Bull. Austral. Math. Soc. 47B38
Vol. 59 (1999) [53–58]

HANKEL OPERATORS FROM THE SPACE OF BOUNDED ANALYTIC FUNCTIONS TO THE BLOCH SPACE

Ruhan Zhao

Boundedness and compactness of little Hankel operators from H^∞ to the Bloch space and the little Bloch space are characterised.

1. Introduction

Let $D = \{z : |z| < 1\}$ denote the unit disk in the complex plane $\mathbb{C}$. Let $A(D)$ be the set of all analytic functions in D. For $1 \leqslant p < \infty$, let $L^p(D)$ denote the Banach space of Lebesgue measurable functions f on D with

$$\|f\|_p = \left(\int_D |f(z)|^p \, dA(z)\right)^{1/p} < \infty,$$

where $dA(z)$ is the normalised area measure on D. The Bergman space L^p_a consists of the analytic functions which lie in $L^p(D)$. Let H^∞ denote the space of all bounded analytic functions f on D with norm $\|f\|_{H^\infty} = \sup_{z\in D}|f(z)|$.

For $f \in L^1(D)$ and $g \in H^\infty$, the (little) Hankel operator is defined by

$$h_f g = P(f\overline{g}), \tag{1}$$

where P denotes the Bergman projection, which is the orthogonal projection from $L^2(D)$ onto L^2_a. Thus for $h \in L^2(D)$,

$$Ph(z) = \int_D \frac{h(w)}{(1-\overline{w}z)^2} \, dA(w). \tag{2}$$

Note that, using (2), we can extend P to a linear operator from $L^1(D)$ into $A(D)$. Recall that the Bloch space B consists of the analytic functions f satisfying

$$\|f\|_B = |f(0)| + \sup_{z\in D}|f'(z)|\left(1-|z|^2\right) < \infty,$$

and the little Bloch space B_0 consists of the analytic functions f satisfying

$$\lim_{|z|\to 1}|f'(z)|\left(1-|z|^2\right) = 0.$$

The following result is well known:

Received 11th May, 1998

https://doi.org/10.1017/S0004972700032597 Published online by Cambridge University Press

Theorem A. *Let $1 < p < \infty$, and let f be analytic on D.*

(i) *The Hankel operator h_f is bounded on L_a^p if and only if $f \in B$;*

(ii) *The Hankel operator h_f is compact on L_a^p if and only if $f \in B_0$.*

See, for example, [**6**, Section 7.6], for the case $p = 2$. For general p, the proof is similar. Note that, in Zhu [**6**], the little Hankel operator h_f is defined by using the orthogonal projection from $L^2(D)$ onto $\overline{L_a^2}$. But, essentially, there is no difference between our Hankel operator defined above and $h_{\overline{f}}$ given in Zhu's book. In fact, it is easy to see that our Hankel operator defined by (1) is the same as $\overline{h}_{\overline{f}}$ in Zhu's book.

Because the Bergman projection is a bounded operator from the space $L^\infty(D)$ onto the Bloch space B, B is the natural limit of L_a^p as p tends to infinity. In view of Theorem A, one may guess that the spaces B and B_0 characterise the bounded and compact Hankel operators h_f from H^∞ to B. Solving this problem is the main purpose of this note. Our main results are the following two theorems.

Theorem 1. *Let $f \in L_a^1$. Then $h_f : H^\infty \to B$ is bounded if and only if $f \in B$.*

This theorem may be compared with [**3**, Theorem 3′ (ii)], which showed, if f is analytic, then the Hankel operator H_f on Hardy spaces is bounded from H^∞ to B if and only if $f \in BMOA$. Here $H_f g = \widehat{P}(f\overline{g})$, where $\widehat{P}$ is the orthogonal projection of $L^2(\partial D)$ onto the Hardy space H^2.

The next theorem deals with compactness. It turns out that the set of compact Hankel operators from H^∞ to the Bloch space B coincides with the set of bounded Hankel operators from H^∞ to the little Bloch space B_0.

