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RESIDUE FREE DIFFERENTIALS AND THE CARTIER OPERATOR FOR ALGEBRAIC FUNCTION FIELDS OF ONE VARIABLE

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1. Introduction. Let K be a field of characteristic $p > 0$ and let A be a separably generated algebraic function field of one variable with K as its exact constant field. Throughout this paper we shall use the following notations to classify differentials of A/K:

$D(A)$: the K-module of all differentials,

$G(A)$: the K-module of all differentials of the first kind,

$R(A)$: the K-module of all residue free differentials in the sense of Chevalley [**2**, p. 48],

$E^*(A)$: the K-module of all pseudo-exact differentials in the sense of Lamprecht [**7**, p. 363], (compare the definition with our Lemma 8).

In the preceding paper [**5**, Satz 1] we proved that for a perfect field K,

$$\dim_K R(A)/E^*(A) = \dim_K G(A)/G(A) \cap E^*(A),$$
$$R(A) = G(A) + E^*(A).$$

This is a natural generalization of the theorem of Kunz [**6**, Satz 3], and this means that the K-dimension of $R(A)/E^*(A)$ is not greater than the genus $g(A/K)$.

In this paper we have two aims. One is to obtain similar formulae with the preceding formulae without the assumption of perfectness on K. Another is to consider a kind of transfer of these formulae on the constant field extension. The former consideration is necessary for the latter one.

In § 2 we shall study a definition and some fundamental properties of the Cartier operator which is an important tool for our aims. In particular we shall make comparison between a differential and the differential obtained by application of the operator on the given differential about their degrees and residues (Lemmas 2 and 3). In § 3 we shall introduce a semi-linear mapping $C^m(\mathrm{Cosp}_{A/A_m}\cdot)$ of $D(A)$ into $D(A_m)$. In relation with the notation A_m we explain some notations now. Let F and S be fields. Then F^{p^n} denotes the field of all elements of p^n-th power of elements in F for an integer n. $F\langle S\rangle$ denotes the extension field of F generated by adjunction of all elements of S to F. Following these notations A_m means $A\langle K_m\rangle$ with $K_m = K^{p^{-m}}$ $(m \geqq 0)$, and Cosp_{A/A_m} is the cotrace from A to A_m. By this mapping we shall define two K-submodules of $R(A)$

$$R_m(A) = \{\omega \in D(A) \mid C^m(\mathrm{Cosp}_{A/A_m}\omega) \in G(A_m)\}$$

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and

$$E_m(A) = \{\omega \in D(A) \mid C^m(\mathrm{Cosp}_{A/A_m}\omega) = 0\}.$$

After some discussions on residues and inseparable poles of differentials we shall characterize elements of $R(A)$ and $E^*(A)$ by the operator $C^m(\mathrm{Cosp}_{A/A_m}\cdot)$, and we shall prove that $R(A) = \bigcup_{m=0}^{\infty} R_m(A) = \bigcup_{m=c^*}^{\infty} R_m(A)$ (Corollary of Lemma 6), and determine an explicit form of a differential in $E^*(A)$ (Lemma 8), where c^* is the minimum integer of such n as $g(A\langle K_n\rangle/K_n) = g_0(A/K)$: the conservative genus of A/K. Further we shall show that linear independency of a finite number of differentials over K mod $E^*(A)$ is transferred into the field obtained by the constant field extension $A\langle L\rangle$ with $L \supset K$ (Lemma 9), for a preparation to the proofs of Theorems 1 and 2. In the last part of the section we shall prove Theorem 1. From this we obtain that the K-dimension of $R(A)/E^*(A)$ is not greater than the conservative genus $g_0(A/K)$.

In § 4 we shall prove Theorem 2 which shows the corresponding transferred formulae by the constant field extension with the formulae in Theorem 1.

As main results we have

THEOREM 1. *Using the same notations mentioned above it holds that*

$$\dim_K R(A)/E^*(A) = \dim_{K_m} G(A_m)/G(A_m) \cap E^*(A_m),$$
$$R(A) = R_m(A) + E^*(A),$$

for any integer $m \geqq c^*$.

THEOREM 2. *Using the same notations mentioned above it holds for a given extension* L *of* K *that*

$$R(A\langle L\rangle) = L\,\mathrm{Cosp}_{A/A\langle L\rangle} R_m(A) + E^*(A\langle L\rangle)$$

for any integer $m \geqq c^*$, *and*

$$\dim_L R(A\langle L\rangle)/E^*(A\langle L\rangle) = \dim_K R(A)/E^*(A).$$

2. The Cartier operator. We begin with the definition of the Cartier operator. Let us denote a separating variable of A over K by x and let us always use it in this sense. With this x, $\{1, x, \ldots, x^{p-1}\}$ forms the so-called p-base of A/B with $B = A^p\langle K\rangle$ and represents any $\omega \in D(A)$ in a unique form $\omega = \sum_{i=0}^{p-1} a_i x^i dx$, with $a_i \in B$. If $a_{p-1}^{p^{-1}}$ is in A, define $C_A(\omega) = a_{p-1}^{p^{-1}} dx$; hence $C_A(\omega)$ is in $D(A)$. Then we say that $C_A(\omega)$ is defined in A and call C_A the Cartier operator for A. We shall simply write C instead of C_A when no confusion might arise.

