement we take $D_1 \subset\subset D$ also in $\mathfrak{V}(p)$ and apply condition $(\star)_0$: there is $D_\bullet \subset D_1$, also in $\mathfrak{V}(p)$ such that, for any $z \in \mathbb{C}$, $\Omega(z) \cap D_\bullet$ is contained in a single component of $\Omega(z) \cap D_1$.

Next we observe that the set $K \doteq \overline{D_1 \cap \Omega}$ is compact and sub-analytic. We then apply a standard result on the theory of subanalytic sets which can be found in ([**Har**], section 8): there is $M > 0$ such that any two points in a component of $\pi^{-1}\{z\} \cap K$ may be joined by a piecewise analytic arc in $\pi^{-1}\{z\} \cap K$ of length $\leq M$.

Hence if t, s belong to $\Omega(z) \cap D_\bullet$ they belong to a component of $\Omega(z) \cap D_1$ and consequently to a component of $\pi^{-1}\{z\} \cap K$. Since $\pi^{-1}\{z\} \cap K \subset \overline{\Omega}(z) \cap D$ the result follows. □

The key point in the argument is the following result:

PROPOSITION VIII.8.2. *Under the same hypotheses as in Proposition VIII.8.1, given $D \in \mathfrak{V}(p)$ there are $D_\star \in \mathfrak{V}(p)$, $D_\star \subset\subset D$ and a constant $C > 0$ such that the following is true: given $u \in \mathcal{C}^\infty(\overline{\Omega} \cap \overline{D}) \cap \mathcal{R}(\Omega \cap D)$ there is $v \in \mathcal{R}(\Omega \cap D_\star)$ such that $\mathrm{d}_t v = \mathrm{d}_t u$ and*

$$\sup_{(z,t)\in\Omega\cap D_\star} |v(z,t)(y - \Phi(x,t))| \le C\|\mathrm{d}_t u\|_{L^\infty(\Omega\cap\overline{D})}\,. \tag{VIII.46}$$

Before we embark on the (rather long) proof of this result, we will show how it can be used to derive our one-sided solvability result.

COROLLARY VIII.8.3. *Assume (VIII.44) and that condition $(\star)_0$ holds. Then given any $D_0 \in \mathfrak{V}(p)$ there is $D_\star \subset\subset D_0$ also belonging to $\mathfrak{V}(p)$ such that for every $f \in C^\infty(\overline{\Omega} \cap D_0, \underline{\Lambda}^1)$ satisfying $\mathrm{D}_1 f = 0$ there is $v \in C^\infty(\Omega \cap D_\star)$ satisfying $\mathrm{D}_0 v = f$ in $\Omega \cap D_\star$ and*

$$\sup_{(z,t)\in\Omega\cap D_\star} |v(z,t)(y - \Phi(x,t))| < \infty.$$

Notice that, in particular, (VIII.44) and condition $(\star)_0$ imply solvability for D near p in degree 1 with respect to Ω (*cf.* Remark VIII.7.3).

PROOF. Write

$$f = f_0\,\mathrm{d}\overline{z} + \sum_{j=1}^{n'} f_j\,\mathrm{d}t_j\,.$$

If we extend f_0 to a smooth function on D_0 and then solve $(\partial v/\partial\overline{z}) = f_0$ in D_0 we obtain a new form $f - \mathrm{D}_0 v \in C^\infty(\overline{\Omega} \cap D_0, \underline{\Lambda}^1)$ which has no $\mathrm{d}\overline{z}$-component. In other words, we can start with $f \in C^\infty(\overline{\Omega} \cap D_0, \underline{\Lambda}^1)$ of the form

$$f = \sum_{j=1}^{n} f_j\,\mathrm{d}t_j\,.$$

Notice that $\mathrm{D}_1 f = 0$ means that $\mathrm{d}_t f = 0$ and that each coefficient f_j is holomorphic in z, that is, $f_j \in \mathcal{R}(\Omega \cap D_0)$.

