CUBIC IDENTITY GRAPHS AND PLANAR GRAPHS DERIVED FROM TREES

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(Received 7 October 1968)

To Bernhard Hermann Neumann on his 60th birthday

Communicated by G. B. Preston

Abstract

The smallest (nontrivial) identity graph is known to have six points and the smallest identity tree seven. It is now shown that the smallest cubic identity graphs have 12 points and that exactly two of them are planar, namely those constructed by Frucht in his proof that every finite group is isomorphic to the automorphism group of some cubic graph. Both of these graphs can be obtained from plane trees by joining consecutive endpoints; it is shown that when applied to identity trees this construction leads to identity graphs except in certain specified cases. In appendices all connected cubic graphs with 10 points or fewer, and all cubic nonseparable planar graphs with 12 points, are displayed.

Introduction

An *identity graph* has no automorphisms other than the identity. (Graph-theoretic definitions not included here may be found in [5].) The trivial graph (consisting of one point and no lines) is an identity graph; in the rest of this paper only nontrivial graphs are considered.

One graph is *smaller* than another if it has fewer points. The smallest identity tree [8] and the smallest identity graphs (see [6]) are shown in Figure 1. The latter two graphs are complementary.

In his pioneering paper [2], Frucht proved that for any finite group F there exists a graph G such that its (automorphism) group $\Gamma(G)$ is isomorphic to F. Subsequently, Frucht [3] demonstrated that there even exists a cubic graph G with $\Gamma(G)$ isomorphic to F. In view of recent efforts ([7], [11],

[1] Work supported in part by the National Science Foundation under Grant GN–2544.

and references cited therein) to find the smallest graph whose group is isomorphic to a given finite group, it seems natural to ask for the smallest cubic graphs with given group. The specialization of this question to the identity group is answered in Theorem 1.

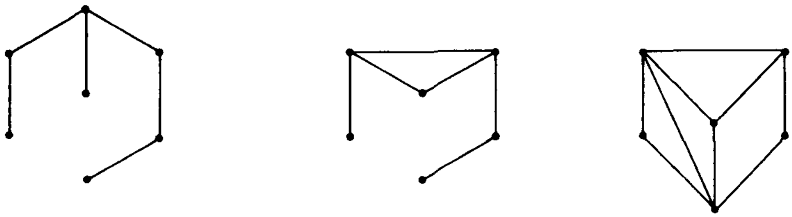


Figure 1

Cubic identity graphs

Unfortunately the present answer has been obtained by exhaustion rather than by theoretical means. It is to be hoped that in the relatively near future a proof by enumeration may be found, perhaps by extending the method in a forthcoming paper by R. W. Robinson on the enumeration of cubic graphs.

THEOREM 1. *The smallest nontrivial cubic identity graphs have* 12 *points. Further, there are exactly two different planar cubic identity graphs with* 12 *points, those shown in Figure* 2.

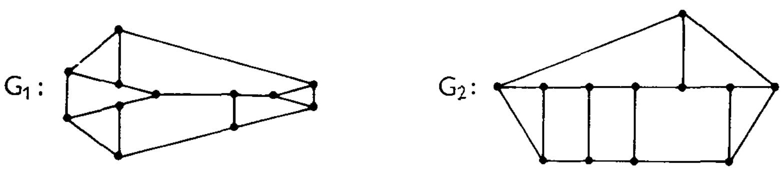


Figure 2

PROOF. Every cubic graph shown in Appendices I and II can be seen to have at least one non-identical automorphism, except for G_1 and G_2.

The graphs G_1 and G_2 were found by Frucht [3] when showing that every finite group is isomorphic to $\Gamma(G)$ for some cubic graph G. Subsequently Sabidussi [10] and others showed that this is rather a weak restriction and that there always exist graphs with given group and possessing a variety of prescribed graph-theoretic properties.