Let L be an extension of K. Then the constant field extension $A\langle L\rangle$ of A by L over K is separably generated over L, as A is separably generated over K, and x remains as a separating variable of $A\langle L\rangle/L$. Hence $\{1, x, \ldots, x^{p-1}\}$ remains as a p-base of $A\langle L\rangle/A^p\langle L\rangle$.

The following fundamental properties of C_A are obvious by making use of facts in Eichler's book [**4**, pp. 160–167].

(1) Whether $C_A(\omega)$ is definable or not, $C_A(\omega)$ itself is independent of the choice of a separating variable.

(2) If $C_A(\omega_1)$ and $C_A(\omega_2)$ are defined for $\omega_1, \omega_2 \in D(A)$, then

$$C_A(y_1^p\omega_1 + y_2^p\omega_2) = y_1C_A(\omega_1) + y_2C_A(\omega_2) \text{ with } y_1, y_2 \in A.$$

(3) If $\omega \in D(A)$ is exact, $C_A(\omega)$ is defined and $C_A(\omega) = 0$. Conversely, if $C_{A\langle L\rangle}(\mathrm{Cosp}_{A/A\langle L\rangle}\omega)$ is defined and is equal to 0 for an extension L of K, then ω is exact (cf. [**2**, p. 118] on the definition of Cosp).

(4) Let $\Omega = \mathrm{Cosp}_{A/A\langle K_m\rangle}\,\omega$ for $\omega \in D(A)$. Then all $C^i(\Omega)$ $(i = 1, \ldots, m)$ are defined in $A\langle K_m\rangle$.

LEMMA 1. *Let $C_A(\omega)$ be defined for a differential $\omega \in D(A)$. If L is an extension of K, then $C_{A\langle L\rangle}(\mathrm{Cosp}_{A/A\langle L\rangle}\omega)$ is defined and*

$$C_{A\langle L\rangle}(\mathrm{Cosp}_{A/A\langle L\rangle}\omega) = \mathrm{Cosp}_{A/A\langle L\rangle}C_A(\omega).$$

Proof. The proof is obvious from the fact that $\mathrm{Cosp}_{A/A\langle L\rangle}dx$ is a non-zero exact differential of $A\langle L\rangle$ and $\mathrm{Cosp}_{A/A\langle L\rangle}ydx = y\ \mathrm{Cosp}_{A/A\langle L\rangle}dx$.

LEMMA 2. *Let $C(\omega)$ be defined for a differential $\omega \in D(A)$. Then*

$$\nu_{\mathfrak{g}}(C(\omega)) + 1 \geqq p^{-1}(\nu_{\mathfrak{g}}(\omega) + 1)$$

holds for any place $\mathfrak{g}$ of A, where $\nu\mathfrak{g}$ is the valuation with respect to a place $\mathfrak{g}$, (cf. [**5**, Hilfssatz 2], when K is perfect).

Proof. If K is perfect, then $K = K^p$, $B = A^p$. Accordingly $C(\omega)$ is defined for any $\omega \in D(A)$. Any uniformizing variable t is a separating variable since the residue field of the place $\mathfrak{g}$ of the uniformizing variable t is separable over K. By property (1) we have

$$\nu_{\mathfrak{g}}(\omega) = \min_i\{\nu_{\mathfrak{g}}(a_it^idt)\} \leqq \nu_{\mathfrak{g}}(a_{p-1}t^{p-1}dt),$$
$$\nu_{\mathfrak{g}}(t^{-1}dt) = -1.$$

From this we get

$$\nu_{\mathfrak{g}}(C(\omega)) + 1 \geqq p^{-1}(\nu_{\mathfrak{g}}(\omega) + 1).$$

From now on we consider a general field K. Let K^* be the smallest perfect field containing K in the algebraic closure of K and let $A^* = A\langle K^*\rangle$ be the constant field extension of A by K^*. Let $\mathfrak{p}^*$ be the place of A^* lying above a place $\mathfrak{g}$ of A and let $e_{\mathfrak{p}^*/\mathfrak{g}}$ be the ramification index of $\mathfrak{p}^*$ w.r.t. $\mathfrak{g}$. If we denote $\Omega = \mathrm{Cosp}_{A/A^*}\omega$, then

$$\nu_{\mathfrak{p}^*}(C(\Omega)) + 1 \geqq p^{-1}(\nu_{\mathfrak{p}^*}(\Omega) + 1).$$