We apply the Approximate Poincaré Lemma: there is $D \in \mathfrak{V}(p)$, $D \subset D_0$ (which is independent of f) and a sequence $u_\nu \in C^\infty(\overline{\Omega} \cap \overline{D})$ such that $\mathrm{D}_0 u_\nu \to f$ in $C^\infty(\overline{\Omega} \cap \overline{D}, \underline{\Lambda}^1)$. Notice that this means

$$\mathrm{d}_t u_\nu \to f \quad \text{in} \quad C^\infty(\overline{\Omega} \cap \overline{D}, \underline{\Lambda}^1)\,; \qquad \frac{\partial u_\nu}{\partial\overline{z}} \to 0 \quad \text{in} \quad C^\infty(\overline{\Omega} \cap \overline{D}).$$

Consider now a linear, continuous extension operator

$$\mathfrak{E} : C^\infty(\overline{\Omega} \cap \overline{D}) \longrightarrow C^\infty(\overline{D})$$

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and if $D = R \times \Theta$ let

$$A_\nu(z,t) = \frac{1}{\pi} \iint_R \mathfrak{E}\left(\frac{\partial u_\nu}{\partial \bar{z}}\right)(z',t) \frac{1}{z-z'} \, dx'dy' .$$

It is easily seen that $A_\nu \to 0$ as $\nu \to \infty$ in $C^\infty(\overline{D})$ and, clearly,

$$\frac{\partial u_\nu}{\partial \bar{z}} = \frac{\partial A_\nu}{\partial \bar{z}} \quad \text{in} \quad \overline{\Omega} \cap \overline{D}.$$

If we substitute $u_\nu - A_\nu$ for u_ν we then obtain a new sequence $u_\nu \in C^\infty(\overline{\Omega} \cap \overline{D})$ such that

$$d_t u_\nu \to f \text{ in } C^\infty(\overline{\Omega} \cap \overline{D}, \underline{\Lambda}^1),\ u_\nu \text{ is holomorphic in } z. \tag{VIII.47}$$

Finally we take $D_\bullet \subset\subset D$ as in Proposition VIII.8.2 and apply its conclusion to $u = u_\nu$: we can find $v_\nu \in \mathcal{R}(\Omega \cap D_\star)$ such that $d_t v_\nu = d_t u_\nu$ and, for some constant $C > 0$,

$$\sup_{(z,t) \in \Omega \cap D_\star} |v_\nu(z,t)(y - \Phi(x,t))| \le C, \quad \forall \nu.$$

But then some subsequence v_{ν_k} converges weakly to a function v which satisfies the required properties. This concludes the proof of Corollary VIII.8.3. □

VIII.9 Proof of Proposition VIII.8.2

We take $D_\bullet = R_\bullet \times \Theta_\bullet \subset\subset D$ as in Proposition VIII.8.1 and start by constructing a suitable covering of $\Omega \cap D_\bullet$. Set $\omega \doteq \pi(\Omega \cap D_\bullet)$ and for each $a \in \mathbb{R}^{n'}$ we set

$$W_a \doteq \{z \in \mathbb{C} : (z,a) \in \Omega \cap D_\bullet\}.$$

Notice that $\{W_a\}$ is an open covering of ω. We also set

$$U_a \doteq \pi^{-1}(W_a) \cap [\Omega \cap D_\bullet]. \tag{VIII.48}$$

Then $\{U_a\}$ is an open covering of $\Omega \cap D_\bullet$ and $(z,t) \in U_a$ implies $(z,a) \in \Omega \cap D_\bullet$. Using the curves given in Proposition VIII.8.1 and the corresponding bound for their lengths we obtain

$$|u(z,t) - u(z,a)| \le M \|d_t u\|_{L^\infty(\Omega \cap D)} \quad (z,t) \in U_a. \tag{VIII.49}$$

The family $\{u(\cdot,a)\}$ defines a holomorphic one-cochain with respect to the open covering $\{W_a\}$ of ω which satisfies

$$|u(z,a) - u(z,b)| \le M \|d_t u\|_{L^\infty(\Omega \cap D)} \quad z \in W_a \cap W_b. \tag{VIII.50}$$

We shall now construct a new one-cochain $w_a \in \mathcal{O}(W_a)$ such that $w_a - w_b = u(\cdot, a) - u(\cdot, b)$ on $W_a \cap W_b$ and for which each term w_a can be estimated, on W_a, in terms of the right-hand side of (VIII.50).

Such a one-chain will be constructed through the following standard argument: start with a partition of unity $\{\psi_j\}$, $0 \le \psi_j \le 1$, subordinate to the covering $\{W_a\}$, that is for each j there corresponds a_j such that $\psi_j \in C_c^\infty(W_{a_j})$ and set

$$g_a(z) = \sum_k \psi_k(z)\,[u(z, a) - u(z, a_k)]\,.$$

Then $g_a \in C^\infty(W_a)$ and $g_a - g_b = u(\cdot, a) - u(\cdot, b)$ in $W_a \cap W_b$. Notice that this last equality implies $\partial g_a/\partial \bar{z} = \partial g_b/\partial \bar{z}$ in $W_a \cap W_b$ and consequently there is $G \in C^\infty(\omega)$, $G = \partial g_a/\partial \bar{z}$ in W_a for every a. Finally we solve

$$\frac{\partial F}{\partial \bar{z}} = G \tag{VIII.51}$$

in ω, with $F \in C^\infty(\omega)$, and set $w_a = g_a - F$.

