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D-modules on spaces of rational maps

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ABSTRACT

Let X be an algebraic curve. We study the problem of parametrizing geometric structures over X which are only generically defined. For example, parametrizing generically defined maps (rational maps) from X to a fixed target scheme Y. There are three methods for constructing functors of points for such moduli problems (all originally due to Drinfeld), and we show that the resulting functors are equivalent in the fppf Grothendieck topology. As an application, we obtain three presentations for the category of D-modules 'on' $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$, and we combine results about this category coming from the different presentations.

Contents

1. Introduction

Let k be an algebraically closed field of characteristic 0, and let X be a smooth, projective and connected algebraic curve over k. Denote by $\mathbb{A}$, $\mathbb{O}$ and K the algebra of adeles, algebra of integer adeles, and the field of rational functions over X, respectively.

In this paper we study the problem of parametrizing geometric structures over X which are only generically defined. The basic example of such a moduli problem is that of generically defined maps (rational maps) from X to a fixed target scheme Y. That is, the starting point is the given set of k-points (in this case it is the set $\mathrm{Hom}(\mathrm{spec}(K), Y)$) and the task at hand is that of constructing a functor of points $\mathbf{Scheme}^{\mathrm{op}} \to \mathbf{Set}$ which describes what is an S-family of such generic maps.

The main example in which we are interested is motivated by the Langlands program. In the classical setting, one encounters sets such as $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$, $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})$ and their relatives. The premise of the geometric program is that these sets are the k-points of some space ('space' interpreted very loosely). The story goes that each point in the set $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$ is to be interpreted as representing a G-bundle on X, together with the data of a reduction to B at the generic point of X.[1] We wish to describe a space parametrizing such data via a functor of points, and as above our starting point is the given set of k-points, and our task is to define what is an S-family of such generic reductions.

The literature (and mathematical folklore) contains three, *a priori* different, constructions of functors of points for such moduli problems (all originally due to Drinfeld). The main theme of this paper is that all three constructions give rise to functors of points which are equivalent in the fppf Grothendieck topology. Consequently, their categories of D-modules are equivalent, as are derivative invariants such as homology groups.

The main result of this paper is Theorem 6.2.4 which is an application of the results in the body of the paper to the geometrization of the set $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$. This space is equipped with a map to Bun_G, and we prove that it has a certain proper-like property (it is not schematic) and that it has homologically contractible fibers.

1.1 Overview

1.1.1 In §2 we present the first construction schema, which we consider to be the conceptually fundamental one. In particular, we construct the presheaves $\mathrm{GMap}(X, Y)$ and $\mathrm{Bun}_G^{H\,(\mathrm{gen})}$ which are the functors of points of the space which classifies rational maps from X to a fixed scheme Y, and of the space which classifies G-bundles on X with a generically defined reduction to a subgroup $H \subseteq G$, respectively. Unfortunately, conceptual appeal notwithstanding, this approach is deficient in the sense that invariants of the spaces so constructed are not easy to describe (directly).

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[1] This is admittedly equivalent to the set of global reductions to B, but this interpretation leads to a different space! See also Remark 2.2.4.

1.1.2 In §§ 3 and 4 we describe an approach for classifying generic maps as degenerations of regular maps. This approach uses the notion of a *quasi-map* and Drinfeld's $\overline{\mathrm{Bun}}_B$ which were first presented by Finkelberg and Mirkovic in [FM99], and have received a fair amount of attention since. In particular, the construction we present is the one used by Gaitsgory in [Gai10].

For a locally closed subscheme $Y \hookrightarrow \mathbb{P}^n$, the space of quasi-maps $\mathrm{QMap}(X, Y)$ is a scheme. The main result of § 3 is given in the following proposition.

PROPOSITION 3.2.2. *The presheaf* $\mathrm{GMap}(X, Y)$ *is a quotient of the scheme* $\mathrm{QMap}(X, Y)$ *by a schematic and proper equivalence relation (up to Zariski sheafification).*

A pleasing corollary of this result is that, while the space of quasi-maps depends on the embedding $Y \hookrightarrow \mathbb{P}^n$, the quotient which yields the space of rational maps depends on the scheme Y alone.

As an application concerning the categories of D-modules we obtain the following corollary.

COROLLARY 3.2.4. *Consider the pullback functors*

$$\mathfrak{Dmod}(\mathrm{QMap}(X,Y)) \xleftarrow{\ f^!\ } \mathfrak{Dmod}(\mathrm{GMap}(X,Y)) \xleftarrow{\ t^!\ } \mathfrak{Dmod}(\mathrm{spec}(k)).$$

The functor $f^!$ *always admits a left adjoint ('!-push-forward'). When* $Y \hookrightarrow \mathbb{P}^n$ *is a closed embedding, the functor* $t^!$ *also admits a left adjoint.*

In § 4 we discuss a relative setup over Bun_G, involving $\overline{\mathrm{Bun}}_B$ and $\mathrm{Bun}_G^{B(\mathrm{gen})}$, with analogous results.

1.1.3 In § 5 we describe an approach for parametrizing generic data using the *Ran space*. This is the approach taken by Gaitsgory in [Gai10, Gai13]. It has the advantage that certain invariants of the spaces so constructed are amenable to computation. Namely, in [Gai13, Theorem 1.8.2] Gaitsgory computes the homology groups of the spaces of rational maps in certain cases (see Theorem 6.2.1 for the statement). The main results in this section are Proposition 5.3.2 and Theorem 5.2.1.

We construct a monad, $\mathcal{M}$, acting on the category of presheaves over the Ran space, $\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$. The essence of Proposition 5.3.2 is that the data of a module for $\mathcal{M}$ is equivalent to that of a moduli space which classifies generic data, as formulated in § 2. In particular, there exists a presheaf $\mathrm{GMap}(X, Y)_{\mathcal{R}\mathrm{an}_X} \in \mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$ classifying generic maps which is naturally a module for $\mathcal{M}$.

The main result of this section is a fully faithful embedding between categories of D-modules, which enables us to apply Gaitsgory's contractibility result to the functors defined in § 2. Namely, Gaitsgory's theorem applies to the presheaf $\mathrm{GMap}(X, Y)_{\mathcal{R}\mathrm{an}_X}$ (for certain choices of Y), and we prove the following theorem.

THEOREM 5.2.1. *There exists a natural map* $\mathrm{GMap}(X, Y)_{\mathcal{R}\mathrm{an}_X} \to \mathrm{GMap}(X, Y)$, *which induces a fully faithful pullback functor*

$$\mathfrak{Dmod}(\mathrm{GMap}(X,Y)_{\mathcal{R}\mathrm{an}_X}) \leftarrow \mathfrak{Dmod}(\mathrm{GMap}(X,Y))$$

on the categories of D-modules. In particular, this map induces an isomorphism on homology groups.

This theorem is a consequence of the structure of the monad $\mathcal{M}$ which acts on the category $\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$, and has little to do with the particular properties of generic maps. The statement in § 5 is given in greater generality.

1.1.4 In §6 we use Gaitsgory's initial homology computation for spaces of rational maps to obtain similar results for additional moduli problems not discussed in [Gai13]. The following theorem concerning $\mathrm{Bun}_G^{B(\mathrm{gen})}$, the space of G-bundles over X equipped with a generic reduction to the Borel subgroup, which is a geometrization of the set $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$, is the main result of the paper.

THEOREM 6.2.4. *There exists an adjunction (*!-forward, !-back*)*

$$\mathfrak{Dmod}(\mathrm{Bun}_G^{B(\mathrm{gen})}) \xrightleftharpoons[\text{!-back}]{\text{!-forward}} \mathfrak{Dmod}(\mathrm{Bun}_G)$$

Moreover, !-back *is fully faithful.*

This theorem is expected to play a role in the understanding of the geometric Eisenstein series functors, as well in the construction of an *extended Whittaker model* for D-modules on Bun_G, as proposed in [Gai10].

2. Moduli spaces of generic data

Notation 2.0.1. Let k be an algebraically closed field of characteristic 0, and let X be a smooth connected and projective curve over k. We denote by $\mathfrak{S}$ the category of finite type schemes over k, and by $\mathfrak{Aff}$ the category of finite type affine schemes over k.

By an ∞-*category* we mean an $(\infty, 1)$-category. We denote by $\mathbf{Cat}_\infty$ the ∞-category of (small) ∞-categories. We denote by $\mathbf{Set}$ and $\mathbf{Gpd}_\infty$ the full subcategories of sets and ∞-groupoids in $\mathbf{Cat}_\infty$, respectively.

For a category C, we let $\mathcal{P}shv(C)$ denote the ∞-category of presheaves, that is, functors $C^{\mathrm{op}} \to \mathbf{Gpd}_\infty$. In the particular case when $C = \mathfrak{Aff}$ we use the term *functor of points* to refer to a presheaf in $\mathcal{P}shv(\mathfrak{Aff})$. When C is equipped with a Grothendieck topology τ, we denote the corresponding ∞-category of sheaves by $\mathcal{S}hv(C; \tau)$ (or omit τ when it is obvious from the context).

If C is a category which has been constructed to classify certain data, we shall often denote an object of C by listing the data which it classifies (and we shall say that the data *presents* the object). For example, in Definition 2.1.1 below, we use the expression $(S, U_S \subseteq S \times X)$ to denote an object of the category Dom_X; it should be clear from the context what type of datum each term in the parenthesis refers to. When the data is required to satisfy certain conditions, these are implicitly assumed to hold and are not reflected in the notation.

2.1 Families of domains
In the interest of motivating Definition 2.1.1, consider the problem of constructing a moduli space of rational functions on X (i.e., generically defined maps to $\mathbb{A}^1$), $K_X : \mathfrak{Aff}^{\mathrm{op}} \to \mathbf{Set}$. An S-point of this functor, $f \in K_X(S)$ should be presented by a rational function on $S \times X$. Let $U \subseteq S \times X$ be the largest open subscheme on which f is defined. In order for K_X to be a functor, we must be able to pull back f along any map of schemes, $T \to S$. Requiring that these pullbacks be defined amounts to the condition that for every $s \in S$, U intersects the fiber $s \times X$. The following definition captures this property of the domain of f.

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DEFINITION 2.1.1. (i) An open subscheme $U_S \subseteq S \times X$ is *universally dense* (in X) if for every map of schemes $T \to S$ the open subscheme $U_T \subseteq T \times X$ formed by the Cartesian square

$$
\begin{array}{ccc}
U_T & \longrightarrow & U_S \\
\downarrow & & \downarrow \\
T \times X & \longrightarrow & S \times X
\end{array}
$$

is dense. It suffices to check this condition at closed points of S; we require that for every closed point $s \in S$, the open subscheme $U_s \subseteq X$ is dense.

(ii) Let Dom_X be the (ordinary) category whose objects are pairs $(S, U_S \subseteq S \times X)$ where $S \in \mathfrak{Aff}$, and $U_S \subseteq S \times X$ is universally dense.

A morphism $(S, U_S \subseteq S \times X) \to (T, U_T \subseteq T \times X)$ is a map of affine schemes $S \xrightarrow{f} T$ which induces a commutative diagram

$$
\begin{array}{ccc}
U_S & \longrightarrow & U_T \\
\downarrow & & \downarrow \\
S \times X & \xrightarrow{f \times \mathrm{id}_X} & T \times X
\end{array}
$$

There exists an evident functor

$$
\mathrm{Dom}_X \xrightarrow{\quad q \quad} \mathfrak{Aff}
$$

$$
(S, U_S \subseteq S \times X) \longmapsto S
$$

which is a Cartesian fibration, and whose fiber, $(\mathrm{Dom}_X)_S$, is the poset of universally dense subschemes in $S \times X$ (a full subcategory of all open subschemes of $S \times X$).

2.2 Abstract moduli spaces of generic data

Notation 2.2.1. A functor between small categories $C \xrightarrow{f} D$ induces, via pre-composition, a functor $\mathcal{P}shv(C) \xleftarrow{f^*} \mathcal{P}shv(D)$. f_* fits into an adjoint triple $(\mathrm{LKE}_f, f_*, \mathrm{RKE}_f)$, where LKE_f and RKE_f are defined on an object of $\mathcal{F} \in \mathcal{P}shv(C)$, by left and right Kan extensions (respectively) along f:

$$
\begin{array}{ccc}
C^{\mathrm{op}} & \xrightarrow{\mathcal{F}} & \mathbf{Gpd}_\infty \\
{\scriptstyle f} \downarrow & \nearrow & \\
D^{\mathrm{op}} & &
\end{array}
$$

The following definition formalizes what we mean by a moduli problem of generic data over X.

DEFINITION 2.2.2. The category of (abstract) *moduli spaces of generic data* is $\mathcal{P}shv(\mathrm{Dom}_X)$. For $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$, its *associated* functor of points is $\mathcal{F} := \mathrm{LKE}_q(\mathcal{F}_{\mathrm{Dom}_X}) \in \mathcal{P}shv(\mathfrak{Aff})$, where q denotes the fibration $\mathrm{Dom}_X \xrightarrow{q} \mathfrak{Aff}$.

Example 2.2.3. Rational functions form a moduli problem of generic data if we set $K_{X, \mathrm{Dom}_X} : (\mathrm{Dom}_X)^{\mathrm{op}} \to \mathrm{Sets}$ to be the functor

$$
(S, U_S \subseteq S \times X) \mapsto \mathrm{Hom}_{\mathfrak{S}}(U_S, \mathbb{A}^1).
$$

Its associated functor of points, $K_X := \mathrm{LKE}_q(K_{X,\mathrm{Dom}_X}) : \mathfrak{Aff}^{\mathrm{op}} \to \mathbf{Set}$, sends

$$S \mapsto \{f \in K(S \times X) : \text{the domain of } f \text{ is universally dense in } X\}$$

where $K(S \times X)$ is the algebra of global sections of the sheaf of total quotients of $\mathcal{O}_{S \times X}$.

Replacing $\mathbb{A}^1$ with an arbitrary target scheme Y, we obtain similarly constructed presheaves classifying generically defined maps from X to Y:

$$\mathrm{GMap}\,(X,Y)_{\mathrm{Dom}_X} : \mathrm{Dom}_X^{\mathrm{op}} \to \mathbf{Set}$$

and its associated functor of points, which we denote

$$\mathrm{GMap}\,(X,Y) : \mathfrak{Aff}^{\mathrm{op}} \to \mathbf{Set}.$$

Remark 2.2.4. Let Y be a projective scheme, and let $\mathrm{Map}(X,Y)$ denote the functor of points which parametrizes families of regular maps. Since X is a curve and Y is projective, every generically defined map from X to Y admits a (unique) extension to a regular map defined across all of X. However, this is no longer the case in families, and consequently the map $\mathrm{Map}(X,Y) \to \mathrm{GMap}\,(X,Y)$ is not an equivalence, despite inducing an isomorphism on the set of k-points. For example, when $Y = \mathbb{P}^1$ the functor $\mathrm{Map}(X,\mathbb{P}^1)$ has infinitely many components (labeled the degree of the map), but $\mathrm{GMap}\,(X,\mathbb{P}^1)$ is connected.

Example 2.2.5 (Reduction spaces). Let $\mathrm{Bun}_G^{B(\mathrm{Dom}_X)} : (\mathrm{Dom}_X)^{\mathrm{op}} \to \mathbf{Gpd}$ be the functor which sends $(S, U_S \subseteq S \times X)$ to the groupoid which classifies the data

$$\left(\mathcal{P}_G, \mathcal{P}_B^{U_S}, \mathcal{P}_B^{U_S} \times_B G \xrightarrow[\cong]{\phi} \mathcal{P}_G\big|_{U_S}\right)$$

where $\mathcal{P}_G$ is a G-torsor over $S \times X$, $\mathcal{P}_B^{U_S}$ is a B-torsor on U_S, and ϕ is an isomorphism of G-bundles over U_S (the data of a reduction of the structure group of $\mathcal{P}_G|_U$ to B). Denote its associated functor of points by

$$\mathrm{Bun}_G^{B(\mathrm{gen})} := \mathrm{LKE}_q(\mathrm{Bun}_G^{B(\mathrm{Dom}_X)}) \in \mathcal{P}shv(\mathfrak{Aff}).$$

The latter is a geometrization of the set $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$. That is, the isomorphism classes of the groupoid $\mathrm{Bun}_G^{B(\mathrm{gen})}(k)$ are in bijection with this set.

More generally, if H is any subgroup of G, we define in a similar way a functor of points $\mathrm{Bun}_G^{H(\mathrm{gen})}$, which classifies families of G-bundles on X with a generically defined reduction to H. In particular, $\mathrm{Bun}_G^{1(\mathrm{gen})}$ is the moduli space of G-bundles equipped with a generic trivialization (equivalently, a generic section).

Notation 2.2.6. We denote objects of $\mathcal{P}shv(\mathrm{Dom}_X)$ using a subscript as in $\mathcal{F}_{\mathrm{Dom}_X}$. Given $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$, we remove the subscript to denote $\mathcal{F} := \mathrm{LKE}_q(\mathcal{F}_{\mathrm{Dom}_X}) \in \mathcal{P}shv(\mathfrak{Aff})$ – the associated functor of points.

Remark 2.2.7. Given a presheaf $(\mathrm{Dom}_X)^{\mathrm{op}} \xrightarrow{\mathcal{F}_{\mathrm{Dom}_X}} \mathbf{Gpd}_\infty$, the points of $\mathcal{F}$ can be computed as follows:

$$
\begin{array}{ccc}
(\mathrm{Dom}_X)^{\mathrm{op}} & \xrightarrow{\mathcal{F}_{\mathrm{Dom}_X}} & \mathbf{Gpd}_\infty \\
{\scriptstyle q^{\mathrm{op}}}\Big\downarrow & \nearrow & \\
\mathfrak{Aff}^{\mathrm{op}} & {\scriptstyle \mathcal{F}=\mathrm{LKE}_q(\mathcal{F}_{\mathrm{Dom}_X})} &
\end{array}
$$

The functor q^{op} is a co-Cartesian fibration, whence it follows that, for every $S \in \mathfrak{Aff}$,

$$\mathcal{F}(S) \cong \mathrm{colim}((\mathrm{Dom}_X)_S^{\mathrm{op}} \xrightarrow{\mathcal{F}_{\mathrm{Dom}_X}} \mathbf{Gpd}_\infty).$$

That is, the passage from $\mathcal{F}_{\mathrm{Dom}_X}$ to $\mathcal{F}$ simply identifies data which agrees on a smaller domain. Ultimately, the object we wish to study is $\mathcal{F} := \mathrm{LKE}_q(\mathcal{F}_{\mathrm{Dom}_X})$ and its invariants; $\mathcal{F}_{\mathrm{Dom}_X}$ itself is no more than a presentation of the former.

We record the following lemma for later use.

