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UNIVERSALITY OF METHODS APPROXIMATING THE DERIVATIVE

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We prove the existence of universal functions for mappings $T_n : C([0,1]) \to L^p([0,1])$, $0 < p < 1$, with $T_n(f) \to f'$ $(n \to \infty)$ on certain subsets of $C^1([0,1])$. As an application we conclude that there are continuous functions $f \in C([0,1])$, such that the derivatives of the Bernstein polynomials

$$\left\{(B_n(f))' : n \in \mathbb{N}\right\}$$

form a dense subset of $L^p([0,1])$ for each $0 < p < 1$.

1. INTRODUCTION

Let $C([0,1])$ denote the Banach space of continuous functions $f : [0,1] \to \mathbb{R}$ endowed with the maximum norm $\|\cdot\|$, let $C^1([0,1])$ denote the normed space of continuously differentiable functions endowed with the norm

$$\|f\|_{1,1} = \|f\| + \int_0^1 |f'(t)|\, dt,$$

and for $0 < p < 1$ let $L^p([0,1])$ denote the F-space of all measurable functions $g : [0,1] \to \mathbb{R}$ with

$$\int_0^1 |g(x)|^p\, dx < \infty$$

(modulo sets of Lebesgue measure zero), endowed with the metric

$$d(g_1, g_2) = \int_0^1 |g_1(x) - g_2(x)|^p\, dx.$$

Let T_n, $n \in \mathbb{N}$ be a family of mappings

$$T_n : C([0,1]) \to L^p([0,1]).$$

We shall give conditions on the mappings T_n, $n \in \mathbb{N}$, satisfied by several classical approximation methods, such that this family of mappings has universal elements.

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2. A universality theorem

Theorem 1. *Let $0 < p < 1$, and let the mappings*

$$T_n : C([0,1]) \to L^p([0,1]), \quad n \in \mathbb{N}$$

have the following properties:

1. *Each mapping*

$$T_n : C([0,1]) \to L^p([0,1]), \quad n \in \mathbb{N}$$

is continuous;

2. *There is a dense subset S of*

$$\left(C^1([0,1]), \|\cdot\|_{1,1}\right)$$

such that $T_n(f) \to f'$ $(n \to \infty)$ for each $f \in S$.

Then the set of functions $f \in C([0,1])$ such that

$$\{T_n(f) : n \in \mathbb{N}\} \text{ is dense in } L^p([0,1])$$

is a dense G_δ subset of $C([0,1])$.

Remarks. 1. If $0 < p_1 \leqslant p_2 < 1$ then $L^{p_2}([0,1]) \subseteq L^{p_1}([0,1])$ and the embedding

$$E : L^{p_2}([0,1]) \to L^{p_1}([0,1]),$$

$E(g) = g$ is dense and continuous. Hence, a standard category argument proves that if the assumptions of Theorem 1 hold for each $0 < p < 1$, then the set of functions $f \in C([0,1])$ such that

$$\{T_n(f) : n \in \mathbb{N}\} \text{ is dense in } L^p([0,1]) \text{ for each } 0 < p < 1$$

is a dense G_δ subset of $C([0,1])$.

2. As will be discussed in Section 6, Theorem 1 cannot be generalised to the case $p \geqslant 1$.

3. Universal elements

To prove Theorem 1 we shall make use of the Universality Criterion of Grosse-Erdmann [4, Theorem 1].

Suppose that Y_1 is a Baire space, Y_2 is second countable, and $T_j : Y_1 \to Y_2$ $(j \in J)$ is a family of continuous mappings. An element $y \in Y_1$ is called universal for this family if $\{T_j y : j \in J\}$ is dense in Y_2. Let U denote the set of all universal elements.