Each of G_1 and G_2 can be constructed by the following method. Consider any tree (see the Appendix of [8] for diagrams of the trees and identity trees with at most 12 points) and draw it in the plane. The result is called a *plane tree*: such trees were counted in [9]. Now draw a polygon joining consecutive endpoints of the plane tree. If the original tree has every point either an endpoint or a point of degree 3, then the resulting planar graph will be cubic. Clearly both G_1 and G_2 can be so constructed; the plane trees from which they can be formed in this way are shown in Figure 3.

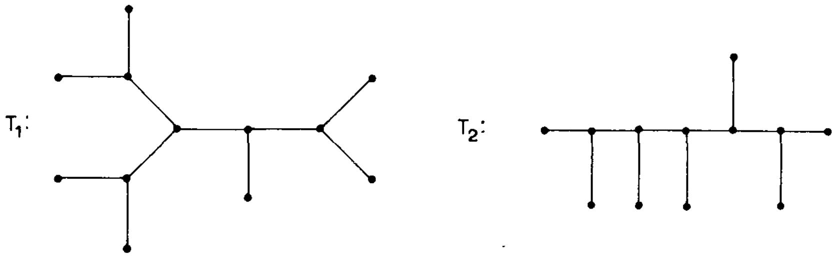


Figure 3

These are not identity trees. Clearly any identity tree has points of degree 2, and therefore cannot lead to a cubic graph. The smallest (nontrivial) identity tree, shown in Figure 1, yields the identity graph of Figure 4, in which the outer polygon is drawn with dashed lines.

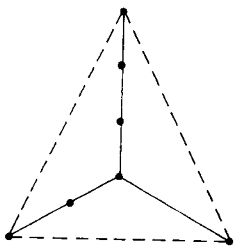


Figure 4

Construction of identity graphs from identity trees

Under the operation described above, not every plane identity tree leads to an identity graph; the simplest contrary example is shown in Figure 5. The mapping taking u into u', v into v', and w into w', and leaving the other points fixed, is a non-identical automorphism of this graph. The same phenomenon is exhibited by any plane tree that can be regarded as made up of a path $v_0 v_1 \cdots v_n v_{n+1}$ $(n \geqq 2)$ together with n disjoint branches $v_m w_{m1} \cdots w_{m i_m}$ $(i_m \geqq 1)$, $m = 1, \cdots, n$, all extending from the same side of the path (see Figure 6): such a plane tree we shall call *exceptional*. (The reader acquainted with the proof in [2] will notice a close similarity to Frucht's original construction; see also Chapter 14 of [5].) It will be an identity tree unless either (i) $i_1 = 1$ or $i_n = 1$, or (ii) $i_m = i_{n-m+1}$

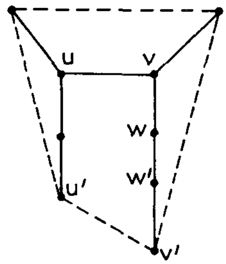


Figure 5

for $m = 1, \cdots, n$. We shall show below that all other plane identity trees lead to identity graphs. We observe as a corollary that since every identity tree with an exceptional plane embedding can also be given a non-exceptional one, by putting the branch $v_1 w_{11} \cdots w_{1i_1}$ on the other side of the path, it can be used to construct an identity graph by the above method.

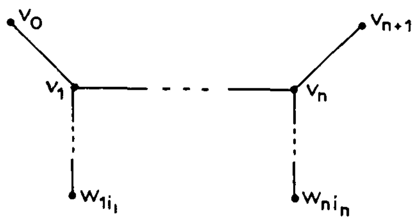


Figure 6

THEOREM 2. *If a plane identity tree T yields a non-identity graph G when consecutive endpoints are joined then it is exceptional.*

PROOF. Let f be a non-identical automorphism of G, and let U denote the outer polygon of G. The image $U' = f(U)$ is a cycle in G which does not coincide with U, because otherwise f would induce a non-identical automorphism of T. Any point of G can be joined to any other by a path in T, of which at most the endpoints can lie on U. The image of this path under f will have a similar property relative to U'. It follows that G lies either wholly inside and on U' or wholly outside and on U'; and the former case is impossible because points of $U - U'$ are certainly outside U'. Thus G *has no points inside* U'. Further we note that *every point of* U' *is of degree* 3 in G (since this is true of U).