Theorem 2. *Let $f \in L_a^1$. Then the following statements are equivalent:*

(i) *$h_f : H^\infty \to B$ is compact;*

(ii) *$h_f : H^\infty \to B_0$ is compact;*

(iii) *$h_f : H^\infty \to B_0$ is bounded;*

(iv) *$f \in B_0$.*

2. Proofs of Theorems

In our proofs, we shall frequently use the facts that the dual space of the little Bloch space B_0 is L_a^1 and the dual space of L_a^1 is the Bloch space B, that is, $B_0^* = L_a^1$ and $(L_a^1)^* = B$, under the following integral pairing

$$\langle f, g \rangle = \int_D f\overline{g}\, dA$$

(See, for example, [**6**, Theorem 5.1.4 and Theorem 5.2.8].) The proof of Theorem 1 is quite elementary.

https://doi.org/10.1017/S0004972700032597 Published online by Cambridge University Press

Proof of Theorem 1: Let $f \in B$, $g \in H^\infty$ and $h \in L^1_a$. A simple application of Fubini's Theorem shows that

$$\langle h, h_f g\rangle = \langle gh, f\rangle. \tag{3}$$

Since $\left(L^1_a\right)^* = B$, we get that

$$\|h_f g\|_B = \sup_{\|h\|_1 \leqslant 1} \left|\langle h, h_f g\rangle\right| = \sup_{\|h\|_1 \leqslant 1} \left|\langle gh, f\rangle\right|. \tag{4}$$

Since $g \in H^\infty$, it is obvious that for $h \in L^1_a$, we have $gh \in L^1_a$ and

$$\|gh\|_1 \leqslant \|g\|_{H^\infty}\|h\|_1.$$

Thus from (4) we have

$$\|h_f g\|_B \leqslant \sup_{\|h\|_1 \leqslant 1} \|gh\|_1\|f\|_B \leqslant \|g\|_{H^\infty}\|f\|_B.$$

Thus $h_f : H^\infty \to B$ is bounded.

Conversely, if $h_f : H^\infty \to B$ is bounded, then $f = h_f 1 \in B$. The proof is complete. ∎

It is easy to see that for $f \in L^1(D)$, $h_f = h_{Pf}$ (see also, [**6**, Proposition 7.6.2]). Thus we get immediately from Theorem 1 the following result.

Corollary 1. *Let $f \in L^1(D)$. Then $h_f : H^\infty \to B$ is bounded if and only if $Pf \in B$.*

The same idea is used for proving Theorem 2. However, more work is needed in this case. The following weak convergence lemma is needed.

Lemma 1. *Let $f \in L^1(D)$. Then the following statements are equivalent.*

(i) *$h_f : H^\infty \to B$ is compact.*

(ii) *If $\{g_n\}$ is a sequence that is bounded on H^∞ and converges to zero uniformly on compact subsets of D, then $\lim_{n\to\infty} \|h_f g_n\|_B = 0$.*

The proof is similar to the proof of the weak convergence theorem for composition operators given in Shapiro's book [**4**, p.29–30]. Note that the above lemma is still valid if we replace the Bloch space B by any *functional Banach space* (see [**2**, p.2] for the definition); these include many well-known function spaces. Note also that the condition $f \in L^1(D)$ is needed in the proof. We leave the details of the proof of Lemma 1 to the reader.

Proof of Theorem 2: We first prove (iv)⇒(iii). Let $f \in B_0$, $g \in H^\infty$ and $h \in L^1_a$. By [**6**,Theorem 5.2.5], there is a function $\varphi \in C_0(D)$ such that $f = P\varphi$, where

$C_0(D)$ denotes the algebra of complex continuous functions f on $\overline{D}$ with $f(z) \to 0$ as $|z| \to 1$. It is easy to see that for any $\psi \in L^1_a$,

$$\langle f, \psi\rangle = \langle P\varphi, \psi\rangle = \langle \varphi, \psi\rangle.$$

Thus from (3) we have

$$\langle h_f g, h\rangle = \langle f, gh\rangle = \langle \varphi, gh\rangle = \langle h_\varphi g, h\rangle. \tag{5}$$

The last equality is from Fubini's Theorem. Since (5) is true for any function $h \in L^1_a$, we see that

$$h_f g = h_\varphi g = P(\varphi\overline{g}).$$

Since $g \in H^\infty$ and $\varphi \in C_0(D)$, it is obvious that $\varphi\overline{g} \in C_0(D)$. Thus, again by [**6**, Theorem 5.2.5], we get that $P(\varphi\overline{g}) \in B_0$. Thus $h_f g \in B_0$ and so the Closed Graph Theorem implies that $h_f : H^\infty \to B_0$ is bounded.