On the other hand the divisor $\mathfrak{M} = (\mathrm{Con}_{A/A^*}\omega)\ (\Omega)^{-1}$ is integral as well-known and this is independent of ω. Since for any $\omega \in D(A)$, $\nu_{\mathfrak{p}^*}(\mathrm{Con}_{A/A^*}\omega) =$

$e_{\mathfrak{p}^*/\mathfrak{g}}\nu_{\mathfrak{g}}(\omega)$, therefore we have

$$\nu_{\mathfrak{g}}(C(\omega)) + 1 \geqq p^{-1}(\nu_{\mathfrak{g}}(\omega) + 1) + e_{\mathfrak{p}^*/f}^{-1}(1 - p^{-1})(\nu_{\mathfrak{p}^*}(\mathfrak{M}) + e_{\mathfrak{p}^*/\mathfrak{g}} - 1) \geqq p^{-1}(\nu_{\mathfrak{g}}(\omega) + 1).$$

This completes our proof.

From Lemma 2 we obtain easily

COROLLARY. *Let ω be in $D(A)$ and let us assume that $C(\omega), \ldots, C^m(\omega)$ are all defined for a large enough integer $m \geqq 0$. Then there exists an integer n^* such that $\nu_{\mathfrak{g}}(C^n(\omega)) \geqq -1$ for any place $\mathfrak{g}$ and integer n with $n^* \leqq n \leqq m$.*

Proof. Using Lemma 2, $\nu_{\mathfrak{g}}(\omega) \geqq -1$ implies $\nu_{\mathfrak{g}}(C(\omega)) \geqq -1$. If $\nu_{\mathfrak{g}}(\omega) \leqq -2$, then $\nu_{\mathfrak{g}}(C(\omega)) > \nu_{\mathfrak{g}}(\omega)$ because $\nu_{\mathfrak{g}}(C(\omega)) - \nu_{\mathfrak{g}}(\omega) \geqq (1 + \nu_{\mathfrak{g}}(\omega))\ (p^{-1} - 1) > 0$. Hence $\nu_{\mathfrak{g}}(C^{n'}(\omega)) \geqq -1$ for $n' \geqq -(1 + \nu_{\mathfrak{g}}(\omega))$. Since the number of poles of ω is finite, let n^* be the maximum of such n''s. Then this n^* satisfies our Corollary.

LEMMA 3. *Let $C(\omega)$ be defined for a $\omega \in D(A)$. Then* $\mathrm{Res}_{\mathfrak{g}}\omega = (\mathrm{Res}_{\mathfrak{g}}C(\omega))^p$ *for any place* $\mathfrak{g}$ (cf. [**5**, Hilfssatz 1] when K is perfect).

Proof. Using the same notations as in the proof of Lemma 2 we have

$$(\mathrm{Res}_{\mathfrak{p}^*}C(\mathrm{Cosp}_{A/A^*}\omega))^p = \mathrm{Res}_{\mathfrak{p}^*}\mathrm{Cosp}_{A/A^*}\omega.$$

If the residue field of $\mathfrak{g}$ is identified with a subfield of the residue field of $\mathfrak{p}^*$, then

$$\mathrm{Res}_{\mathfrak{p}^*}\mathrm{Cosp}_{A/A^*}\omega = \mathrm{Res}_{\mathfrak{g}}\omega \text{ [\textbf{2}, Theorem 11, p. 119]}.$$

Hence we have by Lemma 1

$$(\mathrm{Res}_{\mathfrak{g}}C(\omega))^p = \mathrm{Res}_{\mathfrak{g}}\omega.$$

3. The Cartier operator and residue free differentials. Let us denote the fields $K^{p^{-m}}$, $A\langle K^{p^{-m}}\rangle$ and $B\langle K^{p^{-m}}\rangle$ by K_m, A_m and B_m for any integer $m \geqq 0$. If $\mathfrak{g}$ is a place of A/K whose residue field is separable over the constant field K, we call $\mathfrak{g}$ a separable place (or separable).

We obtain as a simple generalization of [**2**, Corollary 4, p. 123]:

LEMMA 4. *Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ be a finite number of places of A/K. Then there exists a finite purely inseparable extension L of K such that the place $\mathfrak{p}_i$ of $A\langle L\rangle/L$ lying above $\mathfrak{g}_i$ is separable for any $i = 1, \ldots, n$.*

Proof. Using the same notation as in Lemma 2 we denote the place of A^* lying above $\mathfrak{g}_i$ by $\mathfrak{p}_i^*$. By [**2**, Corollary 4, p. 123] there exists an intermediary field M_i between K^* and K having the following properties: M_i is a finite purely inseparable extension of K; the place $\mathfrak{p}_i'$ of $A\langle M_i\rangle$ (which lies above $\mathfrak{g}_i$) lying below $\mathfrak{p}_i^*$ is separable, and $e_{\mathfrak{p}_i^*/\mathfrak{p}_i'} = 1$. If we form the composite field $L = \bigcup_{i=1}^n M_i$, then L is a finite purely inseparable extension over K. If $\mathfrak{p}_i$ is the place of $A\langle L\rangle$ lying below $\mathfrak{p}_i^*$, then $\mathfrak{p}_i$ is lying above $\mathfrak{p}_i'$ and is separable since $\mathfrak{p}_i^*$ is unramified with respect to $A\langle L\rangle$ [**2**, Corollary 3, p. 123].