Observe that such a solution F always exists (every open subset of $\mathbb{C}$ is a domain of holomorphy!) but in order to obtain (VIII.46) we will be forced to make a further contraction in the domain.

We have

$$|g_a(z)| \le \sum_k \psi_k(z)\,|u(z, a) - u(z, a_k)| \le M \|\mathrm{d}_t u\|_{L^\infty(\Omega \cap D)} \quad z \in W_a$$

for every a and thus the proof will be completed if we can show that, for some suitable choice of the partition of unity $\{\psi_j\}$, we can obtain a solution F to (VIII.51) on $R_\star \cap \omega$, with $R_\star \subset R_\bullet$ another square centered at z_0, satisfying

$$|F(z)(y - \Phi(z, t))| \le M_1 \|\mathrm{d}_t u\|_{L^\infty(\Omega \cap D)}, \qquad (z, t) \in \Omega \cap D_\star, \tag{VIII.52}$$

where $D_\star \doteq R_* \times \Theta_\bullet \in \mathfrak{V}(p)$.

Indeed $v \in \mathcal{R}(\Omega \cap D_\star)$, defined on $U_a \cap D_\star$ as

$$u - u(\cdot, a) - w_a = u - u(\cdot, a) - g_a + F,$$

satisfies $\mathrm{d}_t v = \mathrm{d}_t u$ and (VIII.46).

In order to achieve (VIII.52) we start by observing that

$$\left|\frac{\partial g_a}{\partial \bar{z}}(z)\right| \le M \|\mathrm{d}_t u\|_{L^\infty(\Omega \cap D)} \sum_k \left|\frac{\partial \psi_k}{\partial \bar{z}}(z)\right|, \qquad z \in W_a\,, \tag{VIII.53}$$

and take a closer look on the coverings $\{U_a\}$ and $\{W_a\}$. We have

$$\omega = \{z \in R_\bullet : \exists t \in \Theta_\bullet,\ \rho(z, t) > 0\},$$

$$W_a = \{z \in R_\bullet : \rho(z, a) > 0\}.$$

We set

$$\sigma(z) = \sup_{t \in \Theta_\bullet} \rho(z, t) = \max_{t \in \overline{\Theta_\bullet}} \rho(z, t).$$

In particular, σ is Lipschitz continuous. Also

$$\omega = \{z \in R_\bullet : \sigma(z) > 0\}.$$

Set $\lambda(z) = \min\{\sigma(z), \mathrm{dist}\,(z, \partial\omega)\}$ and observe that λ is also Lipschitz continuous. We then recall Lemma IV.3.11:

LEMMA VIII.9.1. *Let $\epsilon > 0$ be arbitrary. There is an open covering of ω by squares Q_j, with sides parallel to the coordinate axes, having the following properties:*

$$\mathrm{diam}\, Q_j \leq \epsilon \inf_{Q_j} \lambda(z). \tag{VIII.54}$$

There are $\psi_j \in C_c^\infty(Q_j)$, $0 \leq \psi_j \leq 1$, such that $\sum \psi_j = 1$ and (VIII.55)

$$\sum_j \left| \frac{\partial \psi_j}{\partial \bar{z}}(z) \right| \leq C\lambda(z)^{-1}.$$

Next we claim that if we take $\epsilon < 1/(2K)$, where K is a Lipschitz constant for ρ, then for each j there is a_j such that $Q_j \subset W_{a_j}$. Indeed let $z_\flat \in Q_j$ and take $t_\flat \in \overline{\Theta_\bullet}$ such that $\sigma(z_\flat) = \rho(z_\flat, t_\flat)$. If $z \in \overline{Q_j}$ we have $|z - z_\flat| \leq \epsilon\sigma(z_\flat)$ and consequently

$$\begin{aligned} \rho(z, t_\flat) &= \rho(z, t_\flat) - \rho(z_\flat, t_\flat) + \rho(z_\flat, t_\flat) \\ &\geq \sigma(z_\flat) - K|z - z_\flat| \\ &\geq \sigma(z_\flat) - \epsilon K \sigma(z_\flat) \\ &\geq \frac{1}{2}\sigma(z_\flat) > 0, \end{aligned}$$

whence our assertion.

