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ON CLASS SUMS IN p-ADIC GROUP RINGS

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1. Introduction. In this note we prove that an isomorphism of p-adic group rings of finite p-groups maps class sums onto class sums. For integral group rings this is a well known theorem of Glauberman (see [**3**; **7**]). As an application, we show that any automorphism of the p-adic group ring of a finite p-group of nilpotency class 2 is composed of a group automorphism and a conjugation by a suitable element of the p-adic group algebra. This was proved for integral group rings of finite nilpotent groups of class 2 in [**5**]. In general this question remains open. We also indicate an extension of a theorem of Passman and Whitcomb. The following notation is used.

G	denotes a finite p-group.
Z	denotes the ring of (rational) integers.
Z_p	denotes the ring of p-adic integers.
Q_p	denotes the p-adic number field.
K	denotes $\bar{Q}_p$, the algebraic closure of Q_p which contains A the field of all algebraic numbers.
$Z_p(G)$	denotes the group ring of G with coefficients from Z_p.
$\{C_i\}$	denotes the class sums of G.
$\{K_i\}$	denotes the class sums of H.
$\{e_i\}$	denotes the primitive central idempotents of $Q_p(G)$.
h_i, k_i	denotes the number of elements in ith conjugacy class of G and H respectively.
$\{\chi_i\}$	denotes the absolutely irreducible characters of $Q_p(G)$.
z_i	denotes the degree of χ_i.

2. Theorem of Glauberman. We state our main theorem.

THEOREM 1. *Let* θ: $Z_p(G) \to Z_p(H)$ *be an isomorphism. Then* $\theta(C_i) = \pm K_i$ *for all* i.

Proof. Replacing $\theta(G)$ by G we can assume that $Z_p(G) = Z_p(H)$. We have to prove that $C_i = \pm K_i$ for all i. At first we claim that

$$K_i = \sum_j a_{ij}C_j \qquad \text{with } a_{ij} \in Z. \tag{1}$$

We know (see [**1**; p. 236]) that

$$e_i = \frac{z_i}{(G:1)}\sum_v \overline{\chi_i(g_v)}\, C_v, \qquad g_v \in C_v, \tag{2}$$

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and that

$$C_i = \sum_v \frac{h_i \chi_v(g_i)}{z_v} e_v, \qquad g_i \in C_i. \tag{3}$$

By the same token we have

$$K_j = \sum_i k_j \frac{\chi_i(x_j)}{z_i} e_i, \qquad x_j \in K_j. \tag{4}$$

Substituting the value of e_i from (2) in (4) we obtain

$$K_j = \frac{1}{(G:1)} \sum_{i,v} k_j \chi_i(x_j)\, \overline{\chi_i(g_v)}\, C_v. \tag{5}$$

Also,

$$K_j = \sum_l a_{jl} C_l \quad \text{for some} \quad a_{jl} \in Z_p. \tag{6}$$

Comparing (4) and (5) we have

$$a_{jv} = \frac{1}{(G:1)} \sum_i k_j \chi_i(x_j) \bar{\chi}_i(g_v). \tag{7}$$

It follows from (7) that $(G:1)\ a_{jv}$ is an algebraic integer. Since the p^mth cyclotomic polynomial over Q_p is irreducible (see [**2**, p. 212]), by taking trace $Q_p(\xi)/Q_p$ where ξ is an appropriate root of unity, we get from (7) that $(G:1)\ a_{jv}$ is a rational number and hence a rational integer. But since a_{jv} is a p-adic integer and $(G:1)$ is a p-power it follows that a_{jv} is a rational integer. Hence (1) is established. Now we use the argument of Glauberman to conclude that $a_{jv} = \pm\delta_{jv}$. This argument consists mainly of assigning a weight

$$w(K_1, \ldots, K_m) = \sum_{i,j} \chi_i(K_j)\, \overline{\chi_i(K_j)}$$

to class sums of every group basis H and observing that

$$w(K_1, \ldots, K_m) = (G:1) \sum_{i,j} k_j a_{ij}^{\,2} \geqq (G:1)^2,$$

with equality if and only if for each i there is exactly one j such that $a_{ij} \neq 0$ and for that j, $a_{ij} = \pm 1$. Hence the class sums of any group basis H have weight $(G:1)^2$ if and only if they are precisely $\{\pm C_i\}$. Reversing the role of G and H one obtains that the only class sums of a group basis with weight $(G:1)^2$ are precisely $\{\pm K_i\}$. It follows therefore that $\{\pm C_i\} = \{\pm K_i\}$.

3. Applications. We state two applications and indicate the proofs briefly as they are well known in the integral case and the proofs in this case are identical.

THEOREM 2. *Let θ be an automorphism of $Z_p(G)$, where G is nilpotent of class 2. Then there exists an automorphism λ of G and a unit γ of $Q_p(G)$ such that*

$$\theta(g) = \pm\gamma g^{\lambda} \gamma^{-1} \textit{ for all } g \in G.$$

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Proof. As in [**5**], the Theorem follows from Propositions 1 and 2.

PROPOSITION 1. *Let θ be an automorphism of $I(G)$ where I is an integral domain with field of quotients F. Suppose that $\theta(C_i) = C_i'$, and that there exists an automorphism σ of G such that $\sigma(C_i) = C_i'$, for all i. Then we can find a unit $\gamma \in F(G)$ such that*

$$\theta(g) = \gamma g^{\sigma} \gamma^{-1} \text{ for all } g \in G.$$

PROPOSITION 2. *Let μ be an automorphism of $Z_p(G)$ where G is nilpotent of class 2. Suppose that $\mu(C_i) = C_i'$, for all i. Then there exists an automorphism σ of G which, when extended to $Z_p(G)$, satisfies $\sigma(C_i) = C_i'$, for all i.*

Proof. Proposition 1 has been proved for $Z(G)$ in [**5**] but the proof is the same for any $I(G)$. For Proposition 2, the existence of such a σ is proved in [**6**]. That $\sigma(C_i) = C_i'$, follows just as in [**5**].

Passman and Whitcomb [**3**; **7**] proved the next Theorem for $Z(G)$.

THEOREM 3. *Let θ: $Z_p(G) \to Z_p(H)$ be an isomorphism. Then there exists a $1 - 1$ correspondence $N \to \phi(N)$ between normal subgroups of G and H. This correspondence satisfies*

(1) $N_1 \subset N_2 \Leftrightarrow \phi(N_1) \subset \phi(N_2)$
(2) $(N : 1) = (\phi(N) : 1)$
(3) $(N_1, N_2) = (\phi(N_1), \phi(N_2))$.

Proof. The correspondence is established due to Theorem 1. The proofs of (1) and (2) are trivial, and (3) follows as in [**4**].

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