INTEGRATION OF SUBSPACES DERIVED FROM A LINEAR TRANSFORMATION FIELD

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1. Introduction. The problem we study is a generalization of a problem first solved by Tonolo **(6)**, then generalized successively by Schouten **(5)**, Nijenhuis **(4)**, Haantjes **(3)**, and Nijenhuis–Frölicher **(2)**. The Tonolo–Schouten approach is distinct from that of Nijenhuis–Haantjes–Frölicher in the sense that the former consider the problem on a Riemannian space, while the latter consider it on a manifold without any further structure.

The object of investigation is the integrability of the distribution θ of vector subspaces θ_p of the tangent space T_p to a manifold M, when θ_p is intrinsically related to a given field h on M, of linear transformations h_p on T_p. The research has so far been restricted to certain types of h. The result, under the weakest restriction, was that of Haantjes, which states that if h is of "type A"* then all the distributions are integrable if and only if the following condition is satisfied:

$$hh[h,h](u,v) + [h,h](hu,hv) - h[h,h](hu,v) - h[h,h](u,hv) = 0$$

where u, v are two vector fields over M, and $[h, h]$ is a vector 2-form introduced by Nijenhuis (cf. § 2).

We free ourselves from any restriction on h. Our result depends entirely on the local factorization of the characteristic polynomial χ of h. To each factor χ_i of χ, there corresponds a distribution θ_i and a projection operator $\epsilon_i(h)$, which is a polynomial in h, and the local integrability condition of θ_i is $(I - \epsilon_i(h))[\epsilon_i(h), \epsilon_i(h)] = 0$ (Theorem 4.2). To each product $\chi_{i_1} \ldots \chi_{i_k}$ of distinct factors of χ, there corresponds a distribution $\theta_{i_1 \ldots i_k}$. The necessary and sufficient condition for these distributions to be all locally integrable is $[\epsilon_i(h), \epsilon_i(h)] = 0$ for all i (Corollary 4.3).

2. Vector forms and projection operators. Let M be a C^∞-manifold and Φ the ring of C^∞-functions on M. By a neighbourhood of a point p in M, we mean an open, connected subset of M containing p.

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*h is said to be of type A if (i) there are functions $\alpha_1, \ldots, \alpha_g$ on M, such that $(\alpha_1)_p, \ldots, (\alpha_g)_p$ are distinct at each p, and give the eigenvalues of h_p, and if (ii) there are vector fields $v_{i1}, \ldots, v_{im_i}$ on M, $i = 1, \ldots, g$, $m_1 + \ldots + m_g = n$ such that $(v_{i1})_p, \ldots, (v_{im_i})_p$ are eigenvectors corresponding to $(\alpha_i)_p$ and are linearly independent.

Definition 2.1. A vector q-form is a C^∞-tensor field over M, skew-symmetric in the covariant part, of covariant degree q, and of contravariant degree 1.

Let h be a vector 1-form. Then we see that h is nothing but a rule which assigns to each point p of M a linear transformation h_p of the tangent space T_p at p to M. Following Nijenhuis **(4, 2)** we introduce a vector 2-form $[h, h]$ defined by

$$\tfrac{1}{2}[h, h](u, v) = [hu, hv] + hh[u, v] - h[hu, v] - h[u, hv], \tag{2.1}$$

where u, v are vector fields over M. That (2.1) does define a tensor, follows from the Φ-linearity in u and v of the right side of (2.1).*

Definition 2.2. A vector 1-form e satisfying $e^2 = e$ on a neighbourhood U is called a *projection operator on* U.

Remark 1. dim $e_q T_q$ is constant for $q \in U$, and we call this constant the *rank* of e. In fact, dim $e_q T_q$, which is an integer, is equal to the trace of e_q, which depends continuously on q, hence is a constant.