For this choice of partition of unity (VIII.53) gives

$$|G(z)| \leq 2MC\|\mathrm{d}_t u\|_{L^\infty(\Omega\cap D)} \lambda(z)^{-1}, \quad z \in \omega. \tag{VIII.56}$$

Since $\rho(z_0, t_0) = 0$ we have $\sigma(z_0) \geq 0$. The case $\sigma(z_0) > 0$ is almost elementary, for we can take $R_\star \subset \omega$ in such a way that $\lambda(z) \geq c > 0$ in $R_\star$ and consequently the solution to (VIII.51), given by F defined via the formula

$$F(z) = \frac{1}{\pi} \iint_{\mathcal{R}_\star} \frac{G(z')}{z - z'} \,\mathrm{d}x'\,\mathrm{d}y'$$

satisfies

$$|F(z)| \le M_1 \|\mathrm{d}_t u\|_{L^\infty(\Omega\cap D)}, \quad z \in R_\star.$$

Let us then assume that $\sigma(z_0) = 0$. We now take $R_\star$ as stated and such that

$$z \in R_\star \implies \lambda(z) = \sigma(z). \tag{VIII.57}$$

Notice that, thanks to (VIII.45), we have

$$\sigma(z) = y - \Psi(x), \quad \Psi(x) = \inf_{t\in\Theta_\bullet} \Phi(x,t)$$

and then

$$\omega \cap R_\star = \{z \in R_* : y > \Psi(x)\}.$$

We now apply the standard identity

$$\frac{1}{z-z'} = \frac{1}{z-\zeta} + \frac{1}{z-z'}\left(\frac{z'-\zeta}{z-\zeta}\right)$$

in order to obtain

$$\begin{aligned}\int_{\omega\cap R_\star} \frac{G(z')}{z-z'}\,\mathrm{d}x'\,\mathrm{d}y' = &\iint_{\omega\cap R_\star} \frac{G(z')}{z-x'-i\Psi(x')}\,\mathrm{d}x'\,\mathrm{d}y' \\ &- i\iint_{\omega\cap R_\star} \frac{G(z')(y'-\Psi(x'))}{(z-z')(z-x'-i\Psi(x'))^2}\,\mathrm{d}x'\,\mathrm{d}y'.\end{aligned}$$

Since the first term on the right-hand side is holomorphic in $\omega\cap R_\star$ we can solve (VIII.51) by taking

$$F(z) = \frac{1}{\pi i}\iint_{\omega\cap R_\star} \frac{G(z')(y'-\Psi(x'))}{(z-z')(z-x'-i\Psi(x'))}\,\mathrm{d}x'\,\mathrm{d}y', \quad z \in \omega\cap R_\star. \tag{VIII.58}$$

It remains to verify (VIII.52). From (VIII.56) and (VIII.57) we obtain

$$\begin{aligned}(y-\Psi(x))|F(z)| \le &M_2\|\mathrm{d}_t u\|_{L^\infty(\Omega\cap D)} \\ \times\iint_{\omega\cap R_\star} \frac{1}{|z-z'|}\cdot\frac{y-\Psi(x)}{|x-x'|+|y-\Psi(x')|}&\mathrm{d}x'\mathrm{d}y', \quad z\in\omega\cap R_\star.\end{aligned} \tag{VIII.59}$$

To conclude we just observe that, since Ψ is Lipschitz,

$$\frac{y-\Psi(x)}{|x-x'|+|y-\Psi(x')|} \le \frac{|y-\Psi(x')|+M_3|x-x'|}{|x-x'|+|y-\Psi(x')|} \le M_3+1$$

and hence (VIII.59) implies

$$(y-\Psi(x))|F(z)| \le M_4 \|\mathrm{d}_t u\|_{L^\infty(\Omega\cap D)}.$$

We have thus proved (VIII.52) since

$$\begin{aligned}(z,t)\in\omega\cap D_\star &\implies z\in R_\star,\ t\in\Theta_\bullet,\ y>\Phi(x,t),\\ &\implies y-\Phi(x,t)\le y-\Psi(x).\end{aligned}$$

The proof of Proposition VIII.8.2 is now complete.