COROLLARY. *Let* $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ *be a finite number of places of* A/K. *Then there exists an integer* $m_0 \geqq 0$ *such that for* $m \geqq m_0$ *the place* $\mathfrak{p}_i$ *of* A_m *lying above* $\mathfrak{g}_i$ *is separable for each* $i = 1, \ldots, n$.

Proof. In the proof of Lemma 4 L is contained in K^* and L is a finite extension of K. Hence there exists the minimum integer m_0 satisfying $K \subseteq L \subseteq K_{m_0}$. Since $\mathfrak{p}_i^*$ is unramified with respect to $A\langle L\rangle$, hence so to A_{m_0}, and therefore $\mathfrak{p}_i^*$ is unramified with respect to A_m for $m \geqq m_0$, so that the place $\mathfrak{p}_i$ of A_m lying below $\mathfrak{p}_i^*$ is separable.

LEMMA 5. *Let* ω *be in* $D(A)$. *Then there exists an integer* $m \geqq 0$ *such that* $\nu_{\mathfrak{p}}(C^m(\text{Cosp}_{A/A_m}\omega)) \geqq -1$ *for all places* $\mathfrak{p}$ *of* A_m/K_m, *and any pole* $\mathfrak{p}$ *of* $\text{Cosp}_{A/A_m}\omega$ *is separable.*

Proof. Using the same notation as in Lemma 4 there exist a finite number of places $\mathfrak{g}$ of A/K lying below all poles $\mathfrak{p}^*$ of $\text{Cosp}_{A/A^*}\omega$. We apply the Corollary of Lemma 4 to these finite number of places. Denoting by r any integer which satisfies the Corollary, the place $\mathfrak{p}'$ of A_r lying below $\mathfrak{p}^*$ is separable. Let $q = -\min\{0, \min \nu_{\mathfrak{p}^*}(\text{Cosp}_{A/A^*}\omega)\}$ for all places $\mathfrak{p}^*$ of A^*/K^*, and let $m = \max\{r, q-1\}$. If $\nu_{\mathfrak{p}}(\Omega) \geqq 0$ for a place of A_m, then $\nu_{\mathfrak{p}}(C^j(\Omega)) \geqq 0$ for $0 \leqq j \leqq m$ by Lemma 2, where $\Omega = \text{Cosp}_{A/A_m}\omega$. If $\nu_{\mathfrak{p}}(\Omega) < 0$, then $\nu_{\mathfrak{p}}(C^j(\Omega)) \geqq -1$ for $m \geqq j \geqq -1 - \nu_{\mathfrak{p}}(\Omega)$ by the Corollary of Lemma 2. The fact that $m \geqq -1 - \nu_{\mathfrak{p}}(\Omega)$ is obvious; as from $e_{\mathfrak{p}^*/\mathfrak{p}} = 1$, it holds that

$$\nu_{\mathfrak{p}}(\Omega) = \nu_{\mathfrak{p}^*}(\text{Con}_{A_m/A^*}\Omega) \geqq \nu_{\mathfrak{p}^*}(\text{Cosp}_{A/A^*}\omega) \geqq -q,$$

hence $m \geqq -1 - \nu_{\mathfrak{p}}(\Omega)$. Since the place $\mathfrak{p}^*$ of A^* lying above any pole $\mathfrak{p}$ of Ω is among the poles of $\text{Cosp}_{A/A^*}\omega$, $\mathfrak{p}^*$ is unramified with respect to A_m, and hence $\mathfrak{p}$ is separable. This completes our proof.

Let us define a set of differentials for any integer $m \geqq 0$:

$$R_m(A) = \{\omega \in D(A);\ C^m(\text{Cosp}_{A/A_m}\omega) \in G(A_m)\}.$$

LEMMA 6. *The mapping* $C^m(\text{Cosp}_{A/A_m}\cdot)$ *is a semi-linear homomorphism of the* K*-submodule* $R_m(A)$ *of* $R(A)$ *onto the* K_m*-module* $G(A_m)$.