LEMMA 2.2.8. *The functor* $\mathcal{P}shv(\mathrm{Dom}_X) \xrightarrow{\mathrm{LKE}_q} \mathcal{P}shv(\mathfrak{Aff})$ *is a left exact localization.*

Proof. For every $S \in \mathfrak{Aff}$, the fiber category $(\mathrm{Dom}_X)_S$ is a filtered poset, hence weakly contractible. Thus for $\mathcal{G} \in \mathcal{P}shv(\mathfrak{Aff})$ the transformation $(\mathrm{LKE}_q \circ q_*)\mathcal{G} \xrightarrow{\cong} \mathcal{G}$ is an equivalence. That is, the functor $\mathcal{P}shv(\mathfrak{Aff}) \xrightarrow{q_*} \mathcal{P}shv(\mathrm{Dom}_X)$ is fully faithful, and the functor LKE_q is a localization. The left exactness of LKE_q follows from [Lur11b, Lemma 2.4.7] after observing that Dom_X admits all finite limits, and that the functor q preserves these. □

2.3 D-modules

For the most part the Langlands program is not as much interested in the set $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$ as it is in the space of functions on this set. The appropriate geometric counterpart of this space of functions should be a suitable category of sheaves on the space chosen as the geometrization of the set. When k is of characteristic 0, this category is expected to be the category of sheaves of D-modules. We now explain how to assign to every $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$ a category of sheaves of D-modules.

We denote by $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$ the ∞-category of stable ∞-categories which are cocomplete, together with colimit preserving functors (equivalently, are left adjoints). In [GR11, §2] Gaitsgory and Rozenblyum construct a functor

$$\mathcal{P}shv(\mathfrak{Aff})^{\mathrm{op}} \xrightarrow{\mathfrak{Dmod}} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$$

whose value on a scheme S is a stable ∞-category $\mathfrak{Dmod}(S)$ such that its homotopy category is the usual triangulated category of sheaves of D-modules on S. The functor $\mathfrak{Dmod}$ carries colimits of presheaves to limits of categories. In particular, for $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$ the natural functor

$$\mathfrak{Dmod}(\mathcal{F}) \to \lim_{S \to \mathcal{F}} \mathfrak{Dmod}(S)$$

is an equivalence.

Remark 2.3.1. For $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$, the category $\mathfrak{Dmod}(\mathcal{F})$ can be presented as a limit over the category $((\mathrm{Dom}_X)_{/\mathcal{F}_{\mathrm{Dom}_X}})^{\mathrm{op}}$

$$\mathfrak{Dmod}(\mathcal{F}) = \mathfrak{Dmod}(\mathrm{LKE}_q(\mathcal{F}_{\mathrm{Dom}_X})) \cong \lim_{(S,U) \to \mathcal{F}} (\mathfrak{Dmod}(S)).$$

The premise of this paper is that a presheaf on Dom_X is the conceptually natural way of classifying structures which are generically defined over X. However, in practice the limit presentation of $\mathfrak{Dmod}(\mathcal{F})$ we obtain as above is unwieldy. In the following sections we shall present more economical presentations of this category.

2.4 Grothendieck topologies on Dom_X

In [GR11, Corollary 3.1.4] it is proven that D-modules may be descended along fppf covers, that is, the functor $\mathfrak{Dmod}$ factors though fppf sheafification. For a presheaf $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$,

this 'continuity' property with respect to the fppf topology can be harnessed to obtain more economical presentations for the category $\mathfrak{Dmod}(\mathcal{F})$, than the one given in Remark 2.3.1. To this end we proceed to define a few natural Grothendieck topologies on Dom_X.

Let τ be the be either the Zariski, étale or fppf Grothendieck topology on $\mathfrak{S}$. We endow Dom_X with a corresponding Grothendieck pulled back from $\mathfrak{S}$ using the functor

$$\mathrm{Dom}_X \longrightarrow \mathfrak{S}$$

$$(S, U_S \subseteq S \times X) \longmapsto U_S.$$

Explicitly, a collection of morphisms in Dom_X, $\{(S_i, U_{S_i} \subseteq S_i \times X) \to (S, U_S \subseteq S \times X)\}$, is a τ-cover in Dom_X if and only if the collection of morphisms $\{U_{S_i} \to U_S\}$ is a τ-cover in $\mathfrak{S}$.

Observe that for all the choices of τ above, the functor $\mathrm{Dom}_X \xrightarrow{q} \mathfrak{Aff}$ is continuous in the sense that every for cover $\{(S_i, U_{S_i} \subseteq S_i \times X) \to (S, U_S \subseteq S \times X)\}$ in Dom_X, its image in $\mathfrak{Aff}$, $\{S_i \to S\}$, is a cover (this follows from the observation that $U_S \to S$ is a cover in all our topologies). Furthermore, Dom_X and $\mathfrak{Aff}$ both admit all finite limits, and q preserves these. By [Lur11b, Lemma 2.4.7], q induces an adjoint pair of functors, $\mathcal{S}hv(\mathrm{Dom}_X) \xleftarrow{(q^*, q_*)} \mathcal{S}hv(\mathfrak{Aff})$ in which the functor q_* is pullback along q, and the functor q^* is the composition of LKE_q followed by sheafification. The functor q^* preserves finite limits.[2]

3. Quasi-maps

Recall Example 2.2.3, in which we constructed a moduli problem $\mathrm{GMap}\,(X, Y)$, classifying generically defined maps from X to Y. In this section we present another approach to constructing a functor of points for this moduli problem using the notion of *quasi-maps.* The latter notion is originally due to Drinfeld, was first described by Finkelberg and Mirkovic in [FM99], and has received a fair amount of attention since. This approach has the advantage of presenting the space of generic maps as a quotient of a scheme by a proper (and schematic) equivalence relation.

The main result of this section is to prove that, up to sheafification in the Zariski topology, both approaches give equivalent functors of points. Consequently, the associated categories of D-modules are equivalent.

For the duration of this section fix $Y \hookrightarrow \mathbb{P}^n$, a scheme Y *together with* the data of a quasi-projective embedding. The space of generic maps constructed using quasi-maps *a priori* might depend on this embedding, however it follows from the equivalence with $\mathrm{GMap}\,(X, Y)$ which we prove that, in fact, it does not (up to sheafification).

3.1 Definitions

First, a minor matter of terminology. Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles over a scheme. We distinguish between two properties of a map of quasi-coherent sheaves $\mathcal{V} \to \mathcal{W}$: The map is called a *subsheaf embedding* if it is an injective map of quasi-coherent sheaves. It is called a *subbundle embedding* if it is an injective map of sheaves whose cokernel is flat (i.e., also a vector bundle). The latter corresponds to the notion of a map between geometric vector bundles which is (fiberwise) injective.

We start by defining the notion of a *quasi-map* from X to $\mathbb{P}^n$. Recall that that a regular map $X \to \mathbb{P}^n$ classifies the data of a line bundle $\mathcal{L}$ on X together with a subbundle embedding

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[2] That is, it is a geometric morphism of topoi.

$\mathcal{L} \xrightarrow{\subseteq} \mathcal{O}_X^{n+1}$. A quasi-map from X to $\mathbb{P}^n$ is a degeneration of a regular map consisting of the data of a line bundle $\mathcal{L}$ on X, together with a subsheaf embedding $\mathcal{L} \hookrightarrow \mathcal{O}_X^{n+1}$ (i.e., it may not be a subbundle). Observe that to any quasi-map we may associate the open subscheme $U \subseteq X$ over which $\mathcal{L}|_U \hookrightarrow \mathcal{O}_U^{n+1}$ is a subbundle determining a regular map $U \to \mathbb{P}^n$. In particular, to every quasi-map we may associate a generically defined map from X to $\mathbb{P}^n$.[3]

DEFINITION 3.1.1. Let $\mathrm{QMap}\,(X, \mathbb{P}^n) : \mathfrak{Aff}^{\mathrm{op}} \to \mathbf{Set}$ be the functor of points whose S-points are presented by the data $(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}^{n+1})$, where $\mathcal{L}$ is a line bundle over $S \times X$, and $\mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}^{n+1}$ is an injection of quasi-coherent sheaves, whose cokernel is S-flat.

If $Y \hookrightarrow \mathbb{P}^n$ is a locally closed subscheme, then a quasi-map from X to Y should be given by the data of a quasi-map from X to $\mathbb{P}^n$, with the additional property that the generic point of X maps to Y. We proceed to define this notion in a way better suited for families.

In the case when $Y \overset{f}{\hookrightarrow} \mathbb{P}^n$ is a closed subscheme, it is defined by a graded ideal $I_Y \subseteq k[x_0, \dots, x_n]$. A regular map $X \xrightarrow{f} \mathbb{P}^n$, presented by the data of a subbundle $\mathcal{L} \subseteq \mathcal{O}_X^{n+1}$, lands in Y if and only if the composition

$$\mathrm{Sym}_X \mathcal{L}^\vee \leftarrow \mathrm{Sym}_X \mathcal{O}_X^{n+1} \cong \mathcal{O}_X \otimes k[x_0, \dots, x_n] \leftarrow \mathcal{O}_X \otimes I_Y$$

vanishes. We degenerate the subbundle requirement to obtain the notion of a quasi-map into Y.

DEFINITION 3.1.2. When $Y \hookrightarrow \mathbb{P}^n$ is closed embedding, we define $\mathrm{QMap}\,(X, Y)$ to be the subfunctor of $\mathrm{QMap}\,(X, \mathbb{P}^n)$ consisting of those points classifying the data $(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}^{n+1})$ such that the composition

$$\mathrm{Sym}_{S \times X} \mathcal{L}^\vee \leftarrow \mathrm{Sym}_{S \times X} \mathcal{O}_X^{n+1} \leftarrow \mathcal{O}_{S \times X} \otimes I_Y$$

vanishes.[4]

We emphasize that the definition of $\mathrm{QMap}\,(X, Y)$ depends on the embedding $Y \hookrightarrow \mathbb{P}^n$, and not on Y alone.

The following lemma is well known (see, for example, [FM99, Lemma 3.3.1]).

LEMMA 3.1.3. $\mathrm{QMap}\,(X, \mathbb{P}^n)$ is representable by a scheme, which is moreover a disjoint (infinite) union of projective schemes.

If $Y \hookrightarrow \mathbb{P}^n$ is a projective embedding, then $\mathrm{QMap}\,(X, Y) \to \mathrm{QMap}\,(X, \mathbb{P}^n)$ is a closed embedding.

DEFINITION 3.1.4. If $U \subseteq \mathbb{P}^n$ is an open subscheme then we define

$$\mathrm{QMap}\,(X, U) \subseteq \mathrm{QMap}\,(X, \mathbb{P}^n)$$

to be the open subscheme which is the complement of $\mathrm{QMap}\,(X, \mathbb{P}^n \backslash U)$ (this is independent of the closed subscheme structure given to $\mathbb{P}^n \backslash U$).

Finally, if $Y \hookrightarrow \mathbb{P}^n$ is an arbitrary locally closed subscheme we define

$$\mathrm{QMap}\,(X, Y) = \mathrm{QMap}\,(X, \overline{Y}) \cap \mathrm{QMap}\,(X, \mathbb{P}^n \backslash (\overline{Y} \backslash Y)).$$

It is a locally closed subscheme of $\mathrm{QMap}\,(X, \mathbb{P}^n)$.

---

[3] Such a map of course admits an extension to a regular map (in terms of bundles, every invertible subbundle $\mathcal{L} \xrightarrow{\phi} \mathcal{O}_X^{n+1}$ extends to a line subbundle $\mathcal{L} \hookrightarrow (\mathrm{Im}(\phi^\vee))^\vee \subseteq \mathcal{O}_X^{n+1})$, but see Remark 2.2.4.

[4] The definition could have been given more economically, by replacing the entire symmetric algebras with their finite-dimensional subspaces containing generators of I_Y.

We point out that a map $S \to \mathrm{QMap}\,(X, \overline{Y})$ lands in the open subscheme $\mathrm{QMap}\,(X, Y)$ if and only if for every geometric point $s \in S(k)$, the corresponding quasi-map carries the generic point of X into Y.

3.1.5 In § 4 we shall need to replace $\mathrm{QMap}\,(X, Y)$ with a relative and twisted version, $\mathrm{QSect}_S\,(S \times X, Y)$, corresponding to a scheme Y over $S \times X$. The details are given at the end of § 3.3.

3.2 Degenerate extensions of generic maps

Recall the presheaves $\mathrm{GMap}\,(X, Y)$ and $\mathrm{GMap}\,(X, Y)_{\mathrm{Dom}_X}$ introduced in Example 2.2.3. There is an evident map

$$\mathrm{QMap}\,(X, Y) \to \mathrm{GMap}\,(X, Y)$$

via which we think of every quasi-map as presenting a generic map. Namely, for every $S \in \mathfrak{Aff}$ it is given by the composition

$$\mathrm{QMap}\,(X, Y)\,(S) \to \coprod_{(S, U) \in \mathrm{Dom}_X} \mathrm{GMap}\,(X, Y)_{\mathrm{Dom}_X}\,(S, U) \to \mathrm{GMap}\,(X, Y)\,(S)$$

where the first map is given by sending a quasi-map presented by $\mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}^{n+1}$ to the open subscheme $U \subseteq S \times X$ where the cokernel is flat, and the regular map it defines on U. However, there is some redundancy in the presentation because a generic map may be presented by several different quasi-maps. We introduce the equivalence relation $\mathcal{E}_Y \subseteq \mathrm{QMap}\,(X, Y) \times \mathrm{QMap}\,(X, Y)$ to be the subfunctor whose S-points are presented by those pairs

$$((\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}^{n+1}), (\mathcal{L}', \mathcal{L}' \hookrightarrow \mathcal{O}_{S \times X}^{n+1})) \in (\mathrm{QMap}\,(X, Y) \times \mathrm{QMap}\,(X, Y))(S)$$

which agree over the intersection of their regularity domains. Observe that the square

$$\begin{array}{ccc} \mathcal{E}_Y & \longrightarrow & \mathrm{QMap}\,(X, Y) \\ \downarrow & & \downarrow \\ \mathrm{QMap}\,(X, Y) & \longrightarrow & \mathrm{GMap}\,(X, Y) \end{array}$$

is Cartesian. The following lemma is well known; we add a proof for completeness.

LEMMA 3.2.1. *The equivalence relation $\mathcal{E}_Y \to \mathrm{QMap}\,(X, Y) \times \mathrm{QMap}\,(X, Y)$ is (representable by) a closed subscheme. The space $\mathcal{E}_Y$ has countably many connected components and the restriction of either projection, $\mathcal{E}_Y \to \mathrm{QMap}\,(X, Y)$, to every component of $\mathcal{E}_Y$ is proper.*

Proof. Let us first consider the case $Y = \mathbb{P}^n$, and show that the subfunctor

$$\mathcal{E}_{\mathbb{P}^n} \to \mathrm{QMap}\,(X, \mathbb{P}^n) \times \mathrm{QMap}\,(X, \mathbb{P}^n)$$

is a closed embedding.

We start by examining when two quasi-maps $S \xrightarrow{\phi, \psi} \mathrm{QMap}\,(X, \mathbb{P}^n)$ are generically equivalent, that is, map to the same S-point of $\mathrm{GMap}\,(X, \mathbb{P}^n)$. Let ϕ and ψ be presented by invertible subsheaves

$$\mathcal{L}_\phi \xhookrightarrow{\kappa_\phi} \mathcal{O}_{S \times X}^{n+1} \quad \text{and} \quad \mathcal{L}_\psi \xhookrightarrow{\kappa_\psi} \mathcal{O}_{S \times X}^{n+1}$$

whose cokernels are S-flat. Let $U \subseteq S \times X$ be the open subscheme where $\mathcal{L}_\phi|_U \hookrightarrow \mathcal{O}_U^{n+1}$ is subbundle, and thus a maximal invertible subbundle. The points ϕ and ψ are generically equivalent if and only if $\mathcal{L}_\psi|_U$ is a subsheaf of $\mathcal{L}_\phi|_U$ (both viewed as subsheaves of $\mathcal{O}_U^{n+1}$).

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Fix a vector bundle, $\mathcal{M}$, on $S \times X$ whose dual surjects on the kernel as indicated below:

$$\mathcal{L}_\phi^\vee \xleftarrow{\ \kappa_\phi^\vee\ } (\mathcal{O}_{S \times X}^{n+1})^\vee \hookleftarrow \mathrm{Ker}(\kappa_\phi^\vee) \twoheadleftarrow \mathcal{M}^\vee.$$

Dualizing and restricting to U, we have

$$
\begin{array}{c}
\mathcal{L}_\psi|_U \\
\downarrow \\
\mathcal{L}_\phi|_U \xrightarrow[\kappa_\phi]{\subseteq} \mathcal{O}_{U_0}^{n+1} \longrightarrow \mathrm{coker}(\kappa_\phi) \xrightarrow{\subseteq} \mathcal{M}|_U
\end{array}
$$

where map $\mathrm{coker}(\kappa_\phi) \to \mathcal{M}$ is injective (in fact a subbundle). Thus, ϕ and ψ are generically equivalent if and only if the composition $\mathcal{L}_\psi|_U \to \mathcal{M}|_U$ vanishes on U if and only if $\mathcal{L}_\psi \to \mathcal{M}$ vanishes on all of $S \times X$ (since $U \subseteq S \times X$ is dense, and both sheaves are vector bundles).

For an arbitrary quasi-projective scheme, $Y \hookrightarrow \mathbb{P}^n$, the lemma now follows from the Cartesianity of the squares below, using the fact that both right vertical maps are proper when restricted to a connected component:

$$
\begin{array}{ccc}
\mathcal{E}_Y & \longrightarrow & \mathcal{E}_{\mathbb{P}^n} \\
\downarrow & & \downarrow \\
\mathrm{QMap}\,(X,Y) \times \mathrm{QMap}\,(X,\mathbb{P}^n) \longrightarrow & \mathrm{QMap}\,(X,\mathbb{P}^n) \times \mathrm{QMap}\,(X,\mathbb{P}^n) \\
\downarrow & & \downarrow{\scriptstyle \pi_1} \\
\mathrm{QMap}\,(X,Y) & \longrightarrow & \mathrm{QMap}\,(X,\mathbb{P}^n)
\end{array}
\qquad \square
$$

Denote the evident simplicial object in $\mathcal{P}shv(\mathfrak{Aff})$

$$
\cdots \ \substack{\longrightarrow \\ \vdots \\ \longrightarrow}\ \mathcal{E}_Y \times_{\mathrm{QMap}(X,Y)} \mathcal{E}_Y \ \substack{\longrightarrow \\ \longleftrightarrow \\ \longrightarrow}\ \mathcal{E}_Y \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\ \mathrm{QMap}\,(X,Y) \tag{3.1}
$$

by

$$\Delta^{\mathrm{op}} \xrightarrow{\ \mathcal{E}_Y^\bullet\ } \mathcal{P}shv(\mathfrak{Aff})$$

$$[n] \longmapsto \mathcal{E}_Y^{(n)}$$

where

$$\mathcal{E}_Y^{(n)} := \underbrace{\mathcal{E}_Y \times_{\mathrm{QMap}(X,Y)} \cdots \times_{\mathrm{QMap}(X,Y)} \mathcal{E}_Y}_{n\text{-times}}.$$

We denote by $\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y$ the functor of points which is the quotient by this equivalence relation – the colimit of this simplicial object. However, in this case it simply reduces to the naive pointwise quotient of sets

$$(\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y)(S) = \mathrm{QMap}\,(X,Y)\,(S)/\mathcal{E}_Y(S)$$

because $\mathcal{E}_Y(S) \subseteq (\mathrm{QMap}\,(X,Y)\,(S))^{\times 2}$ is an equivalence relation in sets.