VIII.10 Solvability for corank one analytic structures

Since the solution v obtained in Corollary VIII.8.3 is holomorphic with respect to z and has tempered growth when $(z,t)\to\partial\Omega\cap D_\star$ the results in Chapter VI show that its boundary value is a well-defined distribution on $\partial\Omega\cap D_\star$ of order ≤ 2. If in addition we also assume the validity of condition $(\star)_0$ at p with respect to $\mathbb{C}\times\mathbb{R}^{n'}\setminus\Omega$, and if we denote by $\mathcal{V}_\bullet$ the bundle spanned by the vector fields $\partial/\partial\bar{z}$, $\partial/\partial t_j$, $j=1,\dots,n$ and by $L\doteq \mathrm{D}_{\partial\Omega}$ the complex induced by the elliptic complex D on $\partial\Omega$, an almost immediate extension of (a) in Theorem VII.12.1 gives:

Given an open neighborhood U of p in $\partial\Omega$ there is another such neighborhood $V\subset U$ such that, given $f\in\mathfrak{U}^{\mathcal{V}_\bullet}(U)$ satisfying $Lf=0$ there is $u\in\mathcal{D}'_{(2)}(V)$ solving $Lu=f$ in V. (VIII.60)

Consider the complex vector fields

$$L_j^\bullet=\frac{\partial}{\partial t_j}-\left(\frac{\partial\rho}{\partial\bar{z}}\right)^{-1}\frac{\partial\rho}{\partial t_j}\frac{\partial}{\partial\bar{z}},\qquad j=1,\dots,n'.$$

Near p the vector fields $L_j^\bullet$ are tangent to $\partial\Omega$ and their restriction to $\partial\Omega$ span $\mathcal{V}_\bullet(\partial\Omega)$. As before we describe $\partial\Omega$ by the equation $y-\Phi(x,t)=0$, with Φ real-analytic and take (x,y,t) as local coordinates in $\partial\Omega$. In these local coordinates the vector fields $L_j\doteq L_j^\bullet|_{\partial\Omega}$ are written as

$$L_j=\frac{\partial}{\partial t_j}-\frac{i\Phi_{t_j}}{1+i\Phi_x}\frac{\partial}{\partial x},\qquad j=1,\dots,n'. \tag{VIII.61}$$

Hence $\mathcal{V}_\bullet(\partial\Omega)$ is exactly the locally integrable structure defined on a neighborhood of the point p in $\mathbb{R}^{n'+1}$ which is orthogonal to the sub-bundle of $\mathbb{C}T^*(\mathbb{R}^{n'+1})$ spanned by $\mathrm{d}Z$, where $Z(x,t)=x+i\Phi(x,t)$.

The reverse argument is also true, that is, any smooth locally integrable structure $\mathcal{V}$ of corank one, say in a neighborhood of the origin in $\mathbb{R}^{n'+1}$, arises from the restriction of the elliptic structure $\mathcal{V}_\bullet$ on $\mathbb{C}\times\mathbb{R}^{n'}$ to a hypersurface Σ in $\mathbb{C}\times\mathbb{R}^{n'}$. Indeed if we choose local coordinates $(x,t)=(x,t_1,\ldots,t_{n'})$ in a neighborhood of the origin in $\mathbb{R}^{n'+1}$ in such a way that the orthogonal of $\mathcal{V}$ is generated by the differential of $Z(x,t)=x+i\Phi(x,t)$, with Φ smooth and real-valued, and if we denote by Σ the image of the imbedding $(x,t)\mapsto(Z(x,t),t)$, it follows easily that $\mathcal{V}=\mathcal{V}_\bullet(\Sigma)$.

Keeping this notation, and recalling Corollary I.10.2, we can (and will) even assume that $\Phi(0,0)=\Phi_x(0,0)=0$. We emphasize that $\mathcal{V}$ is spanned by the pairwise commuting vector fields (VIII.61). We further take a small open neighborhood V of the origin in $\mathbb{C}\times\mathbb{R}^{n'}$ and set

$$\mathrm{V}^+=\{(z,t)\in\mathrm{V}: z=x+iy,\ y>\Phi(x,t)\},$$

$$\mathrm{V}^-=\{(z,t)\in\mathrm{V}: z=x+iy,\ y<\Phi(x,t)\}.$$

Definition VIII.10.1. *We shall say that condition* $(\mathrm{P})_0$ *holds at the origin for the locally integrable structure* $\mathcal{V}$ *if condition* $(\star)_0$ *holds at the origin in* $\mathbb{C}\times\mathbb{R}^{n'}$ *with respect to both* V^+ *and* V^-.