Remark 2. If e is a projection operator on U, so is $e' = I - e$, where I is the identity vector 1-form. We have $e + e' = I$, $ee' = e'e = 0$ and

$$T_q = e_q T_q \oplus e'_q T_q \qquad \text{for} \qquad q \in U.$$

Furthermore we have

$$[e, e] = [e', e']. \tag{2.2}$$

Definition 2.3 A law θ which assigns to each point p in a neighbourhood U of M, an r-dimensional vector subspace θ_p of the tangent space T_p of M at p, is called an *r-dimensional distribution* over U. If at each $p \in U$, we can find a neighbourhood U' of p, U' contained in U, and r C^∞-vector fields $X_1, \ldots, X_r$ over U', such that $(X_1)_q, \ldots, (X_r)_q$ form a basis for θ_q at each $q \in U'$, we say that θ *is* C^∞.

Definition 2.4. Let θ be an r-dimensional C^∞-distribution over a neighbourhood U of p. If there is a neighbourhood U' of p and, for each $q \in U'$, an r-dimensional submanifold N contained in U' and passing through q, such that $\theta_{q'}$ is the tangent space of N at each $q' \in N$, then we say that θ *is integrable in* U', *a neighbourhood of* p.

Definition 2.5. Let θ be a C^∞-distribution over a neighbourhood U of p. If there is a neighbourhood U' of p contained in U such that, for any two C^∞-vector fields X_1, X_2 over U', satisfying $(X_1)_q, (X_2)_q \in \theta_q$ $(q \in U')$, we have $[X_1, X_2]_q \in \theta_q$, then we say that θ *is involutive in* U'.

A C^∞-distribution θ over a neighbourhood U of p is integrable in a neighbourhood U' of p, contained in U, if and only if θ is involutive in the neighbourhood U' of p, Frobenius **(1)**.

Now, if e is a projection operator of rank r over U, then θ, defined by $q \to e_q T_q$, where T_q is the tangent space of M at $q \in U$, is an r-dimensional

*For details of this type of argument, see the proof of Proposition (3.4) in **(2)**.

C^∞-distribution over U. To see that θ is C^∞, choose a co-ordinate system $x_1, \ldots, x_n$ in a neighbourhood of q. Then we can pick r C^∞-vector fields from

$$e\frac{\partial}{\partial x_i}, i = 1, \ldots, n,$$

say,

$$e\frac{\partial}{\partial x_1}, \ldots, e\frac{\partial}{\partial x_r},$$

so that

$$\left(e\frac{\partial}{\partial x_1}\right)_{q'}, \ldots, \left(e\frac{\partial}{\partial x_r}\right)_{q'}$$

are linearly independent, hence form a basis for $e_{q'}T_{q'}$, for q' in a neighbourhood of q.

LEMMA 2.1. *Let e be a projection operator over a neighbourhood U of p, and let θ be the C^∞-distribution defined by $q \to e_q T_q$, $q \in U$. Then θ is integrable in a neighbourhood of p, if and only if $(I - e)[e, e] = 0$ on a neighbourhood of p.*

Proof. If u, v are two C^∞-vector fields over a neighbourhood of p, then we have

$$\begin{aligned}\tfrac{1}{2}(I - e)&[e, e](u, v)\\ &= (I - e)[eu, ev] - (I - e)e[eu, v] - (I - e)e[u, ev] + (I - e)e^2[u, v]\\ &= (I - e)[eu, ev].\end{aligned}$$

If u is a C^∞-vector field over a neighbourhood U' of p, then eu is a C^∞-vector field over U' such that $e_q u_q \in e_q T_q$, $q \in U'$. Conversely, if u is a C^∞-vector field over U' such that $u_q \in e_q T_q$, $q \in U'$, then $u_q = e_q u_q$, hence $u = eu$. Hence, using Frobenius' theroem, we see that θ is integrable in a neighbourhood of p if and only if $[eu, ev]_q \in e_q T_q$ for all q in a neighbourhood U'' of p, and all C^∞-vector fields u, v over U''. This condition is equivalent to $(I - e)[eu, ev]_q = 0$, and the computation above shows that the latter in turn is equivalent to $(I - e)[e, e](u, v)_q = 0$. Q.E.D.