Next we prove further that $h_f : H^\infty \to B_0$ is compact, or (iv)$\Rightarrow$(ii). Let $\{g_n\}$ be a bounded sequence in H^∞ such that $\|g_n\|_{H^\infty} \leqslant 1$ and $g_n(z) \to 0$ uniformly on compact subsets of D. Let $f \in B_0$. Then we have proved that $\{h_f g_n\}$ is a bounded sequence in B_0. By Lemma 1, in order to prove that $h_f : H^\infty \to B_0$ is compact, it is enough to prove that

$$\lim_{n\to\infty} \|h_f g_n\|_B = 0.$$

As before, since $f \in B_0$, there is a function $\varphi \in C_0(D)$, such that $f = P\varphi$. Thus

$$\begin{aligned}
\|h_f g_n\|_B &= \sup_{\|h\|_1 \leqslant 1} \left|\langle h, h_f g_n\rangle\right| = \sup_{\|h\|_1 \leqslant 1} \left|\langle g_n h, f\rangle\right| \\
&= \sup_{\|h\|_1 \leqslant 1} \left|\langle g_n h, \varphi\rangle\right| \\
&\leqslant \sup_{\|h\|_1 \leqslant 1} \left|\int_{\overline{D_r}} g_n h \overline{\varphi}\, dA\right| + \sup_{\|h\|_1 \leqslant 1} \left|\int_{D\setminus\overline{D_r}} g_n h \overline{\varphi}\, dA\right| \\
&= I_1 + I_2,
\end{aligned}$$

where $0 < r < 1$ and $\overline{D_r} = \{z \in D, |z| \leqslant r\}$. Now for any $\varepsilon > 0$, since $\varphi \in C_0(D)$, when r is sufficiently close to 1 we have $|\varphi(z)| < \varepsilon$ whenever $|z| > r$. Thus

$$I_2 \leqslant \sup_{\|h\|_1 \leqslant 1} \varepsilon \|g_n\|_{H^\infty} \int_{D\setminus\overline{D_r}} |h|\, dA < \varepsilon. \tag{6}$$

Since $g_n(z) \to 0$ uniformly on the set $\overline{D_r}$ for a fixed $r \in (0,1)$, we get that when n is big enough,

$$I_1 \leqslant \sup_{\|h\|_1 \leqslant 1} \varepsilon \|h\|_1 \|\varphi\|_{H^\infty} < C\varepsilon. \tag{7}$$

https://doi.org/10.1017/S0004972700032597 Published online by Cambridge University Press

Combining (6) and (7) we see that $\lim_{n\to\infty} \|h_f g_n\| = 0$ and so $h_f : H^\infty \to B_0$ is compact.

After this we can complete the circle among (ii), (iii) and (iv) by observing that (ii) obviously implies (iii) and (iii) implies (iv) since $f = h_f 1 \in B_0$ when $h_f : H^\infty \to B_0$ is bounded.

It is also obvious that (ii) implies (i). Thus what remains is the direction (i)$\Rightarrow$(iv). Let $h_f : H^\infty \to B$ be compact. For $0 < \alpha < 1$, let

$$g_a(z) = \frac{2z\left(1-|a|^2\right)^{1-\alpha}}{\left(1-\bar{a}z\right)^{1-\alpha}}.$$

Then $g_a \in H^\infty$, $\|g_a\|_{H^\infty} \leqslant 2$ and $g_a(z) \to 0$ uniformly on compact subsets of D when $|a| \to 1$. Since $h_f : H^\infty \to B$ is compact, we must have

$$\lim_{|a|\to 1} \|h_f g_a\|_B = 0,$$

which is, in view of the duality relation $\left(L_a^1\right)^* = B$, the same as

$$\lim_{|a|\to 1} \sup_{\|h\|_1 \leqslant 1} \left|\langle h, h_f g_a\rangle\right| = \lim_{|a|\to 1} \sup_{\|h\|_1 \leqslant 1} \left|\langle g_a h, f\rangle\right| = 0.$$