We shall then prove:

Theorem VIII.10.2. *Let* $\mathcal{V}$ *be a corank one, real-analytic, locally integrable structure defined in an open neighborhood of the origin in* $\mathbb{R}^{n'+1}$ *and let* d' *be the associated differential complex. Then* d' *is solvable near the origin in degree one if and only if condition* $(\mathrm{P})_0$ *holds at the origin.*

Proof. The necessity of condition $(\mathrm{P})_0$ follows from Theorem VII.12.1, Corollary VIII.8.3 and Remark VIII.7.3.

We now embark on the proof of the sufficiency. Let us denote by $\mathfrak{W}(0)$ the family of all open neighborhoods of the origin in $\mathbb{R}^{n'+1}$ of the form $U=I\times\Theta$, where I (resp. Θ) is an open interval (resp. ball) centered at the origin in $\mathbb{R}$ (resp. $\mathbb{R}^{n'}$). If $(p,q)\in\mathbb{R}^2$ and if $U\in\mathfrak{W}(0)$ we shall denote by $L^{2,r,s}_{\mathrm{loc}}(U)$ the local Sobolev space of order r with respect to x and of order s with respect to t.

We recall that if we set $M=Z_x^{-1}(\partial/\partial x)$ then the vector fields $L_1,\ldots,L_{n'},M$, (*cf.* (VIII.61)), are pairwise commuting, linearly independent (see (I.38)).

We now make use of (VIII.60). Then there is $(r_0,s_0)\in\mathbb{R}^2$ such that the following is true: given $U\in\mathfrak{W}(0)$ there is $U'=I'\times\Theta'\in\mathfrak{W}(0)$, $U'\subset\subset U$, such that given

$$f(x,t)=\sum_{j=1}^{n'}f_j(x,t)\,\mathrm{d}t_j\in\mathfrak{U}^{\mathcal{V}}(U),\quad Lf=0 \tag{VIII.62}$$

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there is $v \in L^{2,r_0,s_0}_{\mathrm{loc}}(U')$ satisfying $Lv = f$ in U'.

We then fix f as in (VIII.62). Noticing that, for each $k \in \mathbb{N}$, $M^{2k} f$ is also L-closed (here M^{2k} acts componentwise on the one-form f), we can find $v_k \in L^{2,r_0,s_0}_{\mathrm{loc}}(U')$ solving $Lv_k = M^{2k} f$ in U'. Next we solve, in U', $M^{2k} w_k = v_k$. Thus

$$M^{2k}\left[Lw_k - f\right] = 0$$

and consequently we can write

$$Lw_k - f = \sum_{j=0}^{2k-1} g_{jk}(t)\, Z(x,t)^j\,,$$

where g_{jk} are d_t-closed one-forms with distributional coefficients. We can find distributions $G_{jk} \in \mathcal{D}'(\Theta')$ such that $\mathrm{d}_t G_{jk} = g_{jk}$ and hence we have

$$L\left[w_k + \sum_{j=0}^{2k-1} G_{jk}(t)\, Z(x,t)^j\right] = f. \tag{VIII.63}$$

Since $w_k \in L^{2,r_0+2k,s_0}_{\mathrm{loc}}(U')$ it follows that

$$Lw_k - f \in L^{2,r_0+2k-1,s_0-1}_{\mathrm{loc}}(U')$$

and hence $g_{jk} \in L^{2,s_0-1}_{\mathrm{loc}}(\Theta')$. Consequently $G_{jk} \in L^{2,s_0}_{\mathrm{loc}}(\Theta')$ and then, if we set

$$u_k \doteq w_k + \sum_{j=0}^{2k-1} G_{jk}(t)\, Z(x,t)^j$$

we have

$$Lu_k = f, \quad u_k \in L^{2,r_0+2k,s_0}_{\mathrm{loc}}(U'). \tag{VIII.64}$$

Explicitly (VIII.64) means

$$\frac{\partial u_k}{\partial t_j} = \frac{i\Phi_{t_j}}{1+i\Phi_x}\,\frac{\partial u_k}{\partial x} + f_j, \quad j = 1,\ldots,n'.$$

This expression implies that it is possible to trade differentiability with respect to x for differentiability with respect to t_j, $j = 1,\ldots,n'$, that is, we also have $u_k \in L^{2,r_0+k,s_0+k}_{\mathrm{loc}}(U')$.