Proposition 1. (Universality Criterion) *Equivalent are:*

1. *The set U is a dense G_δ-subset of Y_1.*
2. *The set U is dense in Y_1.*
3. *The set $\{(y, T_j y) : y \in Y_1, j \in J\}$ is dense in $Y_1 \times Y_2$.*

4. Dense subsets of $L^p([0,1])$.

First note that $C([0,1])$ is a dense subset of $L^p([0,1])$, see [**3**]. The following propositions prove that functions in $L^p([0,1])$ may be approximated by derivatives of uniformly bounded functions:

Proposition 2. *Let $0 < p < 1$. Then*

$$D_{0,\varepsilon} := \{w' : w \in C^1([0,1]),\ \|w\| \leqslant \varepsilon\}$$

is a dense subset of $L^p([0,1])$ for each $\varepsilon > 0$.

Proof: Fix $\varepsilon > 0$. It is sufficient to approximate continuous functions, so let $g \in C([0,1])$. Let $\varphi \in C^\infty(\mathbb{R}, [0,\infty))$ satisfy $\operatorname{supp}(\varphi) \subseteq [0,1]$ and

$$\int_0^1 \varphi(x)\ dx = 1.$$

Since g is continuous we can choose $m \in \mathbb{N}$ such that $2\|g\|/m \leqslant \varepsilon$. Set

$$\alpha_k = m \int_{k/m}^{(k+1)/m} g(s)\ ds \quad (k = 0, \ldots, m-1).$$

We have

$$\begin{aligned}\beta_k &:= \int_{k/m}^{(k+1)/m} \left| g(t) - \alpha_k \varphi\Big(m\Big(t - \frac{k}{m}\Big)\Big)\right| dt \\ &\leqslant \frac{1}{m}(\|g\| + |\alpha_k|) \leqslant \frac{2\|g\|}{m} \leqslant \varepsilon \quad (k = 0, \ldots, m-1).\end{aligned}$$

Define $v, w : [0,1] \to \mathbb{R}$ by

$$v(x) = -\alpha_k \varphi\Big(m\Big(x - \frac{k}{m}\Big)\Big) \quad \big(x \in [k/m, (k+1)/m],\ \ k = 0, \ldots, m-1\big),$$

and

$$w(x) = \int_0^x g(t) + v(t)\ dt \quad \big(x \in [0,1]\big).$$

Note that $\operatorname{supp}(\varphi) \subseteq [0,1]$ implies that v is continuous (even in C^∞), hence $w \in C^1([0,1])$.

We have $w(k/m) = 0$ $(k = 0, \ldots, m-1)$, since $w(0) = 0$, and

$$\begin{aligned} w\big((k+1)/m\big) - w(k/m) &= \int_{k/m}^{(k+1)/m} g(t) + v(t)\ dt \\ &= \frac{\alpha_k}{m} - \frac{\alpha_k}{m} \int_{k/m}^{(k+1)/m} m\varphi\Big(m\Big(t - \frac{k}{m}\Big)\Big)\ dt = 0,\end{aligned}$$

as

$$\int_{k/m}^{(k+1)/m} m\varphi\Big(m\Big(t-\frac{k}{m}\Big)\Big)\,dt = 1.$$

Let $x \in \big[k/m, (k+1)/m\big]$. Then, by the choice of m,

$$\big|w(x)\big| = \big|w(x) - w(k/m)\big| \leqslant \int_{k/m}^{(k+1)/m} \big|g(t)+v(t)\big|\,dt = \beta_k \leqslant \varepsilon.$$

Hence $w \in D_{0,\varepsilon}$ for each φ with the chosen properties.