Let $\widetilde{h}_a(z) = \left(1-|a|^2\right)^\alpha / (1-\bar{a}z)^{2+\alpha}$. Then, by [**6**, Lemma 4.2.2], we get that

$$\|\widetilde{h}_a\|_1 = \int_D \frac{\left(1-|a|^2\right)^\alpha}{|1-\bar{a}z|^{2+\alpha}}\, dA(z) \leqslant \left(1-|a|^2\right)^\alpha \frac{M}{\left(1-|a|^2\right)^\alpha} = M < \infty.$$

Let $h_a = \widetilde{h}_a / M$. Then $\|h_a\|_1 \leqslant 1$ for any $a \in D$ and so we have

$$\lim_{|a|\to 1} \left|\langle g_a h_a, f\rangle\right| \leqslant \lim_{|a|\to 1} \sup_{\|h\|_1 \leqslant 1} \left|\langle g_a h, f\rangle\right| = 0.$$

Thus

$$
\begin{aligned}
(8) \qquad 0 &= \lim_{|a|\to 1} \left| \int_D g_a(z) h_a(z) \overline{f(z)}\, dA(z) \right| \\
&= \lim_{|a|\to 1} \left| \int_D \frac{\left(1-|a|^2\right)\overline{f(z)} 2z}{M\left(1-\bar{a}z\right)^3}\, dA(z) \right| \\
&= \lim_{|a|\to 1} \frac{1}{M} \left| \int_D \frac{f(z) 2\bar{z}}{\left(1-a\bar{z}\right)^3}\, dA(z) \right| \left(1-|a|^2\right).
\end{aligned}
$$

Since $f \in L_a^1(D)$, we have

$$f(a) = Pf(a) = \int_D \frac{f(z)}{\left(1-a\bar{z}\right)^2}\, dA(z).$$

https://doi.org/10.1017/S0004972700032597 Published online by Cambridge University Press

By taking derivatives with respect ot a on both sides we get

$$f'(a) = \int_D \frac{f(z)2\bar{z}}{(1-a\bar{z})^3}\, dA(z).$$

Thus the integral in the last line of (8) is $f'(a)$ and so we have from (8),

$$\lim_{|a|\to 1} |f'(a)|(1-|a|^2) = 0.$$

Therefore, $f \in B_0$, and the whole proof is complete. ∎

Similarly to Corollary 1, we immediately get from Theorem 2 the following result for a non-analytic function f.

COROLLARY 2. *Let $f \in L^1(D)$. Then the following statements are equivalent:*

(i) $h_f : H^\infty \to B$ *is compact;*
(ii) $h_f : H^\infty \to B_0$ *is compact;*
(iii) $h_f : H^\infty \to B_0$ *is bounded;*
(iv) $Pf \in B_0$.

FINAL REMARK. Although this note is formulated in the case of the unit disk D, the proof goes though for the case of the unit ball of $\mathbb{C}^n$. Thus all results are valid for the unit ball of $\mathbb{C}^n$. All materials which are needed for the proof in this case can be found, for example, in Zhu [5] and Choe [1].

REFERENCES

[1] B.R. Choe, 'Projections, the weighted Bergman spaces, and the Bloch space', *Proc. Amer. Math. Soc.* **108** (1990), 127–136.

[2] C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions* (CRC Press, Boca Raton, 1995).

[3] S. Janson, J. Peetre and S. Semmes, 'On the action of Hankel and Toeplitz operators on some function spaces', *Duke Math. J.* **51** (1984), 937–958.

[4] J. Shapiro, *Composition operators and classical function theory* (Springer-Verlag, Berlin, Heidelberg, New York, 1993).

[5] K. Zhu, 'The Bergman spaces, the Bloch space, and the Gleason's problem', *Trans. Amer. Math. Soc.* **309** (1988), 253–268.

[6] K. Zhu, *Operator theory in function spaces* (Marcel Dekker, New York, 1990).

Department of Mathematics
University of Joensuu
PO Box 111
Fin-80101 Joensuu
Finland

Current address:
Department of Mathematics
Faculty of Science
Kyoto University
Kyoto 606-8502
Japan
e-mail: zhao@kusm.kyoto-u.ac.jp