Let $U_\bullet \in \mathfrak{W}(0)$, $U_\bullet \subset\subset U'$. By the Sobolev imbedding theorem it then follows that for each $\nu \in \mathbb{N}$ we can find a solution $u^\bullet_\nu \in C^\nu(\overline{U_\bullet})$ to the equation $Lu^\bullet_\nu = f$ in $U_\bullet$.

We finally apply, for each $\nu \in \mathbb{N}$, the C^ν-version of the Baouendi–Treves approximation formula (*cf.* Theorem II.1.1). There are $U_1 \in \mathfrak{W}(0)$, $U_1 \subset\subset U_\bullet$, depending only on $U_\bullet$ and on $\mathcal{V}$, and a sequence of holomorphic polynomials $\{p_\nu\} \subset \mathbb{C}[z]$ such that

$$\|u^\bullet_{\nu+1} - u^\bullet_\nu - p_\nu(Z)\|_{C^\nu(\overline{U_1})} \leq \frac{1}{2^\nu}\,. \tag{VIII.65}$$

If we set

$$u_{(1)} = u^\bullet_1, \quad u_{(\nu)} = u^\bullet_\nu - p_1(Z) - \ldots - p_{\nu-1}(Z),\ \nu \geq 2,$$

then (VIII.65) gives

$$\|u_{\nu+1} - u_\nu\|_{C^\nu(\overline{U_1})} \leq \frac{1}{2^\nu}\,.$$

This shows that, for each $p \in \mathbb{N}$, the sequence $(u_\nu)_{\nu \geq p}$ converges to an element $u \in C^p(\overline{U_1})$, of course independent of p, and belonging to $\mathcal{C}^\infty(\overline{U_1})$. Since moreover $Lu_\nu = f$ in U_1 for every ν we also have $Lu = f$ in U_1.

The proof of Theorem VIII.10.2 is complete. □

Notes

Until now, complete answers for local solvability in locally integrable structures, besides the cases $n = 1$ (a situation which has already been discussed in Chapter IV), $\mathcal{V}$ defines an essentially real structure (Section VIII.1) and when $\mathcal{V}$ defines an elliptic structure (Theorem VIII.3.1) are known in the following cases: (i) $\mathcal{V}$ defines a nondegenerate locally integrable CR structure of codimension one ([**AH2**]); (ii) $\mathcal{V}$ defines a nondegenerate real-analytic structure ([**T9**]); (iii) $m = 1$ ([**CorH3**]).

We also mention a necessary condition for nondegenerate CR structures of arbitrary codimension proved in [**AFN**], which was extended to general locally integrable structures with additional nondegeneracy conditions in [**T5**].

The notion of intersection number and the necessary condition given in Theorem VIII.11.4 is due to [**CorT1**].

As far as sufficiency is concerned, we point out the works by Kashiwara–Schapira ([**KaS**]) and Michel ([**Mi**]), which deal with locally integrable CR structures of codimension one and whose Levi form satisfies weaker nondegeneracy conditions.

Locally integrable structures with $m = 1$: for this class of locally integrable structures we have seen in Sections VIII.7 and VIII.8 that condition $(\mathrm{P})_0$ is necessary and (in the real-analytic category) sufficient for local solvability near the origin (*cf.* Corollary VIII.7.5 and Theorem VIII.10.2). This result can be generalized much more. Let us introduce, for each $q = 0, 1, \dots, n-1$, the following property:

$(\mathrm{P})_q$ *Given any open neighborhood V of the origin there is another such neighborhood $V' \subset V$ such that, for every regular value $z_0 \in \mathbb{C}$ of the map Z, either $Z^{-1}\{z_0\} \cap V' = \emptyset$ or else the homomorphism*

$$\tilde{H}_q(Z^{-1}\{z_0\} \cap V') \longrightarrow \tilde{H}_q(Z^{-1}\{z_0\} \cap V)$$

induced by the inclusion map

$$Z^{-1}\{z_0\} \cap V' \subset Z^{-1}\{z_0\} \cap V$$

vanishes identically. Here $\tilde{H}_*$ denotes the reduced homology with complex coefficients.

In 1981 F. Treves proposed the following conjecture: *local solvability near the origin holds for $\mathcal{V}$ if and only if property* $(\mathrm{P})_{q-1}$ *is verified.* Several articles were published towards its verification; see [**MenT**], [**CorH1**], [**CorH2**], [**CorT3**], [**ChT**]. The complete proof of the conjecture was finally achieved in [**CorH3**]. The main point in the proof of Theorem VIII.10.2 that we presented is the use of the special covering (VIII.48), an idea inspired by the work [**H10**].