Since T contains no cycle, being a tree, U' contains at least one edge of U. Let $A = v_0 v_1 \cdots v_n v_{n+1}$ be a path in U' of which just the endpoints belong to U. It will divide U into two parts, say U_1 and U_2. Any point of G can be joined to any other by a path in T, and such a path cannot cross from one side of A to the other, since otherwise it would pass inside U'.

It follows that G lies wholly inside and on one of the cycles $A \cup U_1$ or $A \cup U_2$, say the latter. Therefore U_1 *consists of a single edge* $v_0 v_{n+1}$, *and* U' is simply the cycle $v_0 v_1 \cdots v_{n+1} v_0$ (see Figure 7).

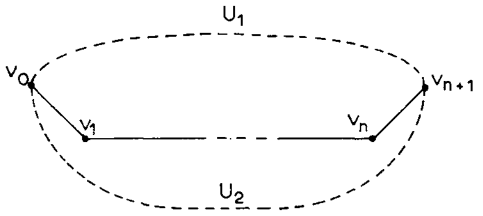


Figure 7

Removal of the edges $v_{m-1}v_m$ and $v_m v_{m+1}$ would disconnect T, and one of the components of the remainder would contain v_m; denote it by T_m $(1 \leqq m \leqq n)$. Since T contains no cycle, the subtrees T_m are mutually disjoint. Each of them contains one or more endpoints of T, all different from v_0 and v_{n+1}. Since U' has the same number of points as U, each T_m contains only one endpoint of T, and therefore is a path. Thus we are in the situation of Figure 6, that is, T is exceptional.

Hamiltonian cubic graphs

It is often of interest to determine the number of hamiltonian graphs of a given type. For planar and nonplanar connected cubic graphs with up to 12 points this information is displayed in Table I, which is based on data in a report by Lederberg [2] and in another by Balaban [1].

TABLE I

Number of points	Hamiltonian		Non-hamiltonian	
	planar	nonplanar	planar	nonplanar
4	1	0	0	0
6	1	1	0	0
8	3	2	0	0
10	8	9	1	1
12	29	51	3	?

[2] J. Lederberg, NASA Report CR 68898, STAR N-66 14074 (DENDRAL 64, a system for computer construction, enumeration and notation of organic molecules as tree structures and cyclic graphs, Part II, Topology of cyclic graphs).

Unsolved problems

1. We show in Table II the *symmetry numbers* of all connected cubic graphs with at most 8 points, that is, the order of the group of each such graph.

TABLE II

Cubic Graph	4.1	6.1	6.2	8.1	8.2	8.3	8.4	8.5
Symmetry Number	24	12	72	16	16	4	8	12

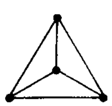
This suggests a problem which sounds opposite to that of this paper. *Determine the maximum symmetry number among all cubic graphs with 2n points.*

2. It has been established that as the number of points becomes very large, almost all graphs are identity graphs. It is now conjectured that in particular, *almost all cubic graphs are identity graphs.*

Appendix I

All connected cubic graphs with 10 points or fewer.

These diagrams are given in Balaban [1]. The cubic graph K_4 with 4 points is called graph 4.1 in this appendix; the two cubic graphs with 6 points are designated 6.1 and 6.2, and so forth. Some of these graphs have special names. For example, 4.1 is the tetrahedron, 6.1 the triangular prism, 6.2. the complete bipartite graph $K_{3,3}$, 8.1 the cube Q_3, 8.2 and 10.13 Möbius ladders [4], 10.14 the pentagonal prism, and 10.19 the well known Petersen graph. All of the planar graphs among these are drawn without intersecting lines.