The functor of points $\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y$ presents another candidate for the 'space of generic maps', *a priori* different from $\mathrm{GMap}\,(X,Y)$. Relative to $\mathrm{GMap}\,(X,Y)$, it has the advantage of being concisely presented as the quotient of a scheme by a proper (schematic) equivalence relation. The following proposition shows that both functors are essentially equivalent, and in particular that (up to Zariski sheafification) $\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y$ is independent of the quasi-projective embedding $Y \hookrightarrow \mathbb{P}^n$.

Proposition 3.2.2. *The map* $\mathrm{QMap}\,(X, Y) \to \mathrm{GMap}\,(X, Y)$ *induces a map of presheaves*

$$\mathrm{QMap}\,(X, Y)\,/\mathcal{E}_Y \to \mathrm{GMap}\,(X, Y)$$

which becomes an equivalence after Zariski sheafification.

We prove this proposition, after some preparations, at the end of this subsection. First, a couple of consequences.

Corollary 3.2.3. *The Zariski sheafification of the presheaf* $\mathrm{QMap}\,(X, Y)\,/\mathcal{E}_Y$ *is independent of the quasi-projective embedding* $Y \hookrightarrow \mathbb{P}^n$.

The main invariant of $\mathrm{GMap}\,(X, Y)$ which we wish to study in this paper is homology, and by extension the category of D-modules (see § 6.1). The following corollary is to be interpreted as providing a convenient presentation of this category of D-modules, and using this presentation to deduce the existence of a de Rham cohomology functor (left adjoint to !-pullback).

Corollary 3.2.4. (i) *Pullback induces an equivalence*

$$\lim_{[n]\in\Delta^{\mathrm{op}}} \mathfrak{Dmod}(\mathcal{E}_Y^{(n)}) \xleftarrow{\cong} \mathfrak{Dmod}(\mathrm{GMap}\,(X, Y)).$$

(ii) *Consider the pullback functors*

$$\mathfrak{Dmod}(\mathrm{QMap}\,(X, Y)) \xleftarrow{f^!} \mathfrak{Dmod}(\mathrm{GMap}\,(X, Y)) \xleftarrow{t^!} \mathfrak{Dmod}(\mathrm{spec}(k)).$$

The functor $f^!$ *always admits a left adjoint ('!-push-forward'). When* $Y \hookrightarrow \mathbb{P}^n$ *is a closed embedding, the functor* $t^!$ *also admits a left adjoint.*

The second assertion above is a kind of 'properness' property of the (non-representable) map $\mathrm{QMap}\,(X, Y) \to \mathrm{GMap}\,(X, Y)$, and the functor of points $\mathrm{GMap}\,(X, Y)$ (when $Y \hookrightarrow \mathbb{P}^n$ is a closed embedding).

The following remark is not used in the rest of the paper. We point it out for future use.

Remark 3.2.5. Corollary 3.2.4 implies that $\mathfrak{Dmod}(\mathrm{GMap}\,(X, Y))$ is compactly generated. Namely, the push-forwards of compact generators of $\mathfrak{Dmod}(\mathrm{QMap}\,(X, Y))$ are generators because $f^!$ is faithful (as is evident from (1)), and are compact because $f^!$ is colimit preserving.

Proof. (1) is an immediate consequence of Proposition 3.2.2.

Regarding (2), it follows from Lemma 3.2.1 that all the maps in $\mathcal{E}_Y^\bullet$ are proper on each component, hence, on the level of D-module categories, each pullback functor admits a left adjoint (a '!-push-forward'). Consequently, the object assignment

$$[n] \in \Delta \mapsto \mathfrak{Dmod}(\mathcal{E}_Y^n) \in \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$$

extends to both a cosimplicial diagram (implicit in the statement of (1)) and a simplicial diagram. In the former, which we denote

$$\mathfrak{Dmod}^!(\mathcal{E}_Y^\bullet) : \Delta \to \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}},$$

functors are given by pullback. In the latter, which we denote

$$\mathfrak{Dmod}_!(\mathcal{E}_Y^\bullet) : \Delta^{\mathrm{op}} \to \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}},$$

the functors are given by the left adjoints to pullback (!-push-forward). When $Y \hookrightarrow \mathbb{P}^n$ is a closed embedding, each of the $\mathcal{E}_Y^{(n)}$ is proper on each component, hence the push-forward diagram is augmented over $\mathfrak{Dmod}(\operatorname{spec}(k))$.

Under the equivalence of (1), the functors whose adjoints we wish to construct are identified with

$$\mathfrak{Dmod}(\mathrm{QMap}\,(X,Y)) \leftarrow \lim_{\Delta^{\mathrm{op}}} \mathfrak{Dmod}^!(\mathcal{E}_Y^{\bullet}) \leftarrow \mathfrak{Dmod}(\operatorname{spec}(k)).$$

The setup above falls into the general framework of *adjoint diagrams* which we describe in the appendix (Lemma A.1.1). In this setup there exists an equivalence,

$$\operatorname*{colim}_{\Delta} \mathfrak{Dmod}_!(\mathcal{E}_Y^{\bullet}) \xrightarrow{\cong} \lim_{\Delta^{\mathrm{op}}} \mathfrak{Dmod}^!(\mathcal{E}_Y^{\bullet}).$$

Under this equivalence, the pair of natural functors

$$\mathfrak{Dmod}(\mathrm{QMap}\,(X,Y)) \leftarrow \lim_{\Delta^{\mathrm{op}}} \mathfrak{Dmod}^!(\mathcal{E}_Y^{\bullet}),$$
$$\mathfrak{Dmod}(\mathrm{QMap}\,(X,Y)) \rightarrow \operatorname*{colim}_{\Delta} \mathfrak{Dmod}_!(\mathcal{E}_Y^{\bullet})$$

are adjoint.

Likewise in the case when $Y \hookrightarrow \mathbb{P}^n$ is a closed embedding we conclude that

$$\lim_{\Delta^{\mathrm{op}}} \mathfrak{Dmod}^!(\mathcal{E}_Y^{\bullet}) \leftarrow \mathfrak{Dmod}(\operatorname{spec}(k)), \quad \operatorname*{colim}_{\Delta} \mathfrak{Dmod}_!(\mathcal{E}_Y^{\bullet}) \rightarrow \mathfrak{Dmod}(\operatorname{spec}(k))$$

are adjoint functors. □

We proceed with the preparations for the proof of Proposition 3.2.2.

3.2.6 *Divisor complements.* Recall that an effective Cartier divisor on a scheme Y is the data of a line bundle $\mathcal{L}$ together with an injection of coherent sheaves $\mathcal{L} \hookrightarrow \mathcal{O}_Y$. The complement of the support of $\mathcal{O}_Y/\mathcal{L}$ is an open subscheme, $U_\mathcal{L} \subseteq Y$. We call an open subscheme arising in this way a *divisor complement.*

LEMMA 3.2.7. *Let* $(S, U) \in \mathrm{Dom}_X$, *and let* $\mathcal{L}_U$ *be a line bundle on* $U \subseteq S \times X$. *There exists a finite Zariski cover*

$$\{(S_i, U_i) \rightarrow (S, U)\}_{i \in I}$$

such that for every i *the open subscheme* $U_i \subseteq S_i \times U_i$ *is a divisor complement. Moreover, we can choose each* U_i *so that* $\mathcal{L}_U|_{U_i}$ *is a trivial line bundle.*

Proof. Since $S \times X$ is quasi-projective, the topology of its underlying topological space is generated by divisor complements. Thus, we may cover U by a finite collection of open subschemes, $\{U_i\}_{i \in I}$, which trivialize $\mathcal{L}_U$, and such that each $U_i \subseteq S \times X$ is a divisor complement. Let $S_i \subseteq S$ be the open subscheme which is the image of $U_i \subseteq S \times X \rightarrow S$. Note that U_i might not be universally dense in $S \times X$, but that it is in $S_i \times X$. Also note that $\{(S_i, U_i) \rightarrow (S, U)\}_{i \in I}$ is a Zariski cover in Dom_X. □

LEMMA 3.2.8. *Let* $\mathcal{V}$ *be a vector bundle over* $S \times X$, *let* $\mathcal{L} \xhookrightarrow{\kappa} \mathcal{V}$ *be an invertible subsheaf, and let* $U \subseteq S \times X$ *be the open subscheme where* κ *is a subbundle embedding. Then, the following two conditions are equivalent:*

(i) *the coherent sheaf* $\mathcal{V}/\mathcal{L}$ *is* S-*flat;*
(ii) *the open subscheme* $U \subseteq S \times X$ *is universally dense relative to* S, *that is, The data* (S, U) *defines a point of* Dom_X.

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In particular, for an effective Cartier divisor, $\mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}$, the open subscheme $U_{\mathcal{L}} \subseteq S \times X$ determines an S-point of Dom_X if and only if the coherent sheaf $\mathcal{O}_{S \times X}/\mathcal{L}$ is S-flat.

Proof. Let p and j denote the maps $U \xrightarrow{j} S \times X \xrightarrow{p} S$. Both conditions may be tested on closed points of S. That is, it suffices to show that for every maximal sheaf of ideals $\mathcal{I}_s \subseteq \mathcal{O}_S$, corresponding to a closed point $s \in S$,

$$\mathrm{Tor}^1_S(\mathcal{V}/\mathcal{L}, \mathcal{I}_s) = 0 \quad \text{iff} \quad U \times_S \{s\} \neq \emptyset.$$

Indeed, $\mathrm{Tor}^1_S(\mathcal{V}/\mathcal{L}, \mathcal{I}_s)$ vanishes if and only if $\mathcal{L}|_{\{s\} \times X} \xrightarrow{\kappa|_{\{s\} \times X}} \mathcal{V}|_{\{s\} \times X}$ is injective if and only if $\kappa|_{\{s\} \times X} \neq 0$ if and only if $U \times_S \{s\} \neq \emptyset$. $\qquad\square$

The following lemma contains the geometric input for the proof of Proposition 3.2.2.

LEMMA 3.2.9. *Assume given:*

– $(S, U) \in \mathrm{Dom}_X$;
– $\mathcal{V}$, *a rank m vector bundle over $S \times X$;*
– $\mathcal{L}_U$, *a line bundle over U together with a subbundle embedding*

$$\mathcal{L}_U \xrightarrow{\kappa_U} \mathcal{V}|_U.$$

Then there exist:

– *a Zariski cover $(\tilde{S}, \tilde{U}) \xrightarrow{p} (S, U)$ in Dom_X;*
– *a line bundle $\mathcal{L}$ on $\tilde{S} \times X$ together with a subsheaf embedding $\mathcal{L} \xhookrightarrow{\kappa} \mathcal{V}|_{\tilde{S} \times X}$ whose cokernel is $\tilde{S}$-flat;*
– *an identification $\mathcal{L}|_{\tilde{U}} \cong \mathcal{L}_U|_{\tilde{U}}$ which exhibits κ as an extension of*

$$\mathcal{L}_U|_{\tilde{U}} \xrightarrow{\kappa_U|_{\tilde{U}}} \mathcal{V}|_{\tilde{U}}.$$

Above, we have used $(-)|_{\tilde{U}}$ to denote pullback along $\tilde{U} \to U$.

Proof. According to Lemma 3.2.7, we may find a Zariski cover in Dom_X, $(\tilde{S}, \tilde{U}) \to (S, U)$, such that:

– $\mathcal{L}_U|_{\tilde{U}}$ is a trivial line bundle;
– the open subscheme $\tilde{U} \subseteq \tilde{S} \times X$ is a divisor complement associated to a Cartier divisor $\mathcal{N} \hookrightarrow \mathcal{O}_{\tilde{S} \times X}$.

We proceed to show that the subbundle embedding

$$(*) \quad \mathcal{L}_U|_{\tilde{U}} \xrightarrow{\kappa_U|_{\tilde{U}}} \mathcal{V}|_{\tilde{U}}$$

admits a degenerate extension across $\tilde{S} \times X$. We point out that the line bundle $\mathcal{N}$ is trivialized over $\tilde{U}$, and we fix identifications $\mathcal{N}|_{\tilde{U}} = \mathcal{O}_{\tilde{U}} \cong \mathcal{L}_U|_{\tilde{U}}$. By a standard lemma in algebraic geometry [Har77, II.5.14], there exist an integer l and a map of coherent sheaves $\mathcal{N}^{\otimes l} \xrightarrow{\kappa} \mathcal{V}|_{\tilde{S} \times X}$ whose restriction to $\tilde{U}$ may be identified with $(*)$. By Lemma 3.2.8, $\mathrm{coker}(\kappa)$ is $\tilde{S}$-flat. $\qquad\square$
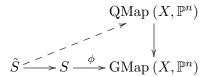
3.2.10 *Proof of Proposition 3.2.2.* The square

$$\begin{array}{ccc} \mathrm{QMap}\,(X,Y) & \longrightarrow & \mathrm{QMap}\,(X,\mathbb{P}^n) \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{QMap}\,(X,Y) & \longrightarrow & \mathrm{GMap}\,(X,\mathbb{P}^n) \end{array}$$

is Cartesian, hence it suffices to prove the proposition for $Y = \mathbb{P}^n$.

It suffices to fix an S-point, $S \to \mathrm{GMap}\,(X,\mathbb{P}^n)$, and show that there exist a Zariski cover $\tilde{S} \to S$, and a lift as indicated by the dashed arrow below:

$$\begin{array}{ccc} & & \mathrm{QMap}\,(X,\mathbb{P}^n) \\ & \nearrow & \downarrow \\ \tilde{S} \longrightarrow S & \xrightarrow{\phi} & \mathrm{GMap}\,(X,\mathbb{P}^n) \end{array}$$

Let ϕ be presented by the data of a point $(S,U) \in \mathrm{Dom}_X$, and a subbundle embedding $\mathcal{L}_U \xrightarrow[\subseteq]{\kappa_U} \mathcal{O}_U^{n+1}$ over U. Lemma 3.2.9 guarantees the existence of a cover $\tilde{S} \to S$, and an invertible subsheaf $\mathcal{L} \xhookrightarrow{\kappa} \mathcal{O}_{\tilde{S} \times X}^{n+1}$, which is an extension of $\kappa_U|_{\tilde{U}}$ to all of $\tilde{S} \times X$. The data associated with κ presents a map, $\tilde{S} \to \mathrm{QMap}\,(X,\mathbb{P}^n)$, which is the sought-after lift.

3.3 Quasi-sections

In the next section we shall need a relative and twisted generalization of the notion of quasi-map, which we now define. All the results in this section proven above could have been stated and proven in this more general setup (at the cost of encumbering the presentation).

Fix $S \in \mathfrak{Aff}$, and let $\mathcal{V}$ be a vector bundle on $S \times X$. Denote the relative projectivization by

$$\mathbb{P}(\mathcal{V}) := \mathrm{proj}_{S \times X}(\mathrm{sym}_{\mathcal{O}_{S \times X}} \mathcal{V}^\vee);$$

it is a locally projective scheme over $S \times X$. We define the space of quasi-sections of $\mathbb{P}(\mathcal{V}) \to S \times X$, relative to S, as in the following definition.

DEFINITION 3.3.1. (i) The functor

$$\mathrm{QSect}_S\,(S \times X, \mathbb{P}(\mathcal{V})) : \mathfrak{Aff}_{/S}^{\mathrm{op}} \to \mathbf{Set}$$

is defined to be the functor of points over S, whose T-points are presented by the data $(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{V}|_{T \times X})$, where $\mathcal{L}$ is a line bundle over $T \times X$, and $\mathcal{L} \hookrightarrow \mathcal{V}|_{T \times X}$ is an injection of quasi-coherent sheaves, whose cokernel is T-flat.

(ii) For a closed embedding $Y \hookrightarrow \mathbb{P}(\mathcal{V})$, defined by a graded sheaf of ideals $\mathcal{I}_Y \subseteq \mathrm{Sym}_{T \times X} \mathcal{V}^\vee$, we define

$$\mathrm{QSect}_S\,(S \times X, Y) \subseteq \mathrm{QSect}_S\,(S \times X, \mathbb{P}(\mathcal{V}))$$

to be the subfunctor consisting of those points presented by the data $(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{V}|_{T \times X})$ such that the composition

$$\mathrm{Sym}_{T \times X} \mathcal{L}^\vee \leftarrow \mathrm{Sym}_{T \times X} \mathcal{V}^\vee \leftarrow \mathcal{I}_Y$$

vanishes.

When $S = \mathrm{spec}(k)$ and $\mathcal{V} = \mathcal{O}_{S \times X}^{n+1}$, this definition reduces to $\mathrm{QMap}\,(X,Y)$.

As for the absolute version, there exists a map

$$\mathrm{QSect}_S\,(S \times X, Y) \to \mathrm{GSect}_S\,(S \times X, Y)$$

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and the counterpart of Proposition 3.2.2 holds. The proof is virtually identical (after adjusting notation), and is omitted.

PROPOSITION 3.3.2. *The map* $\mathrm{QSect}_S(S \times X, Y) \to \mathrm{GSect}_S(S \times X, Y)$ *induces a map of presheaves*

$$\mathrm{QSect}_S([, S) \times X][Y]/\mathcal{E}_Y \to \mathrm{GSect}_S(S \times X, Y)$$

which becomes an equivalence after Zariski sheafification.

4. Drinfeld's compactification of Bun_B

Recall Example 2.2.5, in which we introduced a moduli problem of generic data $\mathrm{Bun}_G^{B(\mathrm{Dom}_X)} \in \mathcal{P}shv(\mathrm{Dom}_X)$, and denoted its associated functor of points by

$$\mathrm{Bun}_G^{B(\mathrm{gen})} := \mathrm{LKE}_q(\mathrm{Bun}_G^{B(\mathrm{Dom}_X)}) \in \mathcal{P}shv(\mathfrak{Aff}).$$

It is a geometrization of $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$. Conceptual appeal notwithstanding, this presentation of $\mathrm{Bun}_G^{B(\mathrm{gen})}$ is too unwieldy to be of much value. Namely, the issue is that using it (directly) to obtain presentations of invariants such as homology or of the category of D-modules is a non-starter.

In the unpublished note [Gai10, § 1.1], Gaitsgory introduces a category denoted $\mathfrak{Dmod}(\mathrm{Bun}_B^{\mathrm{rat}})$, which is cast to play the role of the category of D-modules 'on' $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$. In this section we present the construction of Gatisgory's category, and show that it is equivalent to $\mathfrak{Dmod}(\mathrm{Bun}_G^{B(\mathrm{gen})})$. The discussion parallels that of the previous section.