Solvability in top degree: one of the main questions in the theory is how to generalize condition $(\mathrm{P})_q$ in order to state a plausible conjecture for local solvability for general locally integrable systems. Observe that when $m \geq 2$ the sets $Z^{-1}\{z_0\}$ no longer carry enough information: for instance, in the CR case they are reduced to points.

There is one particular situation where a conjecture can be stated and at least verified in some particular but important cases: this is when $q = n$ (local solvability in maximum degree). Returning to the notation established at the beginning of this chapter, in particular to the vector fields (VIII.4), the equation under study is now

$$\sum_{j=1}^{n} L_j u_j = f, \qquad \text{(VIII.66)}$$

where no compatibility condition occurs. This makes this case, in some sense, the closest to the single equation situation.

Before we introduce the solvability condition for (VIII.66) we introduce the following definition: a real-valued function F defined on a topological space X is said to *assume a local minimum over a compact set $K \subset X$* if there exist $a \in \mathbb{R}$ and $K \subset V \subset X$ open such that $F = a$ on K and $F > a$ on $V \backslash K$.

DEFINITION VIII.10.3. *We shall say that $\mathcal{V}$ satisfies condition $(\mathfrak{P})_{n-1}$ near the origin if there is an open neighborhood $U \subset \Omega$ of the origin such that given any open set $V \subset U$ and given any $h \in C^\infty(V)$ satisfying $L_j h = 0$, $j = 1, \dots, n$, then $\Re h$ does not assume a local minimum over any nonempty compact subset of V.*

By using a classical device due to Lars Hörmander [**H6**], it was proved in [**CorH1**] that local solvability near the origin for (VIII.66) implies condition $(\mathfrak{P}_{n-1})$. This result would be of limited importance if no evidence that $(\mathfrak{P}_{n-1})$ is also a sufficient condition could be presented. This however is not the case, as the discussion that follows will show, and we can even conjecture at this point that *local solvability of (VIII.66) near the origin is equivalent to* $(\mathfrak{P}_{n-1})$.

When $n = 1$ condition $(\mathfrak{P}_0)$ is equivalent to the Nirenberg–Treves condition $(\mathcal{P})$: this result was proved in [**CorH1**] in the analytic category and in the general case in [**T3**]. When $m = 1$ condition $(\mathfrak{P}_{n-1})$ is equivalent to condition $(\mathrm{P})_{n-1}$ ([**CorH1**], [**T3**]). Thus, in these extreme cases, $(\mathfrak{P}_{n-1})$ unifies both known solvability conditions.

Let us pause here to discuss again the case when $m = 1$. The proof of the Treves conjecture in top degree as presented in [**CorH2**] is obtained by proving that $(\mathrm{P})_{n-1}$ implies, when $n \geq 2$, the following property: *there are an open neighborhood U of the origin and constants $C > 0$ and $k \in \mathbb{Z}_+$ such that*

$$\|\phi\|_\infty \leq C \sum_{j=1}^{n} \sum_{|\alpha| \leq k} \|D^\alpha L_j \phi\|_\infty, \quad \phi \in C_c^\infty(U). \tag{VIII.67}$$

Indeed k can be taken any integer $\geq [n/2] + 1$ and equal to zero when the structure is real-analytic ([**CorH1**]). The completion of the argument is quite standard, and holds whatever the value of m: by applying the Hahn–Banach theorem it is easily seen that (VIII.67) implies the existence of weak solutions to (VIII.66), and a general result due to [**T5**] proves the existence of smooth solutions.

For the tube structures it is not difficult to prove that property $(\mathfrak{P}_{n-1})$ implies (VIII.67) and consequently the preceding discussion shows that our conjecture is also satisfied for this particular class.

When $\mathcal{V}$ defines a CR structure of codimension one then condition $(\mathfrak{P}_{n-1})$ is equivalent to the existence of an open neighborhood U of the origin such that at every characteristic point over U the Levi form is not definite. In this case a partial answer was given in [**Mi**], where the existence of *hyperfunction* solutions is proved.

Finally we mention another general class of locally integrable structures that satisfy condition ($\mathfrak{P}_{n-1}$): these are the hypocomplex structures (*cf*. Definition VIII.5.4). For hypocomplex structures it is not still known if ($\mathfrak{P}_{n-1}$) implies the local solvability of (VIII.66). Neverthless, again in this case we can find hyperfunction solutions, as a consequence of more general results proved in [**CorTr**].