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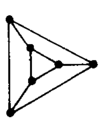

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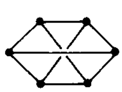

6.2

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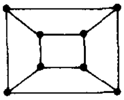
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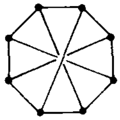
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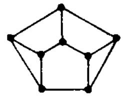
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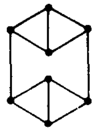
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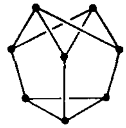
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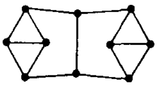
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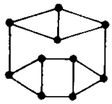
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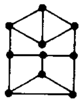
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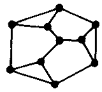
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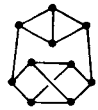
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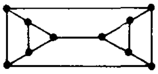
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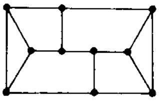
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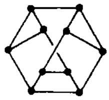
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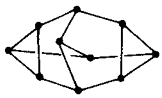
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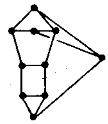
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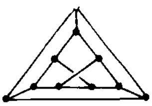
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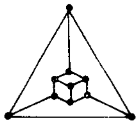
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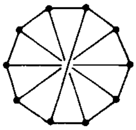
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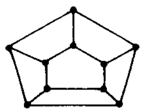
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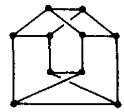
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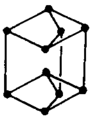
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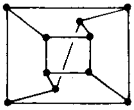
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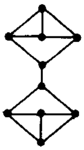
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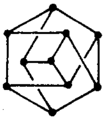
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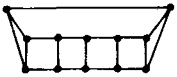
Appendix II

All nonseparable cubic planar graphs with 12 points. These diagrams are given by Grace [3].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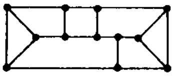
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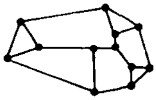
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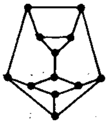
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[3] D. W. Grace, Computer search for non-isomorphic convex polyhedra. Stanford Computation Center Technical Report CS 15 (1956).

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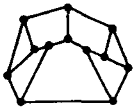
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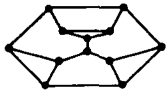
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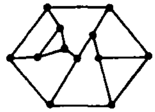
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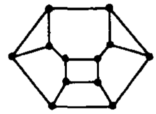
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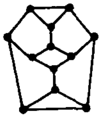
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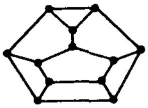
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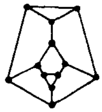
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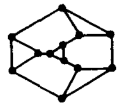
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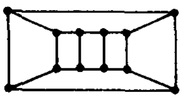
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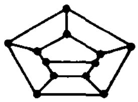
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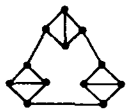
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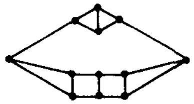
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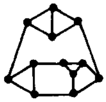
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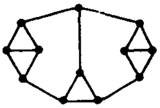
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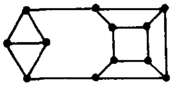
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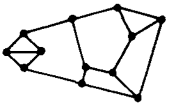
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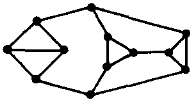
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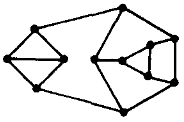
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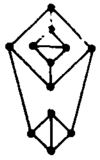
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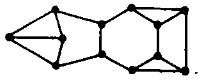
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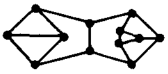
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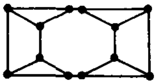
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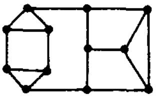
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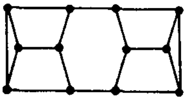
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