Notation 4.0.1. Let G be a connected reductive affine algebraic group. Choose a Borel subgroup B, denote by N the unipotent radical of B, and by $H = B/N$ the canonical Cartan. Choose a root system for G and B, and denote by Λ_G^+ the semigroup of dominant integral weights. For a dominant integral weight λ, let V^λ denote the irreducible representation of G with highest weight λ. For an H-torsor, $\mathcal{P}_H$, we denote by $\lambda(\mathcal{P}_H)$ the $\mathbb{G}_m$-torsor $\mathcal{P}_H \times_\lambda \mathbb{G}_m$ (as well as the associated line bundle – a quasi-coherent sheaf). For a G-torsor, $\mathcal{P}_G$, we denote by $\mathcal{V}_{\mathcal{P}_G}^\lambda$ the vector bundle corresponding to V^λ.

4.1 Constructions

4.1.1 *Plucker data.* Given a scheme Y and a G-bundle $\mathcal{P}_G$ on Y, a convenient way of presenting the data of a reduction of the structure group of $\mathcal{P}_G$ to B is given by specifying an H-bundle, $\mathcal{P}_H$, together with bundle maps for every $\lambda \in \Lambda_G^+$,

$$\lambda(\mathcal{P}_H) \xrightarrow[\subseteq]{\kappa_\lambda} \mathcal{V}_{\mathcal{P}_G}^\lambda,$$

which satisfy the *Plucker relations*. That is, for λ_0 the trivial character, κ^0 is the identity map

$$\mathcal{O} \cong \lambda_0(\mathcal{P}_H) \to \mathcal{V}_{\mathcal{P}_G}^0 \cong \mathcal{O}$$

and for every pair of dominant integral weights the following diagram commutes:

$$\begin{array}{ccc}
(\lambda + \mu)(\mathcal{P}_H) & \xrightarrow{\kappa_{\lambda+\mu}} & \mathcal{V}_{\mathcal{P}_G}^{\lambda+\mu} \\
\downarrow & & \downarrow \\
\lambda(\mathcal{P}_H) \otimes \mu(\mathcal{P}_H) & \xrightarrow{\kappa_\lambda \otimes \kappa_\mu} \mathcal{V}_{\mathcal{P}_G}^\mu \otimes \mathcal{V}_{\mathcal{P}_G}^\lambda
\end{array}$$

From now on, we adopt this Plucker point of view for presenting points of $\mathrm{Bun}_G^{B(\mathrm{gen})}$.

4.1.2 *Degenerate reduction spaces.* Degenerating the data of a reduction of a G-torsor to B, in a similar fashion to the degeneration of a regular map to a quasi-map, we obtain Drinfeld's (relative) compactification of $\mathrm{Bun}_B \to \mathrm{Bun}_G$.

Let $\overline{\mathrm{Bun}_G^B} \in \mathcal{P}shv(\mathfrak{Aff})$ be the presheaf[5] which sends a scheme S to the groupoid which classifies the data

$$(\mathcal{P}_G, \mathcal{P}_H, \lambda(\mathcal{P}_H) \xrightarrow{\kappa_\lambda} \mathcal{V}_{\mathcal{P}_G}^\lambda : \lambda \in \Lambda_G^+)$$

where:

- $\mathcal{P}_G$ is a G-torsor on $S \times X$;
- $\mathcal{P}_H$ is an H-torsor on $S \times X$;
- for every $\lambda \in \Lambda_G^+$, κ_λ is an injection of coherent sheaves whose cokernel is S-flat.

The κ_λ are required to satisfy the Plucker relations. Informally, this is a moduli space of G-bundles on X equipped with a degenerate reduction to B. There is an evident map $\mathrm{Bun}_B \to \overline{\mathrm{Bun}_G^B}$ whose image consists of those points for which the κ_λ are subbundle embeddings. For more details on $\overline{\mathrm{Bun}_G^B}$, see [FM99] or [BG02].

Let $\{\lambda_j\}_{j \in \mathcal{J}} \subseteq \Lambda_G^+$ be a finite subset which generates Λ_G^+ over $\mathbb{Z}_{\geqslant 0}$. The natural map[6]

$$G/B \to \underset{j \in \mathcal{J}}{\times} \mathbb{P}(V^{\lambda_j}) \hookrightarrow \mathbb{P}(\otimes_{j \in \mathcal{J}} V^{\lambda_j})$$

is a closed embedding. For every $j \in \mathcal{J}$ let $\mathcal{V}^{\lambda_j}$ be the vector bundle on $\mathrm{Bun}_G \times X$ corresponding to the representation V^{λ_j}, and let $\mathcal{V} := \otimes_{j \in \mathcal{J}} \mathcal{V}^{\lambda_j}$.

LEMMA 4.1.3 [BG02, Proposition 1.2.2]. *Let $S \to \mathrm{Bun}_G$ classify a G-bundle $\mathcal{P}_G$ on $S \times X$, and denote $(\overline{\mathrm{Bun}_G^B})_S := S \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}_G^B}$. There exists a natural isomorphism*

$$(\overline{\mathrm{Bun}_G^B})_S \xrightarrow{\cong} \mathrm{QSect}_S([,S) \times X][\mathcal{P}_G/B]$$

where the space of quasi-sections is defined via the closed embedding

$$\mathcal{P}_G/B \hookrightarrow \mathbb{P}(\mathcal{V}|_{S \times X}).$$

In particular, $\overline{\mathrm{Bun}_G^B}$ is schematic and proper over Bun_G.

Example 4.1.4. When $G = SL_2$ the presheaf $\overline{\mathrm{Bun}_{SL_2}^B}$ is equivalent to the presheaf which sends a scheme S to the groupoid $\overline{\mathrm{Bun}_{SL_2}^B}(S)$ classifying the data $(\mathcal{L}, \mathcal{V}, \mathcal{L} \hookrightarrow \mathcal{V})$, where $\mathcal{L}$ is a line bundle on $S \times X$, $\mathcal{V}$ is a rank-2 vector bundle on $S \times X$ with trivial determinant, and $\mathcal{L} \hookrightarrow \mathcal{V}$ is an injection of quasi-coherent sheaves whose cokernel is flat over S.

Observe that when $S = \mathrm{spec}(k)$, we may associate to every degenerate reduction $(\mathcal{L}, \mathcal{V}, \mathcal{L} \hookrightarrow \mathcal{V}) \in \overline{\mathrm{Bun}_{SL_2}^B}(k)$ the genuine reduction $(\tilde{\mathcal{L}}, \mathcal{V}, \tilde{\mathcal{L}} \xrightarrow{\subseteq} \mathcal{V}) \in \mathrm{Bun}_B(k)$ where $\tilde{\mathcal{L}}$ is the maximal subbundle, $\mathcal{L} \hookrightarrow \tilde{\mathcal{L}} \subseteq \mathcal{V}$ extending $\mathcal{L}$. However, there may not exist such extension for an arbitrary S-family.

We wish to use $\overline{\mathrm{Bun}_G^B}$ to construct a geometrization for $B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$. Note that on the level of k-points there exists a surjective map

$$\pi_0(\overline{\mathrm{Bun}_G^B}(k)) \to B(K)\backslash G(\mathbb{A})/G(\mathbb{O})$$

but that this map is not bijective.

---

[5] Often denoted by $\overline{\mathrm{Bun}_B}$.

[6] Which maps $1 \in G$ to the highest-weight line in each component.

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4.1.5 Gaitsgory's $\mathfrak{Dmod}(\mathrm{Bun}_B^{\mathrm{rat}})$ of [Gai10, §1.1] may be defined as follows: to every point $P \in \overline{\mathrm{Bun}}_G^B(S)$ we may associate its *regular* domain $U_P \subseteq S \times X$ – the maximal open subscheme where the Plucker data is regular, and hence defines a genuine structure reduction of $\mathcal{P}_G|_{U_P}$ to B.

Define $\mathcal{H} \in \mathcal{P}shv(\mathfrak{Aff})$ be the presheaf which sends S to the groupoid classifying the data

$$(P \in \overline{\mathrm{Bun}_G^B}, P' \in \overline{\mathrm{Bun}_G^B}, \phi)$$

where ϕ is an isomorphism of the underlying G-torsors (defined on all of $S \times X$), which commutes with the κ_λ over $U_P \cap U_{P'}$ (hence induces an isomorphism of B-reductions there). It is evident that $\mathcal{H}$ admits a groupoid structure (in presheaves) over $\overline{\mathrm{Bun}_G^B}$. In [Gai10], $\mathfrak{Dmod}(\mathrm{Bun}_B^{\mathrm{rat}})$ is defined to be the category of equivariant D-modules with respect to this groupoid.

On the level of points, we may define $\overline{\mathrm{Bun}_G^B}^{\mathcal{H}}$ to be the quotient of $\overline{\mathrm{Bun}_G^B}$ by this groupoid (i.e., the colimit of the associated simplicial object in $\mathcal{P}shv(\mathfrak{Aff})$). It follows that $\mathfrak{Dmod}(\mathrm{Bun}_B^{\mathrm{rat}}) \cong \mathfrak{Dmod}(\overline{\mathrm{Bun}_G^B}^{\mathcal{H}})$. After taking this quotient, we do have an identification of sets

$$\pi_0(\overline{\mathrm{Bun}_G^B}^{\mathcal{H}}(k)) \cong B(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

The main result of this section is the following proposition.

PROPOSITION 4.1.6. *There exists a map in* $\mathcal{P}shv(\mathfrak{Aff})$,

$$(\overline{\mathrm{Bun}_G^B})^{\mathcal{H}} \to \mathrm{Bun}_G^{B(\mathrm{gen})},$$

which becomes an equivalence after sheafification in the Zariski topology.

The following corollary is of particular interest in the geometric Langlands program.

COROLLARY 4.1.7. (i) *Pullback along the map constructed in Proposition 4.1.6 gives rise to an equivalence*

$$\lim_{[n]\in\Delta^{\mathrm{op}}}(\mathfrak{Dmod}(\mathcal{H}^{(n)})) \cong \mathfrak{Dmod}((\overline{\mathrm{Bun}_G^B})^{\mathcal{H}}) \leftarrow \mathfrak{Dmod}(\mathrm{Bun}_G^{B(\mathrm{gen})})$$

where

$$\mathcal{H}^{(n)} := \underbrace{\mathcal{H} \times_{\overline{\mathrm{Bun}_G^B}} \cdots \times_{\overline{\mathrm{Bun}_G^B}} \mathcal{H}}_{n\text{-times}}.$$

(ii) *The pullback functors*

$$\mathfrak{Dmod}(\overline{\mathrm{Bun}_G^B}) \leftarrow \mathfrak{Dmod}(\mathrm{Bun}_G^{B(\mathrm{gen})}) \leftarrow \mathfrak{Dmod}(\mathrm{Bun}_G)$$

admit left adjoints ('!-push-forward').

In Theorem 6.2.4 we shall prove that the pullback functor is, moreover, fully faithful. The proof of this corollary is completely analogous to that of Corollary 3.2.4.

4.1.8 *Proof of Proposition 4.1.6.* We proceed to reduce the statement to Proposition 3.3.2. Given $S \to \mathrm{Bun}_G$, we denote

$$(\overline{\mathrm{Bun}_G^B})_S := S \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}_G^B}$$

and denote similarly for $\mathrm{Bun}_G^{B(\mathrm{gen})}$ and $\mathcal{H}$.

It follows from Lemma 4.1.3 that

$$(\overline{\mathrm{Bun}_G^B})_S \cong \mathrm{QSect}_S\left(S \times X, \mathcal{P}_G^S/B\right),$$

and it is evident that

$$(\mathrm{Bun}_G^{B(\mathrm{gen})})_S \cong \mathrm{GSect}_S\left(S \times X, \mathcal{P}_G^S/B\right)$$

and that $\mathcal{H}_S$ is equivalent to the fiber product

$$
\begin{array}{ccc}
\mathcal{H}_S & \longrightarrow & \mathrm{QSect}_S\left(S \times X, \mathcal{P}_G^S/B\right) \\
\downarrow & & \downarrow \\
\mathrm{QSect}_S\left(S \times X, \mathcal{P}_G^S/B\right) & \longrightarrow & \mathrm{GSect}_S\left([,S) \times X][\mathcal{P}_G^S/B]
\end{array}
$$

Thus we obtain maps, for every $S \to \mathrm{Bun}_G$,

$$(\overline{\mathrm{Bun}_G^{B(\mathrm{gen})}})_S/\mathcal{H}_S \xrightarrow{\cong} (\mathrm{Bun}_G^{B(\mathrm{gen})})_S$$

which become equivalences after sheafification in the Zariski topology by Proposition 3.3.2. These maps are all natural in $S \to \mathrm{Bun}_G$, and we conclude the existence of a map of presheaves

$$\overline{\mathrm{Bun}_G^B}^{\mathcal{H}} \cong \operatorname*{colim}_{S \to \mathrm{Bun}_G} (\overline{\mathrm{Bun}_G^B})_S/\mathcal{H}_S \to \operatorname*{colim}_{S \to \mathrm{Bun}_G} (\mathrm{Bun}_G^{B(\mathrm{gen})})_S \cong \mathrm{Bun}_G^{B(\mathrm{gen})}$$

which becomes an equivalence after sheafification in the Zariski topology. □

Remark 4.1.9 (Drinfeld's parabolic structures). In [BG02, 1.3], Braverman and Gaitsgory consider two different notions (attributed to Drinfeld) of a degenerate reduction of a G-torsor (on X) to P. These two notions agree in the case when $P = B$, but differ in general. Correspondingly, they construct two different relative compactifications of the map $\mathrm{Bun}_P \to \mathrm{Bun}_G$, denoted $\overline{\mathrm{Bun}_P}$ and $\widetilde{\mathrm{Bun}_P}$, both schematic and proper over Bun_G. The categories of D-modules, $\mathfrak{Dmod}(\overline{\mathrm{Bun}_P})$ and $\mathfrak{Dmod}(\widetilde{\mathrm{Bun}_P})$, have received a fair amount of attention (e.g., in [BG02, BFGM02]) due to their part in the construction of a geometric 'Eisenstein series' functor

$$\mathfrak{Dmod}(\mathrm{Bun}_G) \xleftarrow{\mathrm{Eis}_M^G} \mathfrak{Dmod}(\mathrm{Bun}_M)$$

where M is the Levi factor of P.

It can be shown that $\overline{\mathrm{Bun}_P}$ and $\widetilde{\mathrm{Bun}_P}$ give rise to two different presentations of $\mathrm{Bun}_G^{P(\mathrm{gen})}$ (up to fppf sheafification) as a quotient of a scheme (relative to Bun_G) by a schematic and proper equivalence relation

$$(\overline{\mathrm{Bun}_P})/\overline{\mathcal{H}_P} \to \mathrm{Bun}_G^{P(\mathrm{gen})} \quad \text{and} \quad (\widetilde{\mathrm{Bun}_P})/\widetilde{\mathcal{H}_P} \to \mathrm{Bun}_G^{P(\mathrm{gen})}.$$

Consequently, we obtain two different presentations for the category of D-modules on $\mathrm{Bun}_G^{P(\mathrm{gen})}$ as a category of equivariant objects,

$$\mathfrak{Dmod}((\overline{\mathrm{Bun}_P}))^{\overline{\mathcal{H}_P}} \xleftarrow{\cong} \mathfrak{Dmod}(\mathrm{Bun}_G^{P(\mathrm{gen})})$$

and

$$\mathfrak{Dmod}((\widetilde{\mathrm{Bun}_P}))^{\widetilde{\mathcal{H}_P}} \xleftarrow{\cong} \mathfrak{Dmod}(\mathrm{Bun}_G^{P(\mathrm{gen})}).$$

5. The Ran space approach to parametrizing domains

In this section we describe an approach to presenting moduli problems of generic data using presheaves over the *Ran space*. This approach has the advantage that D-modules presented this way are amenable to chiral homology techniques. After presenting the framework we prove that it is equivalent, in an appropriate sense, to the approach described in §2 using Dom_X.

5.1 The Ran space

Notation 5.1.1. Let $\mathfrak{Fin}_{\mathrm{sur}}$ denote the category of finite sets with surjections as morphisms.

DEFINITION 5.1.2. The *Ran space,* denoted $\mathcal{R}an_X$, is the colimit of the diagram

$$\mathfrak{Fin}_{\mathrm{sur}}^{\mathrm{op}} \xrightarrow{I \mapsto X^I} \mathcal{P}shv(\mathfrak{Aff})$$

in which a surjection of finite sets $J \twoheadleftarrow I$ maps to the corresponding diagonal embedding $X^J = \mathrm{Maps}(J, X) \hookrightarrow \mathrm{Maps}(I, X) = X^I$.

In the appendix (Proposition A.3.1) it is proven that a point $S \to \mathcal{R}an_X$ is equivalent to the data of a finite subset $F \subset \mathrm{Hom}(S, X)$, that is, $\mathcal{R}an_X(S)$ is the set of finite, non-empty subsets of $\mathrm{Hom}(S, X)$. Note that, as we have defined it, $\mathcal{R}an_X$ is not a sheaf even in the Zariski topology (e.g., it is not separated), and in any case its sheafifications are not representable.

It is common to think of the Ran space as the moduli space for finite subsets[7] of X. Indeed, a closed point $\mathrm{spec}(k) \to \mathcal{R}an_X$ corresponds to a finite subset $F \subset X(k)$. However, since we are concerned with generic data, we take the opposite perspective and interpret such a point as parametrizing the complementary open subscheme $U_F := X \backslash F$. We point out that because X is a curve, every open subscheme is the complement of a finite collection of points, whence we are justified in thinking of $\mathcal{R}an_X$ as a moduli of open subschemes of X.

Notation 5.1.3. (i) Let $S \to \mathcal{R}an_X$ be a point of the Ran space classifying $F = \{f_1, \ldots, f_n\} \subseteq \mathrm{Hom}(S, X)$. We denote by Γ_F the closed subspace $\Gamma_F := \cup \Gamma_{f_i} \subseteq S \times X$, where $\Gamma_{f_i} \subseteq S \times X$ is the graph of $S \xrightarrow{f_i} X$. We denote the complementary open subscheme by

$$U_F := (S \times X) \backslash \Gamma_F.$$

It is universally dense in the sense of Definition 2.1.1.

(ii) Denote the 'category of points' of the Ran space by

$$\mathrm{Ran}_X := \mathfrak{Aff}_{/\mathcal{R}an_X}.$$

Let $\mathrm{Ran}_X \xrightarrow{i'} \mathrm{Dom}_X$ denote the evident functor which is defined on objects by $i'(S, F) = (S, U_F)$.

Remark 5.1.4. The functor i' fits in a commutative triangle

in which the diagonal arrows are Cartesian fibrations. The functor i' preserves Cartesian edges.

---

[7] We emphasize the distinction between finite subsets and finite subschemes.

There are two differences between Ran_X and Dom_X. The first, concerning objects, is that not every family of domains $(S, U) \in \mathrm{Dom}_X$ may be presented using a map $S \to \mathcal{R}\mathrm{an}_X$. That is, Ran_X classifies a restrictive collection of open subschemes in $S \times X$ – *graph complements*. The second difference, concerning morphisms, is that while the fibers Dom_X over $\mathfrak{Aff}$ are posets accounting for the inclusion of open subschemes, the fibers Ran_X are sets and do not account for such inclusions.