Next,

$$\begin{aligned} d(g,w') &= \int_0^1 \big|g(t)-w'(t)\big|^p\,dt \\ &= \int_0^1 \big|v(t)\big|^p\,dt \\ &= \sum_{k=0}^{m-1} |\alpha_k|^p \int_{k/m}^{(k+1)/m} \Big|\varphi\Big(m\Big(t-\frac{k}{m}\Big)\Big)\Big|^p\,dt \\ &= \Big(\sum_{k=0}^{m-1} \frac{|\alpha_k|^p}{m}\Big) \int_0^1 \big|\varphi(t)\big|^p\,dt \\ &= m^{p-1}\Big(\sum_{k=0}^{m-1}\Big|\int_{k/m}^{(k+1)/m} g(t)\,dt\Big|^p\Big)\int_0^1 \big|\varphi(t)\big|^p\,dt =: c\int_0^1 \big|\varphi(t)\big|^p\,dt, \end{aligned}$$

and likewise these equations are valid for each φ with the chosen properties.

Let $\delta > 0$. Since $0 < p < 1$ we can choose φ such that in addition

$$d(g,w') = c\int_0^1 \big|\varphi(t)\big|^p\,dt \leqslant \delta,$$

by choosing $\operatorname{supp}(\varphi)$ sufficiently small. ∎

As a consequence of Proposition 2 we get

Proposition 3. *Let $0 < p < 1$, and let $f \in C\big([0,1]\big)$. Then*

$$D_{f,\varepsilon} := \Big\{w' : w \in C^1\big([0,1]\big),\ \|w-f\| \leqslant \varepsilon\Big\}$$

is a dense subset of $L^p\big([0,1]\big)$ for each $\varepsilon > 0$.

Proof: Fix $\varepsilon > 0$, let $g \in C\big([0,1]\big)$, and let $\delta > 0$. Since $C^1\big([0,1]\big)$ is a dense subset of $C\big([0,1]\big)$ we can choose $u \in C^1\big([0,1]\big)$ such that $\|u-f\| \leqslant \varepsilon/2$. According to Proposition 2 there is a function $v \in D_{0,\varepsilon/2}$ such that

$$d(v'+u', g) = d(v', g-u') \leqslant \delta.$$

Set $w = v + u$. We have $d(w',g) \leqslant \delta$, and

$$\|w-f\| \leqslant \|w-u\| + \|u-f\| = \|v\| + \|u-f\| \leqslant \varepsilon,$$

that is $w \in D_{f,\varepsilon}$. ∎

5. Proof of Theorem 1.

We verify condition 3 of Proposition 1.

Each

$$T_n : C([0,1]) \to L^p([0,1]), \quad n \in \mathbb{N}$$

is continuous, $C([0,1])$ is a Baire space, and $L^p([0,1])$ is separable. Let $f \in C([0,1])$, let $g \in L^p([0,1])$ be without loss of generality in $C([0,1])$, and let $\varepsilon > 0$. Condition 3 of Proposition 1 is verified if we can show that there is a function $q \in C([0,1])$ and a number $n_0 \in \mathbb{N}$ such that

$$\|q - f\| \leqslant \varepsilon \quad \text{and} \quad d(T_{n_0}(q), g) \leqslant \varepsilon.$$

According to Proposition 3 there exists $w \in D_{f,\varepsilon/2}$ such that $d(w', g) \leqslant \varepsilon/3$. Next, since S is dense in $\left(C^1([0,1]), \|\cdot\|_{1,1}\right)$ there exists $q \in S$ such that

$$\|q - w\| \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad d(q', w') \leqslant \frac{\varepsilon}{3},$$

since convergence in $L^1([0,1])$ implies convergence in $L^p([0,1])$, compare [**3**, Lemma 1].

In particular we have $\|q - f\| \leqslant \|q - w\| + \|w - f\| \leqslant \varepsilon$, and

$$d(q', g) \leqslant d(q', w') + d(w', g) \leqslant \frac{2}{3}\varepsilon.$$

Since $q \in S$ we have

$$T_n(q) \to q' \quad (n \to \infty).$$

Hence $d(T_{n_0}(q), q') \leqslant \varepsilon/3$ for some $n_0 \in \mathbb{N}$. Thus

$$d(T_{n_0}(q), g) \leqslant d(T_{n_0}(q), q') + d(q', g) \leqslant \varepsilon. \qquad \square$$