Proof. Since Cosp_{A/A_m} is an A-linear mapping of $D(A)$ into $D(A_m)$, it is obvious from property (2) in § 2 that $R_m(A)$ is a K-module and $C^m(\text{Cosp}_{A/A_m}\cdot)$ is a semi-linear homomorphism of $R_m(A)$ into $G(A_m)$. If $u\text{Cosp}_{A/A_m}dx$ is an element of $G(A_m)$, then $\omega = u^{p^m}x^{p^m-1}dx$ is in $D(A)$ and $C^m(\text{Cosp}_{A/A_m}\omega) = u\text{Cosp}_{A/A_m}dx$. Thus the mapping is surjective. Let $\mathfrak{p}$ be the place of A_m lying above a place $\mathfrak{g}$ of A. If $u\ \text{Cosp}_{A/A_m}dx = C^m(\text{Cosp}_{A/A_m}\omega)$ for $\omega \in R_m(A)$, then by Lemma 3 we have for any place

$$\text{Res}_{\mathfrak{g}}\ \omega = (\text{Res}_{\mathfrak{p}}\ u\ \text{Cosp}_{A/A_m}dx)^{p^m} = 0,$$

as $u\text{Cosp}_{A/A_m}dx$ belongs to $G(A_m)$. Therefore $R_m(A)$ is a K-submodule of $R(A)$.

COROLLARY. $R(A) = \bigcup_{m=0}^{\infty} R_m(A) = \bigcup_{m=c^*}^{\infty} R_m(A)$.

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Proof. We proved in Lemma 6 that $R_m(A) \subseteq R(A)$ for every $m \geqq 0$. Conversely, let $\omega \in R(A)$. By making use of Lemmas 3 and 5 there exists an integer m such that $\nu_{\mathfrak{p}}(C^m(\mathrm{Cosp}_{A/A_m}\omega)) \geqq -1$, $\mathrm{Res}_{\mathfrak{p}}\, C^m(\mathrm{Cosp}_{A/A_m}\,\omega) = 0$ for all places $\mathfrak{p}$ of A_m/K_m and all poles of $C^m(\mathrm{Cosp}_{A/A_m}\,\omega)$ are separable. Therefore $C^m(\mathrm{Cosp}_{A/A_m}\omega)$ is in $G(A_m)$, hence $\omega \in R_m(A)$. From the Corollary of Lemma 4 we can take all $m \geqq c^*$, hence $R(A) = \bigcup_{m=0}^{\infty} R_m(A) = \bigcup_{m=c^*}^{\infty} R_m(A)$.

LEMMA 7. *Let us assume that* $g(A_m/K_m) = g(A_n/K_n)$ *for some two integers* $n \geqq m \geqq 0$. *Then* $R_m(A) \subseteq R_n(A)$.

Proof. From the definition of $R_m(A)$, $C^m(\mathrm{Cosp}_{A/A_m}\omega)$ belongs to $G(A_m)$ for any $\omega \in R_m(A)$. Let $\mathfrak{p}'$ be any place of A_n/K_n and let $\mathfrak{p}$ be the place of A_m/K_m lying below $\mathfrak{p}'$. Then using the Corollary of Lemma 2

$$\begin{aligned}\nu_{\mathfrak{p}'}\,(C^n(\mathrm{Cosp}_{A/A_n}\omega)) &\geqq \nu_{\mathfrak{p}'}\,(C^m(\mathrm{Cosp}_{A/A_n}\omega))\\ &= e_{\mathfrak{p}'/\mathfrak{p}}\nu_{\mathfrak{p}}(C^m(\mathrm{Cosp}_{A/A_m}\omega))\\ &\geqq 0\end{aligned}$$

by the equality of two genera. Hence $C^n(\mathrm{Cosp}_{A/A_n}\omega)$ is contained in $G(A_n)$, which means that $\omega \in R_n(A)$.

From Lemma 7 it holds that

$$R_m(A) \subseteq R_n(A) \text{ if } n \geqq m \geqq c^*.$$

Let us denote the kernel of $C^m(\mathrm{Cosp}_{A/A_m}\,\cdot)$ by $E_m(A)$. Then $E_m(A)$ is contained in $E^*(A)$ in the sense of Lamprecht as shown by the following Lemma.

LEMMA 8. ω *is in* $E_m(A)$ *if and only if* $\omega = \sum_i \alpha_i x_i^{q_i - 1} dx_i$, $q_i = p^{m_i}$, *where all integers* $m_i \leqq m - 1$, *all* $\alpha_i \in K$ *and all* $x_i \in A$.

Proof. If $\omega = \sum_i \alpha_i x_i^{q_i - 1} dx_i$, $q_i = p^{m_i}$, with $m_i \leqq m - 1$, $\alpha_i \in K$ and $x_i \in A$, then $C^m(\mathrm{Cosp}_{A/A_m}\omega) = 0$. We shall prove the "only if" part by induction on m.