In order to minimize cumbersome notation we use the following conventions.

Notation 5.1.5. We implicitly identify the categories $\mathcal{P}shv(\mathrm{Ran}_X) \cong \mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$. We denote objects of this category using a subscript as in $\mathcal{G}_{\mathcal{R}\mathrm{an}_X}$. Under this convention, the functors

$$\mathcal{P}shv(\mathrm{Ran}_X) \xrightarrow{\mathrm{LKE}_s} \mathcal{P}shv(\mathfrak{Aff}) \quad \text{and} \quad \mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X} \xrightarrow{\text{forget}} \mathcal{P}shv(\mathfrak{Aff})$$

are implicitly identified. The left Kan extension of a presheaf $\mathrm{LKE}_s(\mathcal{G}_{\mathcal{R}\mathrm{an}_X}) \in \mathcal{P}shv(\mathfrak{Aff})$ is abusively denoted by $\mathcal{G}_{\mathcal{R}\mathrm{an}_X}$ as well. That is to say, we drop LKE_s from the notation, and the ambient category is to be inferred from the context.

We denote objects of $\mathcal{P}shv(\mathrm{Dom}_X)$ using a subscript as in $\mathcal{F}_{\mathrm{Dom}_X}$. Given $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$, we drop the subscript to denote $\mathcal{F} := \mathrm{LKE}_q(\mathcal{F}_{\mathrm{Dom}_X}) \in \mathcal{P}shv(\mathfrak{Aff})$. We also denote $\mathcal{F}_{\mathcal{R}\mathrm{an}_X} := i'_*(\mathcal{F}_{\mathrm{Dom}_X}) \in \mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$.

Example 5.1.6. As an example of how we use this notation, take $\mathcal{F}_{\mathrm{Dom}_X} = \mathrm{GMap}\,(X, Y)_{\mathrm{Dom}_X}$. Then $\mathrm{GMap}\,(X, Y) := \mathrm{LKE}_q(\mathrm{GMap}\,(X, Y)_{\mathrm{Dom}_X})$ agrees with our previously defined notation. The presheaf $\mathrm{GMap}\,(X, Y)_{\mathcal{R}\mathrm{an}_X}$ classifies the data

$$(S \xrightarrow{F} \mathcal{R}\mathrm{an}_X, U_F \xrightarrow{f} Y)$$

where f is regular map. That is, it classifies a generically defined map together with a domain where it is regular. Depending on the context, it denotes either an object of $\mathcal{P}shv(\mathrm{Ran}_X) \cong \mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$ or the corresponding object of $\mathcal{P}shv(\mathfrak{Aff})$.

5.2 Main statement

The main statement of this section is the following (we use the conventions of Notation 5.1.5).

THEOREM 5.2.1. *Let $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$. There exists a natural map $\mathcal{F}_{\mathcal{R}\mathrm{an}_X} \xrightarrow{\pi} \mathcal{F}$ in $\mathcal{P}shv(\mathfrak{Aff})$. The pullback functor induced by π,*

$$\mathfrak{Dmod}(\mathcal{F}_{\mathcal{R}\mathrm{an}_X}) \xleftarrow{\pi^!} \mathfrak{Dmod}(\mathcal{F}),$$

is fully faithful.

The rest of this section is devoted to the proof of this theorem, which is completed in § 5.5. The reader who is willing to take this theorem on faith can safely skip to § 6.

5.2.2 *Overview.* Let $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$. We think of the associated functor of points $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$ as classifying some generically defined data over X. As observed in Example 5.1.6 above, $\mathcal{F}_{\mathcal{R}\mathrm{an}_X}$ classifies the same type of generic data together with the additional data of a domain where it is regular. We will endow $\mathcal{F}_{\mathcal{R}\mathrm{an}_X}$ with additional structure which reflects the fact that a pair of its points may be parametrizing the same generic data, but with different domains.

The extra structure on $\mathcal{F}_{\mathcal{R}\mathrm{an}_X}$ will be that of a module for a certain monad acting on the category $\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$, which will be constructed in Definition 5.4.1. Theorem 5.2.1 will be

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proved by examining the structure of this monad. At a first approximation, this monad is similar to the one induced by adjunction,

$$\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}an_X} \underset{i'_*}{\overset{\mathrm{LKE}_{i'}}{\rightleftarrows}} \mathcal{P}shv(\mathrm{Dom}_X) \ .$$

However, the monad we shall actually use will be defined using an intermediate domain category, constructed in § 5.3, which is more closely related to $\mathcal{R}an_X$.

5.3 Dom_X^Γ – a more economical category of domains

Recall that objects of Ran_X present open subschemes which are graph complements, but that Ran_X does not include morphisms to account for the inclusions between such subschemes. The category defined below is obtained by adding the appropriate morphisms.

DEFINITION 5.3.1. Let Dom_X^Γ be the category which has the same objects as Ran_X. That is, an object is a pair (S, F) where $S \in \mathfrak{Aff}$ and $F \subseteq \mathrm{Hom}(S, X)$ is a finite non-empty subset.

A morphism $(S, F) \to (T, G)$ is a map of schemes $S \overset{f}{\longrightarrow} T$ such that $\forall g \in G$, $g \circ f \in F$ – that is, pre-composition with f carries G into F.

It is evident that Dom_X^Γ is sandwiched in a commuting diagram

$$
\begin{array}{ccccc}
\mathrm{Ran}_X & \overset{i}{\longrightarrow} & \mathrm{Dom}_X^\Gamma & \overset{p}{\longrightarrow} & \mathrm{Dom}_X \\
 & {\scriptstyle s}\searrow & {\scriptstyle r}\downarrow & {\scriptstyle q}\swarrow & \\
 & & \mathfrak{Aff} & &
\end{array}
\tag{5.1}
$$

in which $p(S, F) = (S, U_F)$. Note that all three diagonal maps are Cartesian fibrations, and that i and p preserve morphisms which are Cartesian over $\mathfrak{Aff}$. We remark that p is not full (though it is faithful).

We endow Dom_X^Γ with the fppf Grothendieck topology pulled back from Dom_X along p. That is, a collection of morphisms

$$\{(S_i, F_i \subseteq \mathrm{Hom}(S_i, X)) \to (S, F \subseteq \mathrm{Hom}(S, X))\}$$

is a cover if and only if the collection of scheme morphisms $\{U_{F_i} \to U_F\}$ is an fppf cover.

PROPOSITION 5.3.2. *The adjunction*

$$\mathcal{P}shv(\mathrm{Dom}_X^\Gamma) \underset{p_*}{\overset{\mathrm{LKE}_p}{\rightleftarrows}} \mathcal{P}shv(\mathrm{Dom}_X)$$

induces mutually inverse equivalences after sheafification in the fppf Grothendieck topology.

The proof is given in § 5.6.

The upshot of this subsection is Corollary 5.3.3 below. The adjunction co-unit for the functors (p_*, LKE_p) induces a natural transformation $\mathrm{LKE}_r \circ p_* \overset{\eta}{\longrightarrow} \mathrm{LKE}_q$ in the following triangle:

$$
\begin{array}{ccc}
\mathcal{P}shv(\mathrm{Dom}_X^\Gamma) & \overset{p_*}{\longleftarrow} & \mathcal{P}shv(\mathrm{Dom}_X) \\
{\scriptstyle \mathrm{LKE}_r}\searrow & & \swarrow{\scriptstyle \mathrm{LKE}_q} \\
 & \mathcal{P}shv(\mathfrak{Aff}) &
\end{array}
$$

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As in Notation 5.1.5, given $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$, we denote

$$\mathcal{F}_{\mathrm{Dom}_X^\Gamma} := p_*(\mathcal{F}_{\mathrm{Dom}_X}) \in \mathcal{P}shv(\mathrm{Dom}_X^\Gamma).$$

COROLLARY 5.3.3. *Let $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$. The natural transformation*

$$\mathrm{LKE}_r(\mathcal{F}_{\mathrm{Dom}_X^\Gamma}) \xrightarrow{\eta} \mathcal{F}$$

becomes an equivalence after fppf sheafification. Consequently, D-module pullback gives rise to an equivalence

$$\mathfrak{Dmod}(\mathrm{LKE}_r(\mathcal{F}_{\mathrm{Dom}_X^\Gamma})) \xleftarrow{\cong} \mathfrak{Dmod}(\mathcal{F}),$$

the point being that when formulating a moduli problem of generic data as functor of points, it suffices to describe its points over Dom_X^Γ, rather than over the much larger category Dom_X.

5.4 The Ran space formulation for moduli problems of generic data

In this subsection we describe the category $\mathcal{P}shv(\mathrm{Dom}_X^\Gamma)$ as the category of modules for a monad acting on $\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$.

DEFINITION 5.4.1. Recall the functor $\mathrm{Ran}_X \xrightarrow{i} \mathrm{Dom}_X^\Gamma$ introduced in (5.1). Let $\mathcal{M}$ be the monad on $\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X} \cong \mathcal{P}shv(\mathrm{Ran}_X)$ induced by the adjunction

$$\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X} \cong \mathcal{P}shv(\mathrm{Ran}_X) \underset{i_*}{\overset{\mathrm{LKE}_i}{\rightleftarrows}} \mathcal{P}shv(\mathrm{Dom}_X^\Gamma) \ .$$

That is, its underlying endofunctor is $i_* \circ \mathrm{LKE}_i$, and its unit and action transformations are induced by the adjunction unit and co-unit. We denote the category of modules for the monad $\mathcal{M}$ by

$$\mathrm{Mod}_{\mathcal{M}} = \mathrm{Mod}_{\mathcal{M}}(\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}).$$

Remark 5.4.2. We describe the action of $\mathcal{M}$ explicitly. Let $\mathcal{G} \in \mathcal{P}shv(\mathrm{Ran}_X)$. The value of $\mathcal{M}(\mathcal{G})$ at $(S, F) \in \mathrm{Ran}_X$ is

$$\mathcal{M}(\mathcal{G})(S, F) \cong \coprod_{G \subseteq F} \mathcal{G}(S, G).$$

The simplicial resolution in the following proposition is the main ingredient for the proof of the full faithfulness assertion in Theorem 5.2.1. Note that the functor $\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X} \xleftarrow{i_*} \mathcal{P}shv(\mathrm{Dom}_X^\Gamma)$ canonically factors through $\mathrm{Mod}_{\mathcal{M}}$.

PROPOSITION 5.4.3. *The canonical functor*

$$\mathrm{Mod}_{\mathcal{M}} \longleftarrow \mathcal{P}shv(\mathrm{Dom}_X^\Gamma)$$

is an equivalence. For every $\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X)$ there exists an augmented simplicial object in $\mathcal{P}shv(\mathfrak{Aff})$,

$$\cdots \; \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \; \mathcal{M}^2(\mathcal{F}_{\mathcal{R}\mathrm{an}_X}) \; \underset{\overset{\longleftarrow}{\longrightarrow}}{\overset{\longleftarrow}{\rightrightarrows}} \; \mathcal{M}(\mathcal{F}_{\mathcal{R}\mathrm{an}_X}) \; \underset{\longrightarrow}{\overset{\longleftarrow}{\rightarrow}} \; \mathcal{F}_{\mathcal{R}\mathrm{an}_X}$$

$$\downarrow$$

$$\mathcal{F}$$

(5.2)

which becomes a colimit diagram, after sheafification in the fppf Grothendieck topology.

In the simplicial complex above, $\mathcal{M}$ refers to the endofunctor $i_* \circ \mathrm{LKE}_i$ underlying the eponymous monad acting on $\mathcal{P}shv(\mathfrak{Aff})_{/\mathcal{R}an_X}$. We remark that despite the vagueness in the existence statement of the simplicial complex, it can be made quite explicit, as will be explained in §5.4.4.

Proof. The first assertion is a consequence of the Barr–Beck–Lurie theorem [Lur11a, Theorem 6.2.0.6], since i_* is conservative and colimit preserving (it admits a right adjoint given by right Kan extension).

We proceed to prove the second assertion. Recall the functors denoted p, q, r, s and i which were introduced in (5.1). Below we use the conventions of Notation 5.1.5. In particular, note that $i_* \mathcal{F}_{\mathrm{Dom}_X^\Gamma} = \mathcal{F}_{\mathcal{R}an_X}$.

The Bar construction for $\mathcal{F}_{\mathrm{Dom}_X^\Gamma} \in \mathcal{P}shv(\mathrm{Dom}_X^\Gamma) \cong \mathrm{Mod}_\mathcal{M}$ yields an augmented simplicial complex in $\mathcal{P}shv(\mathrm{Dom}_X^\Gamma)$,

$$\cdots \; \vdots \; \mathrm{LKE}_i \mathcal{M}^2(\mathcal{F}_{\mathcal{R}an_X}) \rightrightarrows \mathrm{LKE}_i \mathcal{M}(\mathcal{F}_{\mathcal{R}an_X}) \rightrightarrows \mathrm{LKE}_i(\mathcal{F}_{\mathcal{R}an_X})$$
$$\downarrow$$
$$\mathcal{F}_{\mathrm{Dom}_X^\Gamma}$$

which is a colimit diagram [Lur11a, Theorem 4.3.5.8 or Proposition 6.2.2.12].

Applying LKE_r, we obtain an augmented simplicial complex in $\mathcal{P}shv(\mathfrak{Aff})$ which is colimit diagram,

$$\cdots \mathcal{M}^2(\mathcal{F}_{\mathcal{R}an_X}) \rightrightarrows \mathcal{M}(\mathcal{F}_{\mathcal{R}an_X}) \rightrightarrows \mathcal{F}_{\mathcal{R}an_X}$$
$$\downarrow \qquad\qquad\qquad (5.3)$$
$$\mathrm{LKE}_r \mathcal{F}_{\mathrm{Dom}_X^\Gamma}$$

where, as explained in Notation 5.1.5, we also use $\mathcal{M}^n(\mathcal{F}_{\mathcal{R}an_X})$ to denote

$$\mathrm{LKE}_{r \circ i}(\mathcal{M}^n(\mathcal{F}_{\mathcal{R}an_X})) = \mathrm{LKE}_s(\mathcal{M}^n(\mathcal{F}_{\mathcal{R}an_X})) \in \mathcal{P}shv(\mathfrak{Aff}).$$

By Proposition 5.3.2, there exists a map $\mathrm{LKE}_r \mathcal{F}_{\mathrm{Dom}_X^\Gamma} \to \mathcal{F}$ which becomes an equivalence after to fppf sheafification. Composing the augmentation in (5.3) with this map, we obtain the sought-after augmented complex, (5.2). □

5.4.4 We make the simplicial complex appearing in the proposition explicit. For the sake of concreteness, let us consider the case $\mathcal{F}_{\mathrm{Dom}_X} = \mathrm{GMap}\,(X,Y)_{\mathrm{Dom}_X}$. We denote an S-point of $\mathrm{GMap}\,(X,Y)_{\mathcal{R}an_X} \times (\mathcal{R}an_X)^n$ by $(f; F_0, \ldots, F_n)$, where it is understood that each F_i is a finite subset of $\mathrm{Hom}(S,X)$, and that f is a generic map from $S \times X$ to Y, defined on the open subscheme determined by F_0.

Using Remark 5.4.2, we see that the nth term of the simplicial complex (5.2) is the subsheaf

$$\mathcal{M}^n(\mathrm{GMap}\,(X,Y)_{\mathcal{R}an_X}) \subseteq \mathrm{GMap}\,(X,Y)_{\mathcal{R}an_X} \times (\mathcal{R}an_X)^n$$

whose S-points are the tuples $(f; F \subseteq F_1 \cdots \subseteq F_n)$ (i.e., in which the finite subsets are increasing). The maps are given as follows:

(i) for a degeneracy $[n+1] \xrightarrow{d_i} [n]$ (which maps $i, i+1$ to i),

$$\mathcal{M}^{n+1}(\mathrm{GMap}\,(X,Y)_{\mathcal{R}an_X}) \longleftarrow \mathcal{M}^n(\mathrm{GMap}\,(X,Y)_{\mathcal{R}an_X})$$

$$(f; F_0 \subseteq \cdots \subseteq F_i \subseteq F_i \subseteq \cdots \subseteq F_n) \longleftarrow\!\shortmid (f; F_0 \subseteq \cdots \subseteq F_n)$$

(ii) for a face map $[n+1] \xleftarrow{\;s_i\;} [n]$ (skip $i \in [n+1]$),

$$\mathcal{M}^{n+1}(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}) \longrightarrow \mathcal{M}^n(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X})$$
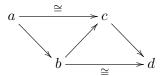
$$(f; F_0 \subseteq \cdots \subseteq F_{n+1}) \longmapsto (f; F_0 \subseteq \cdots \hat{F}_i \ldots \subseteq F_{n+1})$$

where the hat over $\hat{F}_i$ denotes that the ith term has been omitted. We point out that, since $F_i \subseteq F_{i+1}$ the ith term in $(f; F_0, \ldots, \hat{F}_i, \ldots, F_{n+1})$ is the equal to $F_i \cup F_{i+1}$.

Remark 5.4.5. There is another closely related way of describing the category $\mathrm{Mod}_{\mathcal{M}}$. Namely, the presheaf $\mathcal{R}\mathrm{an}_X$ has the structure of a semigroup and the category $\mathrm{Mod}_{\mathcal{M}}$ is equivalent to a certain category of its modules (in $\mathbf{Gpd}_\infty$). This approach is related to Gaitsgory's *unital structures* introduced in [Gai13]. The comparison of these description will be taken up a future note.

5.5 Proof of Theorem 5.2.1

In the proof below we use the following general fact: if C is an ∞-category, then equivalences in C satisfy 'two-out-of-six'. That is, given a commutative diagram in C

in which the horizontal morphisms are equivalences, we may conclude that all the morphisms are equivalences (the sixth being the composition $a \to d$).

Proof of Theorem 5.2.1. By Proposition 5.3.2, it suffices to prove that for $\mathcal{F}_{\mathrm{Dom}_X^\Gamma} = p_*\mathcal{F}_{\mathrm{Dom}_X} \in \mathcal{P}shv(\mathrm{Dom}_X^\Gamma)$ the pullback functor

$$\mathfrak{Dmod}(\mathcal{F}_{\mathcal{R}\mathrm{an}_X}) \leftarrow \mathfrak{Dmod}(\mathrm{LKE}_r\mathcal{F}_{\mathrm{Dom}_X^\Gamma})$$

is fully faithful.