If e_i, $i = 1, \ldots, g$ are projection operators on U, $p \in U$, satisfying

$$\sum_{i=1}^{g} e_i = I, \quad e_i e_j = e_j e_i,$$

then it can be shown that $e_i e_j = 0$ for $i \neq j$, and that $T_q = (e_1)_q T_q \oplus \ldots \oplus (e_g)_q T_q$ for $q \in U$. Let $\theta_{i_1 \ldots i_k}$ be the C^∞-distribution over U defined by

$$q \to (e_{i_1})_q T_q \oplus \ldots \oplus (e_{i_k})_q T_q.$$

Here $i_1, \ldots, i_k$ should be all distinct.

If $\theta_{1 \ldots g-1}$ and $\theta_{2 \ldots g}$ are both integrable in a neighbourhood of p, then using Frobenius' theorem, we see that $\theta_{2 \ldots g-1}$ is integrable in a neighbourhood of p. Repeating this argument, we have: the distributions

$$\theta_{i_1 \ldots i_k}\ (k = 1, \ldots, g - 1; \quad i_s = 1, \ldots, g; \quad s = 1, \ldots, k)$$

are all integrable in a neighbourhood of p if and only if the distributions $\theta_{i_1 \ldots i_{g-1}}$ are all integrable in a neighbourhood of p.

LEMMA 2.2. *The distributions*

$$\theta_{i_1 \ldots i_k} \quad (k = 1, \ldots, g-1; \quad i_s = 1, \ldots, g; \quad s = 1, \ldots, k)$$

are all integrable in a neighbourhood of p if and only if $[e_i, e_i] = 0$ for all i in a neighbourhood of p.

Proof. Notice first that $e_1 + \ldots \overset{i}{\wedge} \ldots + e_g = I - e_i$. Using Lemma 2.1. and (2.2) the integrability of the distribution $\theta_{1 \ldots \overset{i}{\wedge} \ldots g}$ can be expressed as, $e_i[e_i, e_i] = 0$.

Now if all the distributions $\theta_{i_1 \ldots i_k}$ are integrable in a neighbourhood of p, then in particular $\theta_{1 \ldots \overset{i}{\wedge} \ldots g}$ and θ_i are integrable in a neighbourhood of p, so we have $e_i[e_i, e_i] = 0$ and $(I - e_i)[e_i, e_i] = 0$, and hence $[e_i, e_i] = 0$ on a neighbourhood of p.

Conversely if $[e_i, e_i] = 0$ on a neighbourhood of p then of course $e_i[e_i, e_i] = 0$ on a neighbourhood of p, thus the integrability of $\theta_{1 \ldots \overset{i}{\wedge} \ldots g}$.

3. The characteristic polynomial of a vector 1-form. Let h be a vector 1-form on M and let λ be an indeterminate. Suppose $\{x_1, \ldots, x_n\}$ is a co-ordinate system in a neighbourhood of p in M. h has components $h_i^j(x)$ in this neighbourhood, where

$$h \frac{\partial}{\partial x_i} = \sum_{j=1}^{n} h_i^j(x) \frac{\partial}{\partial x_j}. \tag{3.1}$$

We can consider $\chi = \det||\lambda \delta_i^j - h_i^j(x)||$, which is a polynomial in λ of degree n with coefficients which are C^∞-functions of $(x_1, \ldots, x_n)$. It is easy to verify that the coefficients do not depend on the choice of the co-ordinate system, so we have an element χ in $\Phi[\lambda]$. χ is called *the characteristic polynomial of h.*

PROPOSITION 3.1. *Suppose χ_p, the characteristic polynomial of h_p, has a factorization over the reals R:*

$$\chi_p = K_1^{m_1} \ldots K_g^{m_g}, \tag{3.2}$$

where $K_i \in R[\lambda]$, with leading coefficients 1, *and K_i are all distinct and irreducible over R. Then there is a neighbourhood U of p, where χ has a unique factorization*

$$\chi = \chi_1 \ldots . \chi_g \text{ on } U \tag{3.3}$$

satisfying

(i) $\chi_i \in \Phi_U[\lambda]$, *where Φ_U is the ring of C^∞-functions on U;*
(ii) χ_i *has leading coefficient* 1, deg χ_i = deg $K_i^{m_i}$;
(iii) $(\chi_i)_p = K_i^{m_i}$;
(iv) $(\chi_i)_q$ *and* $(\chi_j)_q$ *are relatively prime for $q \in U$, $i \neq j$.*

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For the proof, we apply the following lemma repeatedly.