Let $C(\mathrm{Cosp}_{A/A_1}\omega) = 0$; then ω is exact by the property (3) in § 2. Assume that our lemma is true for all $E_n(A)$ with $n \leqq m - 1$. If $\omega \in E_m(A)$, then

$$C^m(\mathrm{Cosp}_{A/A_m}\omega) = C(\mathrm{Cosp}_{A_{m-1}/A_m} C^{m-1}(\mathrm{Cosp}_{A/A_{m-1}}\omega)) = 0,$$

hence $C^{m-1}(\mathrm{Cosp}_{A/A_{m-1}}\omega) = dz$, $z \in A_{m-1}$. Since A_{m-1} is purely inseparable over A, z has no variable pole. Hence $z = \sum_i \alpha_i^{p^{-(m-1)}} x_i$, where $\alpha_i \in K$, $x_i \in A$, and $dz = \sum_i \alpha_i^{p^{-(m-1)}} \mathrm{Cosp}_{A/A_{m-1}} dx_i$. By this we have

$$C^{m-1}(\mathrm{Cosp}_{A/A_{m-1}}(\omega - \textstyle\sum_i \alpha_i x_i^{p^{m-1}-1} dx_i)) = 0,$$

which means

$$\omega - \textstyle\sum_i \alpha_i x_i^{p^{m-1}-1} dx_i \equiv 0 \bmod E_{m-1}(A).$$

Hence by the induction assumption

$$\omega - \textstyle\sum_i \alpha_i x_i^{p^{m-1}-1} dx_i = \textstyle\sum_j \beta_j y_j^{q_j - 1} dy_j,$$

where $m_j \leqq m - 2$, $\beta_j \in K$, $y_j \in A$. This completes our proof.

Let L be an extension of K and let S be a K-submodule of $D(A)$. Then we define an L-module

$$L\ \mathrm{Cosp}_{A/A\langle L\rangle}S = \{\textstyle\sum_i \alpha_i \mathrm{Cosp}_{A/A\langle L\rangle}\omega_i \mid \alpha_i \in L,\ \omega_i \in S\}.$$

COROLLARY. *If L is algebraic over K, then*

$$E_m(A\langle L\rangle) = L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_m(A).$$

Proof. It is obvious that $L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_m(A) \subseteq E_m(A\langle L\rangle)$. Conversely if $\Omega \in E_1(A\langle L\rangle)$, then $\Omega = dz$, $z \in A\langle L\rangle$. As L is algebraic over K, z is written in such form $z = \sum_i \alpha_i x_i$ with $\alpha_i \in L$, $x_i \in A$. Hence

$$\Omega = \textstyle\sum_i \alpha_i \mathrm{Cosp}_{A/A\langle L\rangle}dx_i \equiv 0 \bmod L\mathrm{Cosp}_{A/A\langle L\rangle}E_1(A).$$

Assume that $E_n(A\langle L\rangle) = L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_n(A)$ for $n \leqq m$.

Let $\Omega = z^{p^m-1}dz$ be a purely pseudo-exact differential of $E_{m+1}(A\langle L\rangle)$, then $C\Omega = z^{p^{m-1}-1}dz \in E_m(A\langle L\rangle)$. By the induction assumption

$$C\Omega = \textstyle\sum_i \beta_i x_i^{q_i-1}\mathrm{Cosp}_{A/A\langle L\rangle}dx_i,\ q_i = p^{m_i},$$

where $m_i \leqq m - 1$, $\beta_i \in L$, $x_i \in A$. From this we have

$$C(\Omega - \textstyle\sum_i \beta_i^p x_i^{r_i-1}\mathrm{Cosp}_{A/A\langle L\rangle}dx_i) = 0,\ r_i = p^{m_i+1},$$

hence

$$\Omega - \textstyle\sum_i \beta_i^p x_i^{r_i-1}\mathrm{Cosp}_{A/A\langle L\rangle}dx_i \equiv 0 \bmod E_1(A\langle L\rangle).$$

Since $E_1(A\langle L\rangle) = L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_1(A)$ as proved already, Ω is contained in $L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_{m+1}(A)$. Since any differential in $E_{m+1}(A\langle L\rangle)$ is written as the linear combination of purely pseudo-exact differentials with the coefficients in L, by Lemma 8 it is contained in $L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_{m+1}(A)$.

If we denote $\bigcup_{m=0}^{\infty}E_m(A)$ by $E^*(A)$, then $E^*(A)$ is a K-submodule of $R(A)$. Since $E_m(A) \subseteq E_n(A)$ for $m \leqq n$, $E^*(A) = \bigcup_{m=c^*}^{\infty}E_m(A)$. Therefore $E^*(A\langle L\rangle) = L\ \mathrm{Cosp}_{A/A\langle L\rangle}E^*(A)$ if L is algebraic over K.

LEMMA 9. *Let L be an extension of K and let $A' = A\langle L\rangle$. If $\omega_1, \ldots, \omega_h$ of $D(A)$ are linearly independent over K* mod $E^*(A)$, *then* $\mathrm{Cosp}_{A/A'}\,\omega_1, \ldots, \mathrm{Cosp}_{A/A'}\,\omega_h$ *are linearly independent over L* mod $E^*(A')$.