We start by reducing to the case when $\mathcal{F}_{\mathrm{Dom}_X^\Gamma}$ is in the essential image if the Yoneda functor $\mathrm{Dom}_X^\Gamma \to \mathcal{P}shv(\mathrm{Dom}_X^\Gamma)$. Denote the Yoneda image of point $(S, F) \in \mathrm{Dom}_X^\Gamma$ by $\mathcal{Y}_{(S,F)} \in \mathcal{P}shv(\mathrm{Dom}_X^\Gamma)$. Present the presheaf $\mathcal{F}_{\mathrm{Dom}_X^\Gamma}$ as the 'colimit of its points',

$$\operatorname*{colim}_{\mathcal{Y}_{(S,F)} \to \mathcal{F}}(\mathcal{Y}_{(S,F)}) \xrightarrow{\cong} \mathcal{F}_{\mathrm{Dom}_X^\Gamma}.$$

Noting that both LKE_i and i_* preserve colimits (since both admit right adjoints), we also have

$$\operatorname*{colim}_{\mathcal{Y}_{(S,F)} \to \mathcal{F}}(\mathrm{LKE}_i(i_*\mathcal{Y}_{(S,F)})) \xrightarrow{\cong} \mathrm{LKE}_i(i_*\mathcal{F}_{\mathrm{Dom}_X^\Gamma}) = \mathrm{LKE}_i\mathcal{F}_{\mathcal{R}\mathrm{an}_X}.$$

Consequently, it suffices to show that the functor

$$\operatorname*{lim}_{\mathcal{Y}_{(S,F)} \to \mathcal{F}} \mathfrak{Dmod}(\mathrm{LKE}_{r\circ i}(i_*\mathcal{Y}_{(S,F)})) \leftarrow \operatorname*{lim}_{\mathcal{Y}_{(S,F)} \to \mathcal{F}} \mathfrak{Dmod}(\mathrm{LKE}_r\mathcal{Y}_{(S,F)})$$

is fully faithful. The latter will follow if we show that for every $(S, F) \in \mathrm{Dom}_X^\Gamma$ the functor

$$\mathfrak{Dmod}(\mathrm{LKE}_{r \circ i}(i_* \mathcal{Y}_{(S,F)})) \leftarrow \mathfrak{Dmod}(\mathrm{LKE}_r \mathcal{Y}_{(S,F)}) = \mathfrak{Dmod}(S)$$

is fully faithful.

The latter functor is induced by the map in $\mathcal{P}shv(\mathfrak{Aff})$,

$$\mathrm{LKE}_{r \circ i} \circ i_* \mathcal{Y}_{(S,F)} = \mathrm{LKE}_s(i_* \mathcal{Y}_{(S,F)}) \to S.$$

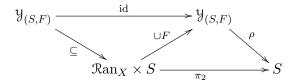The functor of points $\mathrm{LKE}_s(i_* \mathcal{Y}_{(S,F)})$ sends a scheme T to the set

$$\{((T, G), T \xrightarrow{f} S) : G \subset \mathrm{Hom}(T, X) \text{ finite, non-empty}, G \supseteq f^* F\}.$$

'Union with F' gives rise to a map

$$\mathcal{R}an_X \times S \xrightarrow{\quad \cup F \quad} \mathrm{LKE}_s(i_* \mathcal{Y}_{(S,F)})$$

$$((T, G), T \xrightarrow{f} S) \longmapsto (T \xrightarrow{f} S, G \cup f^* F)$$

which fits into the commutative diagram

$$\begin{array}{ccc}
\mathcal{Y}_{(S,F)} & \xrightarrow{\quad \mathrm{id} \quad} & \mathcal{Y}_{(S,F)} \\
& \cup F \nearrow & \downarrow \rho \\
\subseteq \searrow & & \\
& \mathcal{R}an_X \times S \xrightarrow{\quad \pi_2 \quad} & S
\end{array}$$

Passing to D-modules, pullback along the bottom map is fully faithful by [Gai13, Theorem 1.6.5] (or [BD04, Proposition 4.3.3]). We conclude by a 'two-out-of-six' argument: for every pair M, $N \in \mathfrak{Dmod}(S)$, the maps above give rise to a diagram of ∞-groupoids

$$\begin{array}{ccc}
\mathrm{Map}(\rho^! M, \rho^! N) & \xleftarrow{\quad = \quad} & \mathrm{Map}(\rho^! M, \rho^! N) \\
\nwarrow \swarrow & & \nwarrow \\
\mathrm{Map}(\pi_2^! M, \pi_2^! N) & \xleftarrow{\quad \cong \quad} & \mathrm{Map}(M, N)
\end{array}$$

By 'two-out-of-six' for equivalences in $\mathbf{Gpd}_\infty$, it follows that the map

$$\mathrm{Map}(\rho^! M, \rho^! N) \leftarrow \mathrm{Map}(M, N)$$

is an equivalence of ∞-groupoids, so that

$$\mathfrak{Dmod}(i_* \mathcal{Y}_{(S,F)}) \xleftarrow{\quad \rho^! \quad} \mathfrak{Dmod}(S)$$

is fully faithful.

5.6 The proof of Proposition 5.3.2

The following lemma contains the geometric input for the proof of Proposition 5.3.2.

LEMMA 5.6.1. *The functor* $\mathrm{Dom}_X^\Gamma \to \mathrm{Dom}_X$ *has dense image with respect to the fppf topology. That is, every point of* Dom_X *has a cover by points in the essential image of* Dom_X^Γ.

Proof. Let $(S, U) \in \mathrm{Dom}_X$; we must show that it admits a cover by points in the essential image of Dom_X^Γ.

We may assume that S is connected, and by Lemma 3.2.7 we may also assume that $U \subseteq S \times X$ is a divisor complement. Let $\mathcal{L} \to \mathcal{O}_{S \times X}$ be an effective Cartier divisor whose complement is U.

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Since S is connected, the data of the divisor $\mathcal{L} \to \mathcal{O}_{S \times X}$ is equivalent to a map $S \to \mathcal{H}ilb_X^n$ for some n, where $\mathcal{H}ilb_X^n$ is the degree n component of the Hilbert scheme of X.[8] The standard map $X^n \to \mathcal{H}ilb_X^n$ is an fppf cover (it is faithfully flat and finite of index $n!$). Form the pullback

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & X^n \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{H}ilb_X^n \end{array}$$

The components of the top map give rise to a subset $F \subseteq \mathrm{Hom}(\tilde{S}, X)$, which in turn determines a point $(\tilde{S}, F) \in \mathrm{Dom}_X^\Gamma$. Observing that $U_F = \tilde{S} \times_S U$, we get a map $(\tilde{S}, U_F) \to (S, U)$ which is an fppf cover in Dom_X, and whose domain is in the essential image of Dom_X^Γ. $\qquad\square$

Factor p as

$$\mathrm{Dom}_X^\Gamma \xrightarrow{p'} \mathrm{Dom}_X^{00} \xrightarrow{j} \mathrm{Dom}_X$$

where Dom_X^{00} is the essential image of p, the full subcategory of Dom_X consisting of 'graph complements'. We endow Dom_X^{00} with the Grothendieck topology pulled back from the fppf topology on Dom_X. We will prove that j and p' both induce equivalences on sheaf categories, whence Proposition 5.3.2 will follow.

Regarding p', the idea is that every fiber of p' is weakly contractible, and that every map in such a fiber is a cover. Thus, it is reasonable to suspect that p' might be a site equivalence. The necessary accounting is a little involved, and the relevant site-theoretic properties of p', which allow the argument to go through, are embodied in the hypothesis of Lemma 5.6.3. Before stating the lemma, we introduce some notation.

Notation 5.6.2. For a category D and an object $d \in D$, we use $D_{/d}$ to denote the overcategory, and we use $D_{d/}$ to denote the undercategory. We shall denote an object of $D_{/d}$ by $(d', d' \to d)$ where d' is an object of D, and $d' \to d$ is a morphism in D (similarly for undercategories).

If C is another category and $C \xrightarrow{F} D$ is a functor, C_d denotes the *fiber* of f over d, that is, the fibered product $C \times_D \{d\}$ in $\mathbf{Cat}_\infty$. We denote $C_{/d} := C \times_D D_{/d}$; it is a relative overcategory. We denote an object of this category by the data $(c, F(c) \to d)$ where it is implicitly understood that c is an object in C, and that $F(c) \to d$ is a morphism in D. Dually, we denote $C_{d/} = C \times_D D_{d/}$; it is a relative undercategory. This notation is slightly abusive since obviously these categories are dependent on the functor F, and not only on C and d.

LEMMA 5.6.3. *Let C and D be small sites whose underlying categories admit all finite non-empty limits, and whose Grothendieck topologies are generated by finite covers. Let $C \xrightarrow{p} D$ be a functor such that:*

(i) *the Grothendieck topology on C is the pullback of the topology on D;*

(ii) *the functor p is essentially surjective;*

(iii) *for every $c \in C$, and for every morphism in D, $d \xrightarrow{\tilde{f}} p(c)$, there exists a morphism in C, $c' \xrightarrow{\tilde{f}} c$, which lifts[9] f;*

(iv) *p preserves finite limits;*

---

[8] Since X is a curve, $\mathcal{H}ilb_X^n \cong X^{(n)}$, the nth symmetric power.

[9] But we do not assume that a Cartesian lift exists.

(v) *for every $d \in D$, the functor $(C_d)^{\mathrm{op}} \to (C_{d/})^{\mathrm{op}}$ is cofinal;*

(vi) *for every $d \in D$, the category C_d is a cofiltered poset.*

Then the functor

$$\mathcal{S}hv(C) \xleftarrow{\ p_* \ } \mathcal{S}hv(D)$$

is an equivalence, and left Kan extension along p is its inverse (no sheafification necessary).

In §5.6.4 we will show that $\mathrm{Dom}_X^{\Gamma} \xrightarrow{\ p' \ } \mathrm{Dom}_X^{00}$ satisfies the hypothesis of this lemma.

Proof. We will show that the left Kan extension

$$\mathcal{S}hv(C) \xrightarrow{\ \mathrm{LKE}_p \ } \mathcal{P}shv(D)$$

lands in sheaves, and prove that the resulting adjoint functors (LKE_p, p_*)

$$\mathcal{S}hv(C) \underset{p_*}{\overset{\mathrm{LKE}_p}{\rightleftarrows}} \mathcal{S}hv(D)$$

are mutually inverse equivalences.

The following is the key observation: Let $\mathcal{G} \in \mathcal{S}hv(C)$, and let $d \in D$. Then $\mathcal{G}$ is constant on the fiber C_d. First we point out that (4) implies that C_d admits all finite non-empty limits, which may be computed in C. Let $c' \xrightarrow{f} c$ be a morphism in C_d; it is a cover by (1). The value of $\mathcal{G}$ at c may be computed using the Čech complex of f. However, (6) implies that this Cech complex is the constant simplicial object with value c', since $c' \times_c c' = c'$ because C_d is a poset. It follows that $\mathcal{G}(c') \leftarrow \mathcal{G}(c)$ is an equivalence. Since C_d it is weakly contractible (being a cofiltered poset), the observation follows.

Let $\mathcal{G} \in \mathcal{S}hv(C)$, and let us show that the co-unit transformation (*a priori* in $\mathcal{P}shv(C)$)

$$p_* \circ \mathrm{LKE}_p \mathcal{G} \to \mathcal{G}$$

is an equivalence. Fix $c \in C$, and let us prove that the map

$$p_* \circ \mathrm{LKE}_p \mathcal{G}(c) \to \mathcal{G}(c)$$

is an equivalence of groupoids. We compute

$$p_* \mathrm{LKE}_p \mathcal{G}(c) = \mathrm{LKE}_p \mathcal{G}(p(c)) = \mathrm{colim}\left((C_{p(c)/})^{\mathrm{op}} \xrightarrow{\mathcal{G}} \mathbf{Gpd}_\infty\right) \cong .$$

Since $(C_{p(c)})^{\mathrm{op}} \to (C_{p(c)/})^{\mathrm{op}}$ is cofinal by (5),

$$\cong \mathrm{colim}\left((C_{p(c)})^{\mathrm{op}} \xrightarrow{\mathcal{G}} \mathbf{Gpd}_\infty\right) \cong .$$

Because $\mathcal{G}$ is constant on the fibers, and these fibers are weakly contractible, we conclude

$$\cong \mathcal{G}(c).$$

Next we show that for every $\mathcal{G} \in \mathcal{S}hv(C)$, the presheaf $\mathrm{LKE}_p \mathcal{G}$ is in fact a sheaf. Let $d \in D$, and let $\{d_i \to d\}_{i=1}^k$ be a cover in D. Let $c \in C$ be such that $p(c) = d$, and let $\{c_i \xrightarrow{\tilde{f}_i} c\}_{i=1}^k$ be a lift of the f_i; it is a cover of c by (1). For every n-tuple of indexes in $\{1, \ldots, k\}$, $\bar{i}$, we let $c_{\bar{i}}$ and $d_{\bar{i}}$ denote the corresponding n-fold fibered products over c and d, and we note that

$p(c_{\bar{i}}) \cong d_{\bar{i}}$ by (4). Consequently, forming the Čech covers associated with the covers, we obtain a commutative square

$$
\begin{array}{ccc}
\displaystyle\lim_{[n]\in\Delta^{\mathrm{op}}}\left(\coprod_{|\bar{i}|=n}\mathcal{G}(c_{\bar{i}})\right) & \xleftarrow{\;\;\cong\;\;} & \mathcal{G}(c) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} \\
\displaystyle\lim_{[n]\in\Delta^{\mathrm{op}}}\left(\coprod_{|\bar{i}|=n}\mathrm{LKE}_p\mathcal{G}(d_{\bar{i}})\right) & \xleftarrow{\quad\quad} & \mathrm{LKE}_p\mathcal{G}(d)
\end{array}
$$

in which the vertical maps are equivalences by computation above, and the top map is an equivalence because $\mathcal{G}$ is a sheaf. We conclude that the bottom map is an equivalence for every cover of d, thus $\mathrm{LKE}_p\mathcal{G}$ is a sheaf.

We complete the proof of the lemma by observing that we have exhibited adjoint functors

$$
\mathcal{S}hv(C) \xrightleftharpoons[p_*]{\mathrm{LKE}_p} \mathcal{S}hv(D)
$$

for which the co-unit transformation is an equivalence. In addition, Since p is essentially surjective, p_* is conservative, whence we conclude that the unit transformation is also an equivalence. The equivalence of sheaf categories follows. $\qquad\square$

5.6.4 *Proof of Proposition 5.3.2.* Below, all sites are endowed with their (respective) fppf Grothendieck topologies, and we suppress the topology in the notation – for example, $\mathcal{S}hv(\mathrm{Dom}_X) := \mathcal{S}hv(\mathrm{Dom}_X;\mathrm{fppf})$.

Recall the factorization

$$
\mathrm{Dom}_X^{\Gamma} \xrightarrow{p'} \mathrm{Dom}_X^{00} \xrightarrow{j} \mathrm{Dom}_X.
$$

We endow Dom_X^{00} with the Grothendieck topology pulled back from the fppf topology on Dom_X. We treat p' and j separately.

(i) We prove that $\mathcal{S}hv(\mathrm{Dom}_X^{00}) \xleftarrow{j_*} \mathcal{S}hv(\mathrm{Dom}_X)$ is an equivalence by showing that it satisfies the hypothesis of a general criterion for the inclusion of a subsite to induce an equivalence on sheaf categories (often referred to as the 'comparison lemma'). A statement of this criterion is proved in the appendix (Lemma A.2.1).

The functor j has dense image by Lemma 5.6.1. The category Dom_X admits all finite limits. In particular, fibered products in Dom_X are given by squares of the form

$$
\begin{array}{ccc}
(R\times_T S, U\times_T W) & \longrightarrow & (R,W) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\
(S,U) & \xrightarrow{\;\;f\;\;} & (T,V)
\end{array}
$$

From this it is evident that whenever $U \subseteq S \times X$ and $W \subseteq T \times X$ are graph complements (i.e., are points in Dom_X^{00}), then so is

$$
U \times_T W \subseteq R \times_S T \times X,
$$

whence it follows that $(R\times_T S, U\times_T W) \in \mathrm{Dom}_X^{00}$. These are precisely the hypothesis of the comparison lemma (Lemma A.2.1), and we conclude that j_* is an equivalence of sheaf categories.

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(ii) We prove that $\mathcal{S}hv(\mathrm{Dom}_X^\Gamma) \xleftarrow{p'_*} \mathcal{S}hv(\mathrm{Dom}_X^{00})$ is an equivalence by showing that the functor p' satisfies the hypothesis of Lemma 5.6.3. Aside from (5), which we will show, the rest of the hypothesis is immediate.

Fix $(S,U) \in \mathrm{Dom}_X^{00}$. In order to prove that

$$(\mathrm{Dom}_X^\Gamma)_{(S,U)}^{\mathrm{op}} \to ((\mathrm{Dom}_X^\Gamma)_{(S,U)/})^{\mathrm{op}}$$

is cofinal, it suffices to show that, for every point $Q \in ((\mathrm{Dom}_X^\Gamma)_{(S,U)/})^{\mathrm{op}}$, the category

$$((\mathrm{Dom}_X^\Gamma)_{(S,U)}^{\mathrm{op}})_{Q/}$$

is weakly contractible – or equivalently, that its opposite category

$$((\mathrm{Dom}_X^\Gamma)_{(S,U)})_{/Q} \tag{5.4}$$

is weakly contractible. The object Q is presented by the data of

$$
\begin{array}{c}
(T,G) \in \mathrm{Dom}_X^\Gamma \\
\Big\uparrow{\scriptstyle p'} \\
(S,U) \xrightarrow{f:S\to T} (T,U_G) \in \mathrm{Dom}_X^{00}
\end{array}
$$

and the category (5.4) classifies all the ways of lifting f to a 'commutative' square

$$
\begin{array}{ccc}
(S,F) & \longrightarrow & (T,G) \in \mathrm{Dom}_X^\Gamma \\
\Big\uparrow{\scriptstyle p'} & & \Big\uparrow{\scriptstyle p'} \\
(S,U) & \xrightarrow{f:S\to T} & (T,U_G) \in \mathrm{Dom}_X^{00}
\end{array}
$$

This category of 'lifts' is equivalent to the category of all finite subsets $F \subseteq \mathrm{Hom}(S,X)$ such that $U_F = U$, and such that $\{g \circ f : T \to X : g \in G\} \subseteq F$, with morphisms being the opposite of the inclusion of Fs. This category is non-empty, because the assumption that $(S,U) \in \mathrm{Dom}_X^{00}$ implies that it is the image of some (S,F'), and then $(S, F' \cup f^*G)$ completes the square. It also admits finite products (given by the union of Fs), thus is weakly contractible by [Lur11b, Lemma 2.4.6]. $\qquad\square$

6. Some 'homological contractibility' results

In this section we present a few results which relate the D-module categories associated to different moduli spaces of the kind we have been considering. Namely, we prove that certain maps between the spaces induce, via pullback, fully faithful functors on D-module categories. These results are of interest in the geometric Langlands program.

We emphasize the difference between the results we will discuss below, and those discussed in §5. Previously we compared the D-module categories of different functors of points classifying the same type of generic structures in different ways. Below we compare D-module categories associated to moduli problems parametrizing different generic structures.

Full faithfulness of D-module pullback has implications for classical invariants such as homology groups, and we start by pointing these out in §6.1.

6.1 The homology of a functor of points
In this subsection we define the homology groups of an arbitrary functor of points, and relate this invariant to the category of D-modules.