LEMMA 3.2. *Let $\phi \in \Phi[\lambda]$ with leading coefficient* 1. *Suppose for a point p in M, $\phi_p = PQ$; where $P, Q \in R[\lambda]$ (R = the real numbers), with leading coefficients* 1, *and relatively prime to each other. Then there is a neighbourhood U of p, and unique $\mu, \pi \in \Phi_U[\lambda]$ (Φ_U = the ring of C^∞-functions over U), with leading coefficients* 1, *such that $\phi = \mu\pi$ holds over U, and $\mu_p = P$, $\pi_p = Q$. Moreover U can be so chosen that μ_q and π_q are relatively prime at each $q \in U$.*

Proof. Let the degree of P and Q be k and l respectively. Let $x_i (i = 1, \ldots, k)$, $y_j (j = 1, \ldots, l)$, $z_s (s = 1, \ldots, k + l)$ be variables, and $x_0 = y_0 = z_0 = 1$. For $\{x_1, \ldots, x_k\}$, $\{y_1, \ldots, y_l\}$, and $\{z_1, \ldots, z_{k+l}\}$, we write x, y, and z respectively. Let $\bar{P}(x)$, $\bar{Q}(y)$, and $\bar{F}(z)$ be polynomials in λ, defined by

$$\bar{P}(x) = \sum_{i=0}^{k} x_i \lambda^{k-i},\ \bar{Q}(y) = \sum_{j=0}^{l} y_j \lambda^{l-j}, \tag{3.4}$$

and

$$\bar{F}(z) = \sum_{s=0}^{k+l} z_s \lambda^{k+l-s}.$$

Let us take $k + l$ functions of x, y, z defined by

$$G_s(x, y, z) = -z_s + \sum_{i+j=s} x_i y_j \qquad (s = 1, \ldots, k + l). \tag{3.5}$$

Finally, let $a_0 (= 1), a_1, \ldots, a_k$; $b_0 (= 1), b_1, \ldots, b_l$; $c_0 (= 1), c_1 \ldots, c_{k+l}$ be real numbers such that $\bar{P}(a) = P$, $\bar{Q}(b) = Q$, $\phi_p = \bar{F}(c)$. $PQ = \phi_p$ is now $G_s(a, b, c) = 0$, $s = 1, \ldots, k + l$.

The Jacobian

$$J(x, y; z) = \frac{\partial(G_1, \ldots, G_{k+l})}{\partial(x, y)}$$

has the form (3.6), which is nothing but the resultant of the two polynomials in λ, $\bar{P}(x)$ and $\bar{Q}(y)$.

$$J(x, y; z) = \begin{vmatrix}
1 & & & & & 1 & & & & & & \\
y_1 & 1 & & 0 & & x_1 & 1 & & & & & \\
y_2 & y_1 & 1 & & & x_2 & x_1 & 1 & & 0 & & \\
. & & & . & & x_3 & x_2 & x_1 & 1 & & & \\
. & & & & 1 & . & & & & . & & \\
. & & & & y_1 & x_k & & & & & . & \\
. & & & & . & & x_k & & & & & 1 \\
y_l & & & & . & & & x_k & & & & x_1 \\
 & y_l & & & . & & & & . & & & x_2 \\
 & & . & & . & & & 0 & & . & & . \\
 & 0 & & . & . & & & & & & . & . \\
 & & & & y_l & & & & & & & x_k
\end{vmatrix} \tag{3.6}$$

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As $\bar{P}(a) = P$ and $\bar{Q}(b) = Q$ are relatively prime, we have $J(a, b; z) \neq 0$. In particular $J(a, b; c) \neq 0$.