6. Applications.

1. Theorem 1 applies to the derivatives of Bernstein polynomials: Let

$$\big(B_n(f)\big)(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k},$$

and let $T_n : C([0,1]) \to L^p([0,1])$ be defined by $T_n(f) = \big(B_n(f)\big)'$. Obviously each T_n, $n \in \mathbb{N}$ is continuous and condition 2. of Theorem 1 holds for $S = C^1([0,1])$, since $\big(B_n(f)\big)' \to f'$ $(n \to \infty)$ even in $C([0,1])$ for each $f \in C^1([0,1])$, see [**7**, Section 1.8]. Thus, the set of continuous functions $f : [0,1] \to \mathbb{R}$ such that

$$\left\{ \big(B_n(f)\big)' : n \in \mathbb{N} \right\} \text{ is dense in } L^p([0,1]) \text{ for each } 0 < p < 1$$

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is a dense G_δ subset of $C([0,1])$.

2. Theorem 1 applies to the derivatives of Lagrange interpolation polynomials: Let $L_n(f)$ denote the Lagrange interpolation polynomial of f of degree at most n with respect to arbitrary nodes

$$0 \leqslant \xi_0^{(n)} < \xi_1^{(n)} < \cdots < \xi_n^{(n)} \leqslant 1 \quad (n \in \mathbb{N}),$$

and let $T_n : C([0,1]) \to L^p([0,1])$ be defined by $T_n(f) = (L_n(f))'$.

Again, each T_n, $n \in \mathbb{N}$ is continuous and condition 2. of Theorem 1 holds for the set S of all polynomials, since $(L_n(f))' = f'$ if $f \in S$ and $n \geqslant \deg f$. Again, the set of continuous functions $f : [0,1] \to \mathbb{R}$ such that

$$\left\{(L_n(f))' : n \in \mathbb{N}\right\} \text{ is dense in } L^p([0,1]) \text{ for each } 0 < p < 1$$

is a dense G_δ subset of $C([0,1])$.

REMARK. For universal properties of the operators $L_n : C([0,1]) \to L^p([0,1])$ with $p \geqslant 1$ (which depend on the choice of the nodes) see [**5**].

3. Let $(\lambda_n)_{n=1}^\infty$ be a sequence with $|\lambda_n| \in (0,1]$ and with limit 0. Theorem 1 applies to difference quotients: For $f \in C([0,1])$ let $f_e : [-1,2] \to \mathbb{R}$ be the extension defined by

$$f_e(x) = \begin{cases} 2f(1) - f(2-x) & (x \in (1,2]) \\ f(x) & (x \in [0,1]) \\ 2f(0) - f(-x) & (x \in [-1,0)) \end{cases},$$

and let $T_n : C([0,1]) \to L^p([0,1])$ be defined by

$$(T_n(f))(x) = \frac{f_e(x+\lambda_n) - f_e(x)}{\lambda_n}.$$

By standard reasoning each T_n is continuous and $T_n(f) \to f'$ in $C([0,1])$ for each $f \in S := C^1([0,1])$. Once more, the set of continuous functions $f : [0,1] \to \mathbb{R}$ such that

$$\{T_n(f) : n \in \mathbb{N}\} \text{ is dense in } L^p([0,1]) \text{ for each } 0 < p < 1$$

is a dense G_δ subset of $C([0,1])$.

This result is, in a certain sense, the one-dimensional case of Joó's generalisation ([**6**, Theorem I]) of Marcinkiewicz's classical result [**8**] on universal primitives. In [**1**] and [**2**] Bogmér, Sövegjártó and Buczolich proved that there is no universal primitive in $L^1([0,1])$ for $p \geqslant 1$. In particular if $p \geqslant 1$, then there is no $f \in C([0,1])$ such that

$$\{T_n(f) : n \in \mathbb{N}\} \text{ is dense in } L^p([0,1]).$$

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