6.1.1 *Motivation.* To every scheme S of finite type over $\mathbb{C}$, we may associate its analytic topological space, S^{an}. By the *homology* of the scheme S, we mean the topological (singular) homology of S^{an} with coefficients in $\mathbb{C}$.

Let $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$ be any functor of points over $\mathbb{C}$. We define the homotopy type of $\mathcal{F}$ to be the homotopy colimit of all the points of $\mathcal{F}$,

$$\mathrm{type}(\mathcal{F}) := \underset{S \to \mathcal{F}}{\mathrm{hocolim}}(S^{\mathrm{an}}).$$

It is the homology groups of this homotopy type that we are after. The point of the circuitous definition for the homology of $\mathcal{F}$ given below is to have it presented in terms of D-module categories. In Proposition 6.1.7 we prove that (over $\mathbb{C}$) both notions of homology agree.

Notation 6.1.2. For a functor of points, $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$, and a pair of D-modules $M, N \in \mathfrak{Dmod}(\mathcal{F})$ we denote the mapping space (an ∞-groupoid) by

$$\mathrm{Map}_{\mathcal{F}}(M, N) := \mathrm{Map}_{\mathfrak{Dmod}(\mathcal{F})}(M, N).$$

6.1.3 Let $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$ be an arbitrary functor of points, and let

$$\mathcal{F} \xrightarrow{t} \mathrm{spec}(k) =: \mathrm{pt}$$

denote the map to the terminal object. We denote by **Vect** the stable ∞-category of chain complexes of vector spaces over k, mod quasi-isomorphism (whose homotopy category is equivalent to the derived category of the ordinary category of k-vector spaces). We shall identify $\mathfrak{Dmod}(\mathrm{spec}(k)) = \mathfrak{Dmod}(\mathrm{pt}) \cong \mathbf{Vect}$.

A left adjoint, $t_!$, to the pullback functor $\mathfrak{Dmod}(\mathcal{F}) \xleftarrow{t^!} \mathbf{Vect}$ may not be globally defined, but nonetheless makes sense as a partial functor, defined on the full subcategory of those $\mathcal{G} \in \mathfrak{Dmod}(\mathcal{F})$ for which the functor

$$\mathbf{Vect} \longrightarrow \mathbf{Gpd}_{\infty} \tag{6.1}$$

$$V \longmapsto \mathrm{Map}_{\mathcal{F}}(\mathcal{G}, t^! V)$$

is co-representable. For such $\mathcal{G}$, the object $t_! \mathcal{G} \in \mathbf{Vect}$ is such a co-representing object.

DEFINITION 6.1.4. The *canonical sheaf* of a functor of points, $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$, is

$$\omega_{\mathcal{F}} := t^!(k_{\mathrm{pt}}) \in \mathfrak{Dmod}(\mathcal{F}).$$

LEMMA 6.1.5. *Let* $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$. *The partial functor* $t_!$ *is defined on* $\omega_{\mathcal{F}}$.

Proof. Define an object of **Vect**,

$$H := \underset{S \xrightarrow{s} \mathcal{F}}{\mathrm{colim}}(t_! \omega_S),$$

where the index diagram is the category of points of $\mathcal{F}$ (so each S is an affine scheme). We remark that $t_! \omega_S \in \mathbf{Vect}$ is well defined because ω_S is bounded holonomic.

We show that H co-represents the functor (6.1). Indeed,

$$\mathrm{Map}_{\mathcal{F}}(\omega_{\mathcal{F}}, t^! V) \cong \underset{S \xrightarrow{s} \mathcal{F}}{\lim} \mathrm{Map}_S(\omega_S, t^! V) \cong \underset{S \xrightarrow{s} \mathcal{F}}{\lim} \mathrm{Map}_{\mathrm{pt}}(t_! \omega_S, V)$$

$$\cong \mathrm{Map}_{\mathrm{pt}}\left(\underset{S \xrightarrow{s} \mathcal{F}}{\mathrm{colim}}(t_! \omega_S), V \right) = \mathrm{Map}_{\mathrm{pt}}(H, V).$$

We conclude that $t_! \omega_{\mathcal{F}}$ is defined. $\square$

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DEFINITION 6.1.6. We define the *homology* of $\mathcal{F}$ to be

$$\mathrm{H}_\bullet(\mathcal{F};k) := t_!\omega_{\mathcal{F}} \in \mathbf{Vect}.$$

It follows from the proof of Lemma 6.1.5 that

$$\mathrm{H}_\bullet(\mathcal{F};k) \cong \operatorname*{colim}_{S \xrightarrow{s} \mathcal{F}} (t_!\omega_S) = \operatorname*{colim}_{S \xrightarrow{s} \mathcal{F}} \mathrm{H}_\bullet(S;k).$$

The following well-known proposition justifies our use of the word 'homology' (we include a proof for completeness).

PROPOSITION 6.1.7. *Assume* $k = \mathbb{C}$, *and let* $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$. *Then*

$$\mathrm{H}_\bullet(\mathcal{F};\mathbb{C}) \cong \mathrm{H}_\bullet^{\mathrm{top}}(\mathrm{type}(\mathcal{F});\mathbb{C})$$

where $\mathrm{H}_\bullet^{\mathrm{top}}$ *denotes topological homology.*

Proof. Since both homology theories are the left Kan extensions from affine schemes (equivalently, they are colimit preserving), it suffices to consider the case when $\mathcal{F}$ is representable by an affine scheme S.

For an affine scheme S, ω_S is a bounded holonomic complex, and using the Riemann–Hilbert correspondence we obtain an equivalence

$$\mathrm{H}_\bullet(S;\mathbb{C}) = t_!t^!\mathbb{C}_{\mathrm{pt}} \cong t_!^c t_c^!\mathbb{C}_{\mathrm{pt}} \cong$$

where $t_c^!$ and $t_!^c$ denote the !-functors on the (derived) category of constructible sheaves of vector spaces on S^{an}. Denote the duality functor on constructible sheaves by $\mathbb{D}$, and topological cohomology by $\mathrm{H}_{\mathrm{top}}^\bullet$. By Verdier duality we have an equivalence

$$\cong t_!^c t_c^! \mathbb{D}\mathbb{C}_{\mathrm{pt}} \cong \mathbb{D} t_*^c t_c^* \mathbb{C}_{\mathrm{pt}} = \mathbb{D}(\mathrm{H}_{\mathrm{top}}^\bullet(S^{\mathrm{an}};\mathbb{C})) \cong .$$

Using the universal coefficient theorem (and that S^{an} has finite-dimensional cohomologies), we conclude

$$\cong \mathrm{H}_\bullet^{\mathrm{top}}(S^{\mathrm{an}};\mathbb{C})$$

as claimed. $\square$

Remark 6.1.8. If a map between functors of points $\mathcal{F} \xrightarrow{f} \mathcal{G}$ induces a fully faithful pullback functor on

$$\mathfrak{Dmod}(\mathcal{F}) \xleftarrow[\supseteq]{f^!} \mathfrak{Dmod}(\mathcal{G})$$

then

$$\mathrm{H}_\bullet(\mathcal{F};k) \cong \mathrm{H}_\bullet(\mathcal{G};k)$$

since

$$\mathrm{Map}_{\mathrm{pt}}(t_!\omega_{\mathcal{F}},k) \cong \mathrm{Map}_{\mathcal{F}}(\omega_{\mathcal{F}},\omega_{\mathcal{F}}) \xleftarrow[\cong]{f^!} \mathrm{Map}_{\mathcal{G}}(\omega_{\mathcal{G}},\omega_{\mathcal{G}}) \cong \mathrm{Map}_{\mathrm{pt}}(t_!\omega_{\mathcal{G}},k).$$

In the particular case of $\mathcal{F} \xrightarrow{t} \mathrm{spec}(k)$, the full faithfulness of $t^!$ is equivalent to $\mathrm{H}_\bullet(\mathcal{F};k) \xrightarrow{\cong} \mathrm{H}_\bullet(k_{\mathrm{pt}};k) = k$ being an equivalence, that is, to $\mathcal{F}$ being *homologically contractible.*

6.2 Back to D-modules

The following theorem of Gaitsgory is the prototype for the main result of this section, as well as its foundation.

THEOREM 6.2.1 [Gai13, Theorem 1.8.2]. *Let Y be a connected affine scheme which can be covered by open subschemes U_α, each of which is isomorphic to an open subscheme of the affine space $\mathbb{A}^n$ (for some integer n). Then, the pullback functor*

$$\mathfrak{Dmod}(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}) \xleftarrow{\;t^!\;} \mathfrak{Dmod}(\mathrm{spec}(k)) = \mathbf{Vect}$$

is fully faithful.

In particular, we conclude that under the assumptions of the theorem

$$\mathrm{H}_\bullet(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}\,;k) \cong k.$$

In this section we use Theorem 6.2.1 to obtain additional results of a similar nature.

Recall that the *a priori* premise of this paper is that 'the correct' (from a conceptual point of view) space of generic maps is presented by the functor of points $\mathrm{GMap}\,(X,Y)$, introduced in Example 2.2.3.

COROLLARY 6.2.2. *Let Y be as in Theorem 6.2.1. The pullback functor*

$$\mathfrak{Dmod}(\mathrm{GMap}\,(X,Y)) \leftarrow \mathfrak{Dmod}(\mathrm{spec}(k)) = \mathbf{Vect}$$

is fully faithful.

Proof. Consider the pullback functors

$$\mathfrak{Dmod}(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}) \xleftarrow{\;\pi^!\;} \mathfrak{Dmod}(\mathrm{GMap}\,(X,Y)) \xleftarrow{\;t^!\;} \mathfrak{Dmod}(\mathrm{spec}(k))\,.$$

The composition is fully faithful by Theorem 6.2.1. $\pi^!$ is fully faithful by Theorem 5.2.1. We conclude that $t^!$ is fully faithful. $\qquad\square$

The next result is a minor extension of Theorem 6.2.1, in which we remove the requirement that the target be affine.

THEOREM 6.2.3. *Let Y be a connected and separated scheme which can be covered by open subschemes U_α, each of which is isomorphic to an open subscheme of the affine space $\mathbb{A}^n$ (for some integer n). Then the pullback functor*

$$\mathfrak{Dmod}(\mathrm{GMap}\,(X,Y)) \xleftarrow{\;t^!\;} \mathfrak{Dmod}(\mathrm{spec}(k)) = \mathbf{Vect}$$

is fully faithful. When Y is projective, $t^!$ admits a (globally defined) left adjoint.

The main examples to consider for Y (aside from $\mathbb{A}^n$), are $\mathbb{P}^n$, a connected affine algebraic group G, and its flag variety G/B. We prove this theorem in § 6.2.7.

Recall the functors of points $\mathrm{Bun}_G^{H(\mathrm{gen})}$ and $\mathrm{Bun}_G^{1(\mathrm{gen})} \in \mathcal{P}shv(\mathfrak{Aff})$ which were introduced in Example 2.2.5. The following theorem is the main result of this section.

THEOREM 6.2.4. *Let G be a connected reductive algebraic group. Let H be a subgroup of G such that G/H is rational (e.g., $H = 1$, N, or any parabolic subgroup). Then the pullback functor*

$$\mathfrak{Dmod}(\mathrm{Bun}_G^{H(\mathrm{gen})}) \leftarrow \mathfrak{Dmod}(\mathrm{Bun}_G)$$

is fully faithful. When $H = B$, this pullback functor admits a (globally defined) left adjoint.

Theorem 6.2.4 is proven in § 6.2.11, after some preparations.

The existence of the left adjoint can be extended to include any parabolic subgroup, if the statement (and proof) of Proposition 4.1.6 is extended accordingly.

We remark that the existence of the left adjoint above (and in Theorem 6.2.7) is a proper-like property of the map $\mathrm{Bun}_G^{B(\mathrm{gen})} \to \mathrm{Bun}_G$ ($\mathrm{GMap}\,(X,Y) \to \mathrm{spec}(k)$), though this map is not schematic. We also emphasize, as a concrete application, that pullback full faithfulness implies that the homology groups of the spaces in question are isomorphic (see Remark 6.1.8).

The rest of this section contains the proofs (and supporting lemmas) of Theorems 6.2.3 and 6.2.4.

By a *Zariski cover* of presheaves we mean a morphism of presheaves, which becomes an effective epimorphism after sheafification in the Zariski Grothendieck topology.

LEMMA 6.2.5. *The functor*

$$\mathrm{GMap}\,(X,-) : \mathfrak{S} \to \mathcal{P}shv(\mathfrak{Aff})$$

carries Zariski covers to Zariski covers.

Proof. Let Y be a scheme, and $\{Y_i \to Y\}_{i \in I}$ its finite cover by open subschemes. We must show that for every point $S \xrightarrow{s} \mathrm{GMap}\,(X,Y)$, there exist a Zariski cover $\tilde{S} \to S$, and a lift

$$
\begin{array}{ccc}
& & \coprod_{i \in I} \mathrm{GMap}\,(X,Y_i) \\
& \nearrow & \downarrow \\
\tilde{S} \hookrightarrow S & \xrightarrow{s} & \mathrm{GMap}\,(X,Y)
\end{array}
$$

The point s is presented by a point $(S,U) \in \mathrm{Dom}_X$, together with a regular map $U \to Y$. For every $i \in I$, let $U_i := U \times_{Y_i} Y \subseteq U$ (it is an open subscheme of U), and let $S_i \subseteq S$ be the open subscheme which is the image of $U_i \to S \times X \to S$. The composition $U_i \to U \to Y$ lands in Y_i, and thus determines a lift

$$
\begin{array}{ccc}
& & \mathrm{GMap}\,(X,Y_i) \\
& \nearrow & \downarrow \\
S_i \hookrightarrow S & \xrightarrow{s} & \mathrm{GMap}\,(X,Y)
\end{array}
$$

Taking $\tilde{S} = \coprod S_i$, the map $\coprod S_i \to \coprod \mathrm{GMap}\,(X,Y_i)$ is the sought-after lift of s. $\square$

LEMMA 6.2.6. *The functor*

$$\mathrm{GMap}\,(X,-) : \mathfrak{Aff} \to \mathcal{P}shv(\mathfrak{Aff})$$

preserves finite limits.

Proof. $\mathrm{GMap}\,(X,-)$ is the composition

$$\mathfrak{Aff} \xrightarrow{\mathrm{GMap}(X,-)_{\mathrm{Dom}_X}} \mathcal{P}shv(\mathrm{Dom}_X) \xrightarrow{\mathrm{LKE}_q} \mathcal{P}shv(\mathfrak{Aff})$$

$\mathrm{GMap}\,(X,-)_{\mathrm{Dom}_X}$ preserves (all) limits, and LKE_q preserves finite limits. $\square$

6.2.7 *Proof of Theorem 6.2.3.* The theorem is now an almost immediate result of Lemmas 6.2.6 and 6.2.5.

Let $\{U_i \to Y\}_{i \in I}$ be a cover of Y by its affine open subschemes, which are each isomorphic to an open subscheme of $\mathbb{A}^n$. We note that since Y is separated, every intersection of the U_i has the same property.

Construct the Čech complex corresponding to the cover

$$\Delta^{\mathrm{op}} \xrightarrow{\quad U_\bullet \quad} \mathfrak{Aff}$$

$$[n] \longmapsto \coprod_{|\bar{i}|=n} U_{\bar{i}}$$

where $\bar{i} = (i_1, \ldots, i_n)$ is a multi-index of elements in I, and $U_{\bar{i}} = \bigcap_{k=1}^n U_{i_k}$. We have that $Y = \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} (\coprod_{|\bar{i}|=n} U_{\bar{i}})$. By Lemma 6.2.6, the simplicial object

$$\Delta^{\mathrm{op}} \xrightarrow{\quad \mathrm{GMap}(X, U_\bullet) \quad} \mathcal{P}shv(\mathfrak{Aff})$$

$$[n] \longmapsto \coprod_{|\bar{i}|=n} \mathrm{GMap}\,(X, U_{\bar{i}})$$

is the Čech nerve of $\{\mathrm{GMap}\,(X, U_i) \to \mathrm{GMap}\,(X, Y)\}_{i \in I}$, which is a Zariski cover by Lemma 6.2.5. We conclude that the homology of $\mathrm{GMap}\,(X, Y)$ is isomorphic to that of a point, being the colimit

$$\mathrm{H}_\bullet(\mathrm{GMap}\,(X, Y); k) \cong \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} \mathrm{H}_\bullet\left(\coprod_{|\bar{i}|=n} \mathrm{GMap}\,(X, U_{\bar{i}}); k\right)$$

$$\cong \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} \mathrm{H}_\bullet\left(\coprod_{|\bar{i}|=n} \mathrm{spec}(k); k\right) \cong \mathrm{H}_\bullet(\mathrm{spec}(k); k).$$

Finally, the equivalence $\mathrm{H}_\bullet(\mathcal{F}) \cong \mathrm{H}_\bullet(\mathrm{spec}(k); k)$ implies the full faithfulness of $t^!$ (see Remark 6.1.8).

Regarding the existence of the left adjoint, when Y is projective, this is a restatement of Corollary 3.2.4(2). $\qquad\square$

We continue with the preparations for the proof of the Theorem 6.2.4. The following is a corollary of Lemma 6.2.6.

COROLLARY 6.2.8. *Let be G an algebraic group.*

(i) $\mathrm{GMap}\,(X, G)$ *is a group object in* $\mathcal{P}shv(\mathfrak{Aff})$.

(ii) *If Y is a scheme acted on by G, then $\mathrm{GMap}\,(X, Y)$ is acted on by $\mathrm{GMap}\,(X, G)$.*

DEFINITION 6.2.9. A map of presheaves $\mathcal{E} \to \mathcal{B}$ is an *fppf-locally trivial fibration with fiber $\mathcal{F}$*, if there exist an fppf cover $\mathcal{B}' \to \mathcal{B}$ (i.e., a morphism of presheaves which becomes an effective epimorphism after fppf sheafification), and a map

$$\mathcal{B}' \times_\mathcal{B} \mathcal{E} \to \mathcal{F}$$

which exhibits the former as a product $\mathcal{B}' \times_\mathcal{B} \mathcal{E} \cong \mathcal{F} \times \mathcal{B}'$.

LEMMA 6.2.10. *Let $\mathcal{E} \xrightarrow{p} \mathcal{B}$ be an fppf-locally trivial fibration with fiber $\mathcal{F}$. If $\mathfrak{Dmod}(\mathcal{F}) \xleftarrow{t^!} \mathfrak{Dmod}(\mathrm{spec}(k))$ is fully faithful, then $\mathfrak{Dmod}(\mathcal{E}) \xleftarrow{p^!} \mathfrak{Dmod}(\mathcal{B})$ is fully faithful.*

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Proof. Let $M, N \in \mathfrak{Dmod}(\mathcal{B})$. We must show that

$$\mathrm{Map}_{\mathcal{E}}(p^! M, p^! N) \leftarrow \mathrm{Map}_{\mathcal{B}}(M, N) \qquad (*)$$

is an equivalence of ∞-groupoids.