Furthermore, as $G_s(a, b, c) = 0$, $s = 1, \ldots, k + l$, we can use the implicit function theorem to find (i) an open neighbourhood V of c in R^{k+l}, the $(k + l)$-dimensional euclidean space, and (ii) a unique set of C^∞-functions f_i, g_j, $i = 1, \ldots, k$, $j = 1, \ldots, l$, defined on V and satisfying (A) and (B):

(A) $G_s(f_1(z), \ldots, f_k(z), g_1(z), \ldots, g_l(z), z) = 0$ for $z \in V$

(B) $f_i(c) = a_i$, $g_j(c) = b_j$; $i = 1, \ldots, k$; $j = 1, \ldots, l$.

Now, let

$$\phi = \sum_{s=0}^{k+l} \phi_s \lambda^{k+l-s},$$

where $\phi_s \in \Phi$, $\phi_0 = 1$. By ψ we denote the C^∞-mapping $M \to R^{k+l}$ defined by $q \to (\phi_1(q), \ldots, \phi_{k+l}(q))$. Take U to be the connected component of $\psi^{-1}(V)$, containing p. If we let $\alpha_i = f_i \circ \psi$ and $\beta_j = g_j \circ \psi$, then our desired elements of $\Phi_U[\lambda]$ are

$$\mu = \sum_{i=0}^{k} \alpha_i \lambda^{k-i}$$

and

$$\pi = \sum_{j=0}^{l} \beta_j \lambda^{l-j},$$

where $\alpha_0 = \beta_0 = 1$.

As $J(a, b; c) \neq 0$ and as $J(q) = J(\alpha_1(q), \ldots, \alpha_j(q), \beta_1(q), \ldots, \beta_l(q); \phi_1(q), \ldots, \phi_{k+l}(q))$ is a continuous function of q in U, we can take a neighbourhood U' of p, contained in U, such that $J(q) \neq 0$ for $q \in U'$. Then for $q \in U'$, $\mu_q = \bar{P}(\alpha_1(q), \ldots, \alpha_k(q))$ and $\pi_q = \bar{Q}(\beta_1(q), \ldots, \beta_l(q))$ are relatively prime. Q.E.D.

Remark 1. If we let (3.2) to be the factorization of χ_p into irreducible factors over the complex numbers C, then all K_i are linear in λ, and we obtain $\chi_i \in \tilde{\Phi}_U[\lambda]$, where $\tilde{\Phi}_U$ is the ring of complex-valued C^∞-functions over U. However, this result does not appear to be necessary for our purpose.

Remark 2. If $m_i > 1$, one might expect to obtain a further factorization of

$$\chi_i = \chi_{i1}\chi_{i2};\ \chi_{i1},\ \chi_{i2} \in \Phi_{U'}[\lambda]$$

for a neighbourhood U' of p, contained in U. But the following example shows that this is not necessarily the case.

Let ϕ be a polynomial in λ, with coefficients depending on two real parameters x and y, and having the form

$$\phi = \lambda^4 - 2x\lambda^2 + (x^2 + y^2). \tag{3.7}$$

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Then $\phi_{(0,0)} = 0$ has $\lambda = 0$ as a root of multiplicity 4. The solution of $\phi_{(x,y)} = 0$ has four roots $\pm r^{\frac{1}{2}}(\cos \frac{1}{2}\theta \pm i \sin \frac{1}{2}\theta)$, where $x = r\cos\theta$, $y = r \sin\theta$, and we pick fixed branches for $\cos\frac{1}{2}\theta$ and $\sin\frac{1}{2}\theta$. So (3.7) has a unique factorization over R at any point $(x_0, y_0) \neq (0, 0)$

(3.8) $$\phi = (\lambda^2 - 2r_0^{\frac{1}{2}} \cos \tfrac{1}{2}\theta_0\lambda + r_0)(\lambda^2 + 2r_0^{\frac{1}{2}} \cos \tfrac{1}{2}\theta_0 \lambda + r_0).$$

If we want to extend this factorization over a small neighbourhood of (x_0, y_0) we have

(3.9) $$\phi = (\lambda^2 - 2r^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\lambda + r)(\lambda^2 + 2r^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\lambda + r).$$

This extension is uniquely determined by requiring the coefficients in the factors of (3.9) to be continuous in a small neighbourhood of (x_0, y_0). However, (3.9) will not give a factorization in a neighbourhood of $(0, 0)$ because in a neighbourhood of $(0, 0)$, $\cos\frac{1}{2}\theta$ is not a single-valued function.