Proof. We shall show that if $\mathrm{Cosp}_{A/A'}\omega_1, \ldots, \mathrm{Cosp}_{A/A'}\omega_h$ are linearly dependent over L mod $E_m(A')$, then $\omega_1, \ldots, \omega_h$ are linearly dependent over K mod $E_m(A)$. This is sufficient to our proof. We shall prove it by the induction on m.

The case $m = 1$ has been proven by Rosenlicht [**8**, Lemma 5]. Hence assume that our lemma is true for all m less than n. Let $\sum_i \alpha_i \mathrm{Cosp}_{A/A'}\omega_i \equiv 0 \bmod E_n(A')$ for L elements $\{\alpha_i\}$, not all of which are zero. Then using the Corollary of Lemma 8 we have

$$\textstyle\sum_i \alpha_i^{p^{-1}}\mathrm{Cosp}_{A_1/A_1'}C(\mathrm{Cosp}_{A/A_1}\omega_i) \equiv 0 \bmod E_{n-1}(A_1').$$

By the induction assumption

$$\sum_i a_i^{p-1} C(\mathrm{Cosp}_{A/A_1}\omega_i) \equiv 0 \bmod E_{n-1}(A_1)$$

for K elements $\{a_i\}$, not all of which are zero. Using the Corollary of Lemma 8 again and Lemma 8 we have

$$\sum_i a_i^{p-1} C(\mathrm{Cosp}_{A/A_1}\omega_i) = \sum_i b_i^{p-1} \mathrm{Cosp}_{A/A_1} u_i^{p^{n(i)-1}-1} du_i,$$

where $n(i) \leqq n - 1$, $b_i \in K$ and $u_i \in A$. Hence

$$C(\mathrm{Cosp}_{A/A_1}(\sum_i a_i\omega_i - \sum_i b_i u_i^{p^{n(i)}-1} du_i)) = 0,$$

therefore we have $\sum_i a_i \omega_i \equiv 0 \bmod E_n(A)$. This completes our proof.

COROLLARY. *If $g(A/K) = g(A'/L)$, then it holds that*

$$\dim_K G(A)/G(A) \cap E^*(A) = \dim_L G(A')/G(A') \cap E^*(A').$$

In particular if $m, n \geqq c^*$, then

$$\dim_{K_m} G(A_m)/G(A_m) \cap E^*(A_m) = \dim_{K_n} G(A_n)/G(A_n) \cap E^*(A_n).$$

Proof. Let $\omega_1, \ldots, \omega_h$ be a base of $G(A)$ mod $G(A) \cap E^*(A)$ over K. If $g(A/K) = g(A'/L)$, then $\mathrm{Cosp}_{A/A'}\omega_1, \ldots, \mathrm{Cosp}_{A/A'}\omega_h$ are contained in $G(A')$ and by Lemma 9 they are linearly independent over L mod $G(A') \cap E^*(A')$. Hence we have

$$\dim_K G(A)/G(A) \cap E^*(A) \leqq \dim_L G(A')/G(A') \cap E^*(A').$$

Conversely it is obvious that a linear dependency of $\omega_1, \ldots, \omega_h$ mod $E^*(A)$ over K implies a linear dependency of $\mathrm{Cosp}_{A/A'}\omega_1, \ldots, \mathrm{Cosp}_{A/A'}\omega_h$ mod $E^*(A')$ over L. Hence the first equality holds. The second one follows easily from $g(A_m/K_m) = g(A_n/K_n)$.

The proof of Theorem 1. Let $R(A) = \bigcup_{m=c^*}^{\infty} R_m(A)$ and $E^*(A) = \bigcup_{m=c^*}^{\infty} E(A)$. Since $C^m(\mathrm{Cosp}_{A/A_m}\cdot)$ is the semi-linear homomorphism of $R_m(A)$ onto $G(A_m)$ whose kernel is $E_m(A)$, we have

$$R_m(A)/E_m(A) \cong G(A_m), \dim_K R_m(A)/E_m(A) = \dim_{K_m} G(A_m) = g_0(A/K).$$

At the same time, by this mapping $R_m(A) \cap E^*(A)$ is mapped onto $G(A_m) \cap E^*(A_m)$, hence by the Corollary of Lemma 9 we obtain

$$\dim_K R_m(A)/R_m(A) \cap E^*(A) = \dim_{K_m} G(A_m)/G(A_m) \cap E^*(A_m)$$

and this dimension is equal to $\dim_{K_{c^*}} G(A_{c^*})/G(A_{c^*}) \cap E^*(A_{c^*})$ for any $m \geqq c^*$. On the other hand we have

$$R(A)/E^*(A) = \bigcup_{m=c^*}^{\infty} (R_m(A) + E^*(A))/E^*(A),$$

$$\dim_K (R_m(A) + E^*(A))/E^*(A) = \dim_K R_m(A)/R_m(A) \cap E^*(A).$$

As for $n \geqq m$,

$$R_n(A) + E^*(A)/E^*(A) \supseteq R_m(A) + E^*(A)/E^*(A);$$

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by Lemma 7, these two modules are identical and

$$R(A)/E^*(A) \cong R_m(A)/R_m(A) \cap E^*(A)$$

for any $m \geqq c^*$. Therefore we obtain the two equalities in Theorem 1.