Fix a Cartesian square

$$
\begin{array}{ccc}
\mathcal{F} \times \mathcal{B}_0 & \longrightarrow & \mathcal{E} \\
\downarrow{\scriptstyle \ulcorner} & & \downarrow \\
\mathcal{B}_0 & \longrightarrow & \mathcal{B}
\end{array}
$$

in which $\mathcal{B}_0 \to \mathcal{B}$ is an fppf cover. Denote the Čech simplicial complex associated with the cover $\mathcal{B}_0 \to \mathcal{B}$ by

$$\Delta^{\mathrm{op}} \xrightarrow{\;\mathcal{B}_\bullet\;} \mathcal{P}shv(\mathfrak{Aff})$$

$$[n] \longmapsto \mathcal{B}_n := \underbrace{\mathcal{B}_0 \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathcal{B}_0}_{n\text{-times}}$$

and the one associated with the cover $\mathcal{B}_0 \times \mathcal{F} \to \mathcal{E}$ by

$$\Delta^{\mathrm{op}} \xrightarrow{\;(\mathcal{B}_0 \times \mathcal{F})_\bullet\;} \mathcal{P}shv(\mathfrak{Aff})$$

$$[n] \longmapsto (\mathcal{B}_0 \times \mathcal{F})_n := \underbrace{(\mathcal{B}_0 \times \mathcal{F}) \times_{\mathcal{E}} \cdots \times_{\mathcal{E}} (\mathcal{B}_0 \times \mathcal{F})}_{n\text{-times}}$$

There exist equivalences of stable ∞-categories

$$\mathfrak{Dmod}(\mathcal{B}) \cong \lim_{[n] \in \Delta} \mathfrak{Dmod}(\mathcal{B}_n) \quad \text{and} \quad \mathfrak{Dmod}(\mathcal{E}) \cong \lim_{[n] \in \Delta} \mathfrak{Dmod}((\mathcal{B}_0 \times \mathcal{F})_n)$$

and $p^!$ is induced by a transformation of the cosimplicial diagrams.

Let M_n and N_n denote the images of M and N in $\mathfrak{Dmod}(\mathcal{B}_n)$. Let $(p^! M)_n$ and $(p^! N)_n$ denote the images of M and N in $\mathfrak{Dmod}((\mathcal{B}_0 \times \mathcal{F})_n)$. We have equivalences of ∞-groupoids

$$\mathrm{Map}_{\mathcal{B}}(M, N) \cong \lim_{[n] \in \Delta} \mathrm{Map}_{\mathcal{B}_n}(M_n, N_n)$$

and

$$\mathrm{Map}_{\mathcal{E}}(p^! M, p^! N) \cong \lim_{[n] \in \Delta} \mathrm{Map}_{(\mathcal{B}_0 \times_{\mathcal{B}} \mathcal{E})_n}((p^! M)_n, (p^! N)_n).$$

We have that $(p^! M)_n \cong p_n^! N_n$, where p_n is the map $(\mathcal{B}_0 \times_{\mathcal{B}} \mathcal{E})_n \xrightarrow{p_n} \mathcal{B}_n$. Furthermore, the map $(*)$ is the limit of the maps

$$\mathrm{Map}_{(\mathcal{B}_0 \times_{\mathcal{B}} \mathcal{E})_n}(p_n^! M_n, p_n^! N_n) \leftarrow \mathrm{Map}_{\mathcal{B}_n}(M_n, N_n'). \qquad (**)$$

Finally, for each n we have a commuting diagram

$$
\begin{array}{ccc}
(\mathcal{B}_0 \times_{\mathcal{B}} \mathcal{E})_n & \xrightarrow{\;\cong\;} & \mathcal{F} \times \mathcal{B}_n \\
\downarrow{\scriptstyle p_n} & & \downarrow \\
\mathcal{B}_n & \xrightarrow{\;=\;} & \mathcal{B}_n
\end{array}
$$

Consequently, for every $[n] \in \Delta$, the functor $p_n^!$ is fully faithful, and we see that map $(**)$ is an equivalence. We conclude that the map $(*)$ is an equivalence. $\qquad \square$

The proof below proceeds by showing that the map $\mathrm{Bun}_G^{H(\mathrm{gen})} \to \mathrm{Bun}_G$ is a fibration with contractible fibers. After the proof we indicate a strategy for another proof, similar to that of Theorem 6.2.3.

6.2.11 *Proof of Theorem 6.2.4.* Observe that there exists a Cartesian square

$$
\begin{array}{ccc}
\mathrm{GMap}\,(X, G/H) \times \mathrm{Bun}_G^{1(\mathrm{gen})} & \longrightarrow & \mathrm{Bun}_G^{H(\mathrm{gen})} \\
\downarrow & & \downarrow \\
\mathrm{Bun}_G^{1(\mathrm{gen})} & \longrightarrow & \mathrm{Bun}_G
\end{array}
$$

The functor $\mathrm{Bun}_G^{1(\mathrm{gen})} \to \mathrm{Bun}_G$ becomes an effective epimorphism after étale sheafification. Indeed, if $\mathcal{P}_G$ is G-torsor on $S \times X$ then, by the Drinfeld–Simpson theorem [DS95, Theorem 2], there exists an étale base change $S' \to S$ such that $\mathcal{P}_G \times_S S'$ is Zariski locally trivial, hence admits a generic trivialization.

Our assumptions on G/H imply that it may be covered by open subschemes which are isomorphic to open subschemes of affine space. Thus by Theorem 6.2.3,

$$
\mathfrak{Dmod}(\mathrm{GMap}\,(X, G/H)) \leftarrow \mathfrak{Dmod}(\mathrm{spec}(k))
$$

is fully faithful. The full faithfulness of the pullback functor

$$
\mathfrak{Dmod}(\mathrm{Bun}_G^{H(\mathrm{gen})}) \leftarrow \mathfrak{Dmod}(\mathrm{Bun}_G)
$$

now follows from Lemma 6.2.10.

In the case when $H = B$, the existence of a left adjoint is a restatement of Corollary 4.1.7(2). □

Remark 6.2.12. A different proof of the theorem may be deduced from the following statement.

Let $Y \to S \times X$ be an fppf fiber bundle with fiber F, which becomes Zariski locally trivial, after a suitable fppf base change $\tilde{S} \to S$, and such that $\mathfrak{Dmod}(F) \leftarrow \mathfrak{Dmod}(\mathrm{spec}(k))$ is fully faithful. Then the pullback functor

$$
\mathfrak{Dmod}(\mathrm{GSect}_S\,(S \times X, Y)) \leftarrow \mathfrak{Dmod}(S)
$$

is fully faithful.

A. Appendix

A.1 Limits and colimits of adjoint diagrams
Let

$$
I \xrightarrow{\ G\ } \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}
$$

$$
i \longmapsto C_i
$$

be a small diagram. If for every morphism, $i \xrightarrow{f} j$ in I, the functor $C_i \xrightarrow{G(f)} C_j$ admits a left adjoint,[10] then there exists a unique diagram (up to contractible ambiguity)

$$I^{\mathrm{op}} \xrightarrow{F} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$$

$$i \longmapsto C_i$$

such that for every morphism, $i \xrightarrow{f} j$, the functor $C_j \xrightarrow{F(f)} C_i$ is left adjoint to $G(f)$. Let us call the pair of diagrams, F and G, *adjoint*.

The following lemma appears in [Gai11, 1.3.3], where it is attributed to J. Lurie.

LEMMA A.1.1. *Let F and G be adjoint I-diagrams as above.*

(i) *There exists an equivalence of stable ∞-categories*

$$\operatorname*{colim}_{i \in I^{\mathrm{op}}} F(i) \cong \lim_{i \in I} G(i)$$

   (the colimit and limit are taken within $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$).

(ii) *For every $j \in I$, the natural functors*

$$C_j \xrightarrow{\delta_j} \operatorname{colim}_{i \in I^{\mathrm{op}}} F(i),$$

$$C_j \xleftarrow{\pi_j} \lim_{i \in I} G(i)$$

   become adjoint, (δ_j, π_j), after identifying the categories on the right via (1).

(iii) *Let $C \in \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$, and let $I^{\triangleleft} \xrightarrow{G^{\triangleleft}} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$ be a co-augmentation of C over of G, such that for each i, the functor $C \to C_i$ admits a left adjoint. Then the natural functors*

$$\operatorname{colim}_{i \in I^{\mathrm{op}}} F(i) \longrightarrow C,$$

$$\lim_{i \in I} G(i) \longleftarrow C$$

   become adjoint, after identifying the categories on the left via (1).

Proof. (i) The categories $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$ and $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,R}}$ both admit small limits, and the inclusion into $\widehat{\mathbf{Cat}}_\infty$ preserves these ([Lur09, 5.5.3.5, 5.5.3.18] and [Lur11a, 1.1.4.4]). Consequently, since the diagram G lands in both categories (viewed as subcategories of $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex}}$), we have an equivalence

$$\lim(I \xrightarrow{G} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}) \cong \lim(I \xrightarrow{G} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,R}}).$$

There exists a duality[11]

$$\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,R}} \xrightarrow{\cong} (\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}})^{\mathrm{op}}$$

which is the identity on objects, and carries each functor to its left adjoint. It carries a limit cone $I^{\triangleleft} \xrightarrow{G^{\lim}} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,R}}$ for G, to a colimit cone $(I^{\mathrm{op}})^{\triangleright} \xrightarrow{F^{\mathrm{colim}}} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$ for F, supported on the

---

[10] Not to be confused with the right adjoint it admits by virtue of being a morphism in $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$.

[11] Thus, $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$ and $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,R}}$ admit colimits as well, but these are not (in general) preserved by the inclusion into $\widehat{\mathbf{Cat}}_\infty$.

same objects. In particular, restricting to the cone point, we get an equivalence

$$\lim(I \xrightarrow{G} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,R}}) \cong \mathrm{colim}\,(I^{\mathrm{op}} \xrightarrow{F} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}).$$

(ii) In the limit and colimit diagrams above, the functors $C_j \xleftarrow{\pi_j} \lim_{i \in I} G(i)$ and $C_j \xrightarrow{\delta_j} \mathrm{colim}_{i \in I^{\mathrm{op}}} F(i)$ correspond under the duality, hence are adjoint.

(iii) By the same argument as in the first part of (1), the induced functor $C \to \lim_{i \in I} G(i)$ admits a left adjoint, thus can be thought of as a map in $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,R}}$. Dualizing, we get an augmented I^{op}-diagram, $(I^{\mathrm{op}})^\triangleright \xrightarrow{F^\triangleright} \widehat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$, whence the assertion follows. $\qquad\square$

A.2 A comparison lemma for sites
The following lemma is an analog of the 'comparison lemma' [Joh02, Theorem 2.2.3], which applies to sheaves of sets (cf. [Lur09, Warning 7.1.1.4]).

LEMMA A.2.1. *Let C be a small category with a Grothendieck topology, and let $C^0 \xrightarrow[\subseteq]{j} C$ be a full subcategory. Assume that:*

(i) *C admits all finite limits;*
(ii) *for any fibered product in C, $c_1 \times_c c_2$, if $c_1, c_2 \in C^0$ then $c_1 \times_c c_2 \in C^0$ (we do not assume that $c \in C^0$);*
(iii) *C^0 is dense in C – that is, every object in C has a cover by objects in C^0.*

Then the restriction functor

$$\mathcal{S}hv(C^0) \xleftarrow{j_*} \mathcal{S}hv(C)$$

is an equivalence, where the topology of C^0 is the pullback of the topology of C.

For example, the full subcategory $\mathrm{Dom}_X^{00} \subseteq \mathrm{Dom}_X$ satisfies the assumptions of this lemma (see § 5.6.4).

The proof uses the slice category notation introduced in Notation 5.6.2. We start with the following claim.

CLAIM A.2.2. *In the context of Lemma A.2.1, the right Kan extension functor*

$$\mathcal{S}hv(C^0) \xrightarrow{\mathrm{RKE}_j} \mathcal{P}shv(C)$$

lands in $\mathcal{S}hv(C)$.

Proof of Lemma A.2.1. Assuming Claim A.2.2, it suffices to prove that the resulting adjoint functors (j_*, RKE_j)

$$\mathcal{S}hv(C) \underset{j}{\overset{j_*}{\rightleftarrows}} \mathcal{S}hv(C^0)$$

are inverse equivalences. Indeed, it is immediate that the co-unit transformation $j_* \circ \mathrm{RKE}_j \to 1_{C^0}$ is an equivalence.

It remains to show that the unit transformation is also an equivalence. Let $\mathcal{F} \in \mathcal{S}hv(C)$, let $c \in C$, and let us prove that

$$\mathcal{F}(c) \to \mathrm{RKE}_j \circ j_* \mathcal{F}(c)$$

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is an equivalence of ∞-groupoids (we emphasize that $\mathcal{F}$ is assumed to be a sheaf, and not an arbitrary presheaf). It is *a priori* true that this map is an equivalence whenever $c \in C^0$. For general $c \in C$, let $c^0 \xrightarrow{f} c$ be a cover with $c^0 \in C^0$; since C is assumed to admit all limits, $\mathcal{F}(c)$ may be calculated using the Čech complex associated to f. By assumption (2), all the terms in this complex belong to C^0 and the assertion that the unit transformation is an equivalence follows. $\qquad\square$

Proof of Claim A.2.2. Let $\mathcal{F}_0 \in \mathcal{S}hv(C^0)$, and denote $\mathcal{F} := \mathrm{RKE}_j(\mathcal{F}_0)$. Let $c \in C$, and let $\mathcal{S}_c \subseteq C_{/c}$ be a covering sieve. We must show that
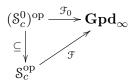
$$\lim(\mathcal{S}_c^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathbf{Gpd}_\infty) \leftarrow \mathcal{F}(c) \tag{A.1}$$

is an equivalence.

The categories $C_{/c}^0$, and $\mathcal{S}_c$ are both full subcategories of $C_{/c}$, and we denote their intersection

$$\mathcal{S}_c^0 := C_{/c}^0 \cap \mathcal{S}_c.$$

The triangle

$$
\begin{array}{ccc}
(\mathcal{S}_c^0)^{\mathrm{op}} & \xrightarrow{\mathcal{F}_0} & \mathbf{Gpd}_\infty \\
{\scriptstyle\subseteq}\downarrow & \nearrow{\scriptstyle\mathcal{F}} & \\
\mathcal{S}_c^{\mathrm{op}} & &
\end{array}
$$

is a right Kan extension, since for every $d \to c \in \mathcal{S}_c$ we have that $C_{/d}^0 \xrightarrow{\cong} (\mathcal{S}_c^0)_{/d \to c}$. Thus it suffices to show that

$$\lim((\mathcal{S}_c^0)^{\mathrm{op}} \xrightarrow{\mathcal{F}_0} \mathbf{Gpd}_\infty) \leftarrow \lim((C_{/c}^0)^{\mathrm{op}} \xrightarrow{\mathcal{F}_0} \mathbf{Gpd}_\infty) \cong \mathcal{F}(c)$$

is an equivalence. In turn, the latter equivalence will follow if we show that the following triangle is a right Kan extension:

$$
\begin{array}{ccc}
(\mathcal{S}_c^0)^{\mathrm{op}} & \xrightarrow{\mathcal{F}_0} & \mathbf{Gpd}_\infty \\
{\scriptstyle\subseteq}\downarrow & \nearrow{\scriptstyle\mathcal{F}_0} & \\
(C_{/c}^0)^{\mathrm{op}} & &
\end{array}
\tag{A.2}
$$

We now use our assumptions on the relation between C and C^0. Let $c^0 \xrightarrow{f} c$ where $c^0 \in C^0$. Using hypothesis (3), conclude that $\mathcal{S}_c^0$ generates a covering sieve over c, in C. It is always true that the maps

$$\{c_i \times_c c^0 \to c^0 : (c_i \to c) \in \mathcal{S}_c^0\}$$

generate a covering sieve, over c^0, in C. However, according to hypothesis (2), each of the fiber products belongs to C^0, so that the latter maps also generate a covering sieve in C^0 (over c_0), which is simply the fibered product

$$
\begin{array}{ccc}
(\mathcal{S}_c^0)_{/(c^0 \to c)} & \longrightarrow & \mathcal{S}_c^0 \\
{\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\
C_{/c^0}^0 & \xrightarrow{\circ f} & C_{/c}^0
\end{array}
$$

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Finally, since $\mathcal{F}_0$ is a sheaf on C^0 we have an equivalence

$$\lim(((\mathcal{S}_c^0)_{/(c^0 \to c)})^{\mathrm{op}} \xrightarrow{\mathcal{F}_0} \mathbf{Gpd}_\infty) \xleftarrow{\cong} \mathcal{F}_0(c^0)$$

implying that (A.2) is a right Kan extension. Tracing back, we conclude that $\mathrm{RKE}_j\mathcal{F}_0 \in \mathcal{P}shv(C)$ is a sheaf. $\qquad\square$

A.3 Points of the Ran space

PROPOSITION A.3.1. *The functor of points* $\mathfrak{Aff}^{\mathrm{op}} \xrightarrow{\mathcal{R}an_X} \mathbf{Gpd}_\infty$ *takes values in sets. Namely, for every* $S \in \mathfrak{Aff}$,

$$\mathcal{R}an_X(S) = \{F \subseteq \mathrm{Hom}(S,X) : F \text{ finite, non-empty}\}.$$

Proof. Consider the augmented $\mathfrak{Fin}_{\mathrm{sur}}^{\mathrm{op}}$ diagram

$$(\mathfrak{Fin}_{\mathrm{sur}} \cup \{\emptyset\})^{\mathrm{op}} \longrightarrow \mathbf{Set} \subseteq \mathbf{Gpd}_\infty$$

given by

(the circular arrows represent the action of the respective symmetric groups on n elements, S_n). By definition, $\mathcal{R}an_X(S)$ is the colimit in $\mathbf{Gpd}_\infty$ of the top row,[12] so we must show that diagram is a colimit diagram.

It suffices to prove that for every $F \in \{F \subseteq \mathrm{Hom}(S,X) : F \text{ finite}\}$, the following homotopy fiber is contractible:

Since colimits in $\mathbf{Gpd}_\infty$ are universal, this fiber is the colimit of the $\mathfrak{Fin}_{\mathrm{sur}}^{\mathrm{op}}$ diagram in $\mathbf{Gpd}_\infty$

where $\{n\}$ denotes a finite set with n elements and $\mathrm{Surj}(\{n\},F)$ is the set of surjections $\{n\} \twoheadrightarrow F$. We prove that this colimit is contractible. Applying the Grothendieck unstraightening construction, we get the Cartesian fibration

---

[12] We emphasize that we want to show this diagram is a homotopy colimit. That this is a colimit diagram in sets is obvious.

(where F is now considered as an abstract finite set). The homotopy type we are after is the weak homotopy type of the total space, $(\mathfrak{Fin}^{\mathrm{op}}_{\mathrm{sur}})_{/F}$, which is evidently contractible since it has a terminal element. $\qquad\square$

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