Remark 3. If in (3.3) we have $\chi_i = (\lambda - \alpha_i)^{m_i}$ for some i, then $(\chi_i)_q = 0$ has only one root of multiplicity m_i, for $q \in U$. If $\chi_i = (\lambda^2 + \beta_i\lambda + \beta_i')^{m_i}$ for some i, then $(\chi_i)_q = 0$ has two distinct complex roots, each of multiplicity m_i, for $q \in U'$, where the neighbourhood U' is chosen sufficiently small with $p \in U' \subset U$. In both cases it is easy to see that $\alpha_i \in \Phi_U$ or β_i, $\beta_i' \in \Phi_U$ (for example, by expanding $(\lambda - \alpha_i)^{m_i}$ or $(\lambda^2 + \beta_i\lambda + \beta_i')^{m_i}$ and using the fact that the coefficients in the expansion are C^∞-functions on U).

Remark 4. Although it may not be possible to factor χ_i any further into polynomials in λ with C^∞-coefficients over some neighbourhood, it is well known that the roots of $(\chi_i)_q = 0$, for each i, are continuous (multivalued) functions of q. In particular, the roots of $(\chi_i)_q = 0$ are close to those of $K_i = 0$ if q is close to p.

4. Integration. Let A be a linear transformation on a finite dimensional vector space V over the reals R. Let λ be an indeterminate, and consider V as an $R[\lambda]$-module by letting $Fv = F(A)v$ for $F \in R[\lambda]$, $v \in V$, where, if

$$F = \sum_{i=1}^{m} a_i\lambda^i,$$

$F(A)$ denotes the linear transformation

$$\sum_{i=1}^{m} a_iA^i.$$

Let K be the characteristic polynomial of A, and suppose $K = FG$ where $F, G \in R[\lambda]$; $\deg F$, $\deg G < \deg K$; F and G have leading coefficients 1 and are relatively prime over R. Then there exist unique $P, Q \in R[\lambda]$, with $\deg P < \deg G$, $\deg Q < \deg F$, satisfying

(4.1) $$PF + QG = 1.$$

Because $KV = 0$, we have from (4.1), $(PF)^2v = (PF)v$ for all $v \in V$, Let $V_F = (QG)V$ and $V_G = (PF)V$, then we have

(4.2) $$V = V_F \oplus V_G.$$

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It is also easy to see that, $V_F = \{v \in V \mid Fv = 0\}$ and $V_G = \{v \in V \mid Gu = 0\}$. In fact let $V_F' = \{v \in V \mid Fv = 0\}$. Then, as $FV_F = 0$, we have $V_F \subset V_F'$. Conversely, if $v \in V_F'$, then $0 = (PF)v = (1 - QG)v$, hence $v = (QG)v \in V_F$, so $V_F \supset V_F'$. Finally, F is the characteristic polynomial of $A|V_F$, and $\dim V_F = \deg F$; G is the characteristic polynomial of $A|V_G$, and $\dim V_G = \deg F$.

Now, if we take A to be h_p and B to be T_p in the argument above, (4.2) gives a decomposition of T_p. We want to extend this decomposition to each T_q, for q in a neighbourhood of p, with the help of the factorization (3.3) of the characteristic polynomial χ of h. For this purpose we first prove a lemma.