COROLLARY. *If A/K is conservative, then*

$$\dim_K R(A)/E^*(A) = \dim_K G(A)/G(A) \cap E^*(A),$$
$$R(A) = G(A) + E^*(A).$$

(cf. [**5**, Satz 1] and [**6**, Satz 3]).

4. Residue free differentials and constant field extension. Let L be an extension of K and let $A' = A\langle L\rangle$.

LEMMA 10. *It holds that*

$$R_m(A') \subseteq L\ \mathrm{Cosp}_{A/A'}R_m(A) + E_m(A').$$

In particular if $g(A_m/K_m) = g(A'_m/L_m)$, then equality holds.

Proof. By definition, a differential $\Omega \in D(A')$ is in $R_m(A')$ if and only if $C^m(\mathrm{Cosp}_{A'/A_m'}\ \Omega)$ is in $G(A_m')$. Any differential in $G(A_m')$ is a linear combination of differentials in $\mathrm{Cosp}_{A_m/A_m'}G(A_m)$ with the coefficients in L_m. Hence $C^m(\mathrm{Cosp}_{A'/A_m'}\Omega)$ is written as follows:

$$C^m(\mathrm{Cosp}_{A'/A_m'}\Omega) = \textstyle\sum_i \lambda_i^{p-m}\mathrm{Cosp}_{A_m/A_m'}\Omega_i,$$

where $\Omega_i \in G(A_m)$, $\lambda_i \in L$. On the other hand $C^m(\mathrm{Cosp}_{A/A_m}\ \cdot)$ is the mapping of $R_m(A)$ onto $G(A_m)$, hence $\Omega_i = C^m(\mathrm{Cosp}_{A/A_m}\omega_i)$ with $\omega_i \in R_m(A)$. Therefore

$$C^m(\mathrm{Cosp}_{A'/A_m'}(\Omega - \textstyle\sum_i \lambda_i \mathrm{Cosp}_{A/A'}\omega_i)) = 0,$$

and hence $\Omega - \sum_i \lambda_i \mathrm{Cosp}_{A/A'}\omega_i \equiv 0 \mod E_m(A')$. This proves the first assertion.

Assume $g(A_m/K_m) = g(A_m'/L_m)$ and let $\Omega = \sum_i \lambda_i \mathrm{Cosp}_{A/A'}\omega_i$ be in $L\ \mathrm{Cosp}_{A/A'}R_m(A)$. Then

$$C^m(\mathrm{Cosp}_{A'/A_m'}\Omega) = \textstyle\sum_i \lambda_i^{p-m}\mathrm{Cosp}_{A_m/A_m'}C^m(\mathrm{Cosp}_{A/A_m}\omega_i)$$

with $C^m(\mathrm{Cosp}_{A/A_m}\omega_i) \in G(A_m)$. As $C^m(\mathrm{Cosp}_{A/A_m'}\omega_i)$ is in $G(A_m')$, $C^m(\mathrm{Cosp}_{A'/A_m'}\Omega) \in G(A_m')$, which means that Ω is in $R_m(A')$. Since $R_m(A') \supseteq E_m(A')$, we have

$$R_m(A') = L\ \mathrm{Cosp}_{A/A'}R_m(A) + E_m(A').$$

Proof of Theorem 2. By Theorem 1, $R(A') = R_m(A') + E^*(A')$ for $m \geqq c'^*$, where c'^* is in the same meaning with c^*. If $m \geqq c^*$, then $g(A_m'/L_m) = g(A_m/K_m)$ and by Lemma 10 we have

$$R_m(A') = L\ \mathrm{Cosp}_{A/A'}R_m(A) + E_m(A').$$

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Since $R(A) = R_m(A) + E^*(A)$, it holds that

$$R(A') = L\,\mathrm{Cosp}_{A/A'}R_m(A) + E^*(A').$$

On the assertion of the dimensions, using the Corollary of Lemma 9 we have

$$\dim_{K_m} G(A_m)/G(A_m) \cap E^*(A_m) = \dim_{L_m} G(A_m')/G(A_m') \cap E^*(A_m')$$

for $m \geqq c^*$. Therefore by Theorem 1

$$\dim_K R(A)/E^*(A) = \dim_L R(A')/E^*(A').$$

This completes the proof of Theorem 2.

COROLLARY. $R(A') = L\,\mathrm{Cosp}_{A/A'}R(A) + E^*(A')$.

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