On the Arithmetic and Geometric Means Inequality.

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Let a and b be positive and unequal; and let x be their arithmetic mean. Then we have the following equality and inequality

$$a - x = x - b$$

$$\frac{a}{x} < \frac{x}{b}.$$

The inequality is tantamount to the fact that the G.M. of a and b is less than the A.M.

Instead of the single arithmetic mean, let us consider $p+q-1$ arithmetic means inserted between a and b. Let these be $x_1, x_2, \ldots x_{p+q-1}$. Then we have by the above

$$a - x_1 = x_1 - x_2 = \ldots = x_{p+q-1} - b \qquad (1)$$

and

$$\frac{a}{x_1} < \frac{x_1}{x_2} < \ldots\ldots\ldots < \frac{x_{p+q-1}}{b}. \qquad (2)$$

(1) and (2) each contain $p+q$ members. The A.M. of the first q members in (1) is equal to the A.M. of the remaining p members, since all the members are equal. Hence

$$\frac{a - x_q}{q} = \frac{x_q - b}{p}.$$

The G.M. of the first q members in (2) is less than the G.M. of the remaining p members, since each of the former is less than each of the latter. Hence

$$\left(\frac{a}{x_q}\right)^{\frac{1}{q}} < \left(\frac{x_q}{b}\right)^{\frac{1}{p}}. \qquad (4)$$

From (3) we get $x_q = (pa+qb)/(p+q)$ and from (4) $x_q > (a^p b^q)^{1/(p+q)}$. Hence we have

$$\frac{pa+qb}{p+q} > (a^p b^q)^{\frac{1}{p+q}}. \qquad (5)$$

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Having proved this, we could now generalise and show that if $p, q, \ldots t$ be any n positive rational numbers and $a, b, \ldots k$ any n positive quantities, not all equal, then

$$\frac{pa+qb+\ldots+tk}{p+q+\ldots+t} > (a^p b^q \ldots k^t)^{\frac{1}{p+q+\ldots+t}}.$$

If y be the geometric mean of a and b we have

$$\frac{a}{y}=\frac{y}{b}, \quad a-y>y-b.$$

Starting from this equality and inequality and proceeding in like manner, we get result (5). In this way we have another demonstration of the arithmetic and geometric means inequality.

The trick employed here is of a more general application. The following inequalities, for example, can be deduced in this way from the particular cases of them when $n=2$, in which case they are easily shown to be true.

(i) If A, B, ... K be n positive acute angles, not all equal, then

$$(1) \quad \frac{\sin A+\sin B+\ldots+\sin K}{n} < \sin\frac{A+B+\ldots+K}{n}$$

$$(2) \quad \frac{\cos A+\cos B+\ldots+\cos K}{n} < \cos\frac{A+B+\ldots+K}{n}$$

$$(3) \quad \frac{\tan A+\tan B+\ldots+\tan K}{n} > \tan\frac{A+B+\ldots+K}{n}.$$

(ii) If A, B, ... K be n positive angles, each less than half a right angle, and not all equal, then

$$\tan\frac{A+B+\ldots+K}{n} > \sqrt[n]{\tan A \,.\, \tan B \ldots \tan K}.$$

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