LEMMA 4.1. *If ϕ and ψ are elements of $\Phi[\lambda]$ with leading coefficients* 1 *and degree k and l respectively, and if at each point q in a neighbourhood U, ϕ_q and ψ_q are relatively prime, then there exist unique $\mu, \pi \in \Phi_U[\lambda]$ of degree $\leqslant l-1$, $k-1$ respectively, satisfying*

$$\mu\phi + \pi\psi = 1 \quad over \quad U. \tag{4.3}$$

Proof. Let

$$\phi = \sum_{i=0}^{k} \alpha_i \lambda^{k-i}, \ \psi = \sum_{i=0}^{l} \beta_i \lambda^{l-i},$$

where $\alpha_i, \beta_i \in \Phi$, and $\alpha_0 = \beta_0 = 1$. Let

$$\mu = \sum_{i=1}^{l} \mu_i \lambda^{l-i}, \ \pi = \sum_{i=1}^{k} \pi_i \lambda^{k-i}.$$

Substituting these expressions in (4.3), we see that finding the required μ, π is equivalent to solve (4.4) for the μ_i and π_i's:

$$\begin{cases} \sum_{i+j=p} \alpha_i \mu_j + \sum_{i+j=p} \beta_i \pi_j = 0 & 1 \leqslant p \leqslant k+l-1 \\ \alpha_k \mu_l + \beta_l \pi_k = 1. \end{cases} \tag{4.4}$$

The determinant D of the coefficients of the left member of (4.4) is

$$D = \begin{vmatrix} 1 & & & & 1 & & & \\ & & 0 & & & & 0 & \\ \alpha_1 & 1 & & & \beta_1 & 1 & & \\ \cdot & & \cdot & & \cdot & & \cdot & \\ \cdot & & & 1 & \cdot & & & 1 \\ \alpha_k & & & \alpha_1 & \beta_l & & & \beta_1 \\ & \cdot & & \cdot & & \cdot & & \cdot \\ & 0 & \cdot & \cdot & & 0 & \cdot & \cdot \\ & & & \alpha_k & & & & \beta_l \end{vmatrix} \tag{4.5}$$

D_q, $q \in U$, is the resultant of two polynomials ϕ_q, ψ_q in $R[\lambda]$, and as ϕ_q and ψ_q are relatively prime, we have $D_q \neq 0$. Hence we can solve (4.4) for μ_i and π_i over U (the solution is unique) and find them as rational functions of α_i

and β_i, with non-zero denominator over U. Hence $\mu_i, \pi_i \in \Phi_U$. Thus $\mu, \pi \in \Phi_U[\lambda]$ are uniquely determined. Q.E.D.

Now, if χ is the characteristic polynomial of h and if

$$\chi = \chi_1\chi_2 \ldots \chi_g \quad \text{over a neighbourhood } U \text{ of } p$$

is the factorization (3.3), then χ_i and $\hat{\chi}_i = \chi_1 \ldots \overset{i}{\wedge} \ldots \chi_g$ are relatively prime at each point of U. By Lemma 4.1 we have $\mu_i, \pi_i \in \Phi_U[\lambda]$ satisfying

$$\mu_i\chi_i + \pi_i\hat{\chi}_i = 1 \text{ on } U. \tag{4.6}$$

As before, using $\chi_q T_q = 0$ for $q \in U$, and (4.6), we see that $[(\pi_i\hat{\chi}_i)(h)]^2 = (\pi_i\hat{\chi}_i)(h)$. Let us denote $\pi_i\hat{\chi}_i \in \Phi_U[\lambda]$ by ϵ_i, and $(\pi_i\hat{\chi}_i)_q T_q$ by $T_q(\chi_i)$. Then $\epsilon_i(h)$ is a projection operator on U, and $\dim T_q(\chi_i) = \deg \chi_i$. As $T_q = T_q(\chi_i) \oplus \ldots \oplus T_q(\chi_g)$, we have

$$\sum_{i=1}^{g} \epsilon_i(h) = 1.$$

Furthermore θ_i defined by $q \to T_q(\chi_i)$, $q \in U$, is a C^∞-distribution over U. Using Lemma 2.1 we have:

THEOREM 4.2. *The distribution θ_i is integrable in a neighbourhood of p if and only if*

$$(I - \epsilon_i(h))[\epsilon_i(h), \epsilon_i(h)] = 0 \tag{4.7}$$

holds on a neighbourhood of p.

As in § 2, if we define

$$\theta_{i_1 \ldots i_k} \text{ by } q \to T_q(\chi_{i_1}) \oplus \ldots \oplus T_q(\chi_{i_k}),$$

we have, by Lemma 2.2:

COROLLARY 4.3. *The distributions $\theta_{i_1 \ldots i_k}$ $(k = 1, \ldots, g - 1; i_s = 1, \ldots, g; s = 1, \ldots, k)$ are all integrable in a neighbourhood of p, if and only if $[\epsilon_i(h), \epsilon_i(h)] = 0$ holds on a neighbourhood of p for all i.*

The important feature of the projection operator $\epsilon_i(h)$ is that $\epsilon_i(h)$ is a polynomial in h with coefficients in Φ_U. This property essentially characterizes $\epsilon_i(h)$, as shown below.

PROPOSITION 4.4. *Let the characteristic polynomial χ of h have the factorization (3.2) $\chi_p = K_1^{m_1} \ldots K_g^{m_g}$ at p, and (3.3) $\chi = \chi_1 \ldots \chi_g$ on a neighbourhood U of p; and let $\epsilon_i(h)$ be the projection operator on U corresponding to χ_i. If e is a projection operator on U such that $e = \epsilon(h)$, $\epsilon \in \Phi_U[\lambda]$, then on U we have*

$$e = \epsilon(h) = \sum_{i=1}^{g} \delta_i\epsilon_i(h) \quad \text{where} \quad \delta_i = 0 \quad \text{or} \quad 1. \tag{4.8}$$

First we prove a lemma.

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LEMMA 4.5. *Let A be a linear transformation on a real vector space V of finite dimension, and suppose that the characteristic polynomial of A is of the form K^m, where $K \in R[\lambda]$ is irreducible over R. Then if for $P \in R[\lambda]$, $P(A)^2 = P(A)$, then $P(A) = I$ or 0.*

Proof. Let $Q = 1 - P$, then $Q(A)^2 = Q(A)$. If $P(A) \neq I$ and $P(A) \neq 0$, then V has a decomposition $V = V_1 \oplus V_2$; $V_1, V_2 \neq \{0\}$, where $V_1 = PV$, $V_2 = QV$. As

$$P(A|V_2) = P(AQ(A)) = P(A)Q(A) = 0,$$

P is divisible by the minimal polynomial of $A|V_2$, which in turn is equal to $K^{m'}$ for some m', $1 \leqslant m' < m$. Hence P is divisible by K. Similarly Q is divisible by K. But P and $Q = 1 - P$ are relatively prime, so we have a contradiction. Q.E.D.

Proof of the Proposition 4.4. As e is a polynomial in h with coefficients in Φ_U, $e_q T_q(\chi_i) \subset T_q(\chi_i)$. We can define projection operators e_i over U by letting $e_i = e\epsilon_i(h)$. Then $e_i e_j = 0$ for $i \neq j$ and $e = \sum_{i=1}^{g} e_i$ over U. We want to prove either $e_i = 0$ or $e_i = \epsilon_i(h)$ for each i.

We first notice that $e_i \mid T_p(\chi_i) = \epsilon(h \mid T_p(\chi_i))$, and that $h \mid T_p(\chi_i)$ has characteristic polynomial $K_i^{m_i}$. Hence, using Lemma 4.5, we see that either $e_i(T_p(\chi_i)) = \{0_p\}$ or $e_i(T_p(\chi_i)) = T_p(\chi_i)$. But, as $e_i(T_q(\chi_j)) = \{0_q\}$ for $j \neq i$, $q \in U$, and, as e_i has constant rank over U, we conclude that (i) if $e_i(T_p(\chi_i)) = \{0_p\}$ then $e_i(T_q(\chi_i)) = \{0_q\}$ for all $q \in U$, and that (ii) if $e_i(T_p(\chi_i)) = T_p(\chi_i)$ then $e_i(T_q(\chi_i)) = T_q(\chi_i)$ for all $q \in U$. In the first case $e_i = 0$; in the second case $e_i = \epsilon_i(h)$. Q.E.D.

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