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GAUSSIAN ESTIMATES FOR THE HEAT KERNEL OF THE WEIGHTED LAPLACIAN AND FRACTAL MEASURES

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ABSTRACT. Let $0 < w$ be a smooth function on a complete Riemannian manifold M^n, and define $L = -\Delta - \nabla(\log w)$ and $R_w = \mathrm{Ric} - w^{-1}\,\mathrm{Hess}\, w$. In this paper we show that if $R_w \geq -nK$, $(K \geq 0)$, then the positive solutions of $(L + \partial/\partial t)u = 0$ satisfy a gradient estimate of the same form as that obtained by Li and Yau ([LY]) when L is the Laplacian. This is used to obtain a parabolic Harnack inequality, which in turn, yields upper and lower Gaussian estimates for the heat kernel of L. The results obtained are applied to study the L^p mapping properties of $t \longrightarrow e^{-tL}\mu$ for measures μ which are α-dimensional in a sense that generalises the local uniform α-dimensionality introduced by R. S. Strichartz ([St2], [St3]).

0. **Introduction and notations.** Let (M^n, g) be an n-dimensional, complete Riemannian manifold and $0 < w \in C^\infty(M^n)$ a given function on M^n, the weight function. We will denote (*cf.* [S]) by R_w the symmetric tensor

$$R_w = \mathrm{Ric} - w^{-1}\,\mathrm{Hess}\, w,$$

and by L the operator

$$L = -\Delta - \nabla(\log w).$$

L is induced by the quadratic form

$$Q(f) = \int_{M^n} |\nabla f|^2 w\, dV, \quad f \in C_c^\infty(M^n),$$

and extends to a self-adjoint operator on $L^2(w\,dV)$, where dV denotes the standard Riemannian measure on M^n (*cf.* [Bk1]). Therefore the heat semigroup $\exp(-tL)$ can be defined via the spectral theorem and, since $Q(f)$ is a Dirichlet form, $\exp(-tL)$ induces a positivity preserving contraction semigroup on $L^p(w\,dV)$ for all $1 \leq p \leq \infty$ ([RS], [F]). Moreover Strichartz's proof of the existence of the heat kernel for Δ ([St1]) can be adapted to show that $\exp(-tL)$ has a smooth strictly positive symmetric heat kernel $h(x, y, t)$. The paper is divided in two parts. In the first part we obtain upper and lower bounds for $h(x, y, t)$, which extend results obtained for the Laplacian by Li and Yau ([LY]), and further investigated, using different methods, by Davies ([D1]–[D5]) and Varopoulos ([V]). Li and Yau's estimates depend on the hypothesis that the Ricci

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curvature of M^n is bounded below. This is replaced by the assumption, in force throughout the paper, that R_w is bounded from below, and the estimates that can be deduced from it (Theorems 8 and 11) are exactly of the same form as Li and Yau's. As for the Laplacian, the fundamental step towards obtaining diagonal and off-diagonal estimates is establishing a parabolic Harnack inequality.

In the second part of the paper we apply the results of the first part to study measures on Riemannian manifolds with weight which are α-dimensional in a sense that generalises the concept of local uniform α-dimensionality introduced by R. S. Strichartz for measures on $\mathbf{R}^n$, and, more generally, on manifolds with bounded geometry ([St2], [St3]). Extending results of Strichartz ([St3]) and Lau ([La]), we relate various notions of α-dimensionality to L^p mapping properties of of $t \longrightarrow e^{-tL}\mu$. The main result is that if a locally finite (complex) measure ν satisfies

$$\sup_{0<r\leq 1} r^{(n-\alpha)/p'} \left\| \frac{|\nu|(B(x,r))}{\mathrm{vol}_w(B(x,r))} \right\|_{L^p(w\,dV)} \leq C, \quad p^{-1} + p'^{-1} = 1,$$

for given $1 \leq p \leq \infty$ and $0 \leq \alpha \leq 1$, then

$$\sup_{0<t\leq 1} t^{(n-\alpha)/2p'} \|e^{-tL}\nu\|_{L^p(w\,dV)} \leq C_1,$$

where vol_w denotes the volume with respect to $w\,dV$. Moreover the converse holds if ν is a positive measure.

The paper is organised as follows. In §1, using a suitable generalisation of Bishop's comparison theorem ([S], Theorem 4.1), we derive pointwise and distributional inequalities for Lr, where $r(x) = d(x_0, x)$ is the distance from some fixed point x_0, and in §2 the comparison theorem is used to prove relative volume estimates for the $w\,dV$-measure of geodesic balls in M^n, which extend those in Proposition 4.1 of [CGT]. §3 is devoted to the proof of the analogue of Li and Yau's gradient estimate for positive solutions of the equation $(L + \partial/\partial t)u = 0$, and §4 to the parabolic Harnack inequality for L. Adapting ideas of Varopoulos ([V]), the parabolic Harnack inequality and the results of the first two sections are used in §5 to derive Gaussian upper bounds for h, and in §6, to obtain comparable lower bounds. In §7 we present an example that shows that some hypothesis on R_w is necessary for the kind of estimates obtained. §8 is devoted to applications to α-dimensional measures. In this section the relative w-volume estimate of §2 will play a central role.

1. **The effect of L on the distance function.** Given $x_0 \in M^n$, let (r, ξ) be spherical geodesic coordinates at x_0, and denote by $\sqrt{g}(r, \xi)$ the area element, so that the Riemannian volume element is given locally by $dV = \sqrt{g}(r, \xi)\,dr\,d\xi$, $d\xi$ being the standard measure on the unit sphere $ST_{x_0}M^n$ of $T_{x_0}M^n$. We denote by $c(\xi)$ the distance along the geodesic $\gamma_\xi(t) = \exp_{x_0} t\xi$ from x_0 to its cut locus $\mathrm{Cut}(x_0)$. Note that $D_{x_0} = \{(r, \xi) \in \mathbf{R}^+ \times ST_{x_0}M^n : 0 < r < c(\xi)\}$ is the domain of the normal coordinates at x_0. If $r(x)$ is

the distance from the point x_0, using the equivalent definition of $L = -w^{-1}\mathrm{div}(w\nabla\cdot)$, a computation in the coordinates (r, ξ) shows that

$$-Lr = (w\sqrt{g})^{-1}\frac{\partial(w\sqrt{g})}{\partial r}(r,\xi), \quad 0 < r < c(\xi). \tag{1}$$

Then we have the following analogue of the Laplacian comparison theorem:

LEMMA 1. *If $R_w \geq nK$, then*

$$-Lr \leq n\frac{C_K}{S_K}, \tag{2}$$

where

$$S_K(r) = \begin{cases} (\sqrt{-K})^{-1}\sinh(\sqrt{-K}r) \\ r \\ (\sqrt{K})^{-1}\sin(\sqrt{K}r) \end{cases} \qquad C_K(r) = \begin{cases} \cosh(\sqrt{-K}r) & \text{for } K < 0 \\ 1 & \text{for } K = 0 \\ \cos(\sqrt{K}r) & \text{for } K > 0 \end{cases}$$

The inequality holds pointwise in the complement of the cut locus of x_0 and in the distributional sense in $M^n \setminus x_0$.

PROOF. By Theorem 4.1 in [S], $R_w \geq nK$ implies

$$(w\sqrt{g})^{-1}\frac{\partial(w\sqrt{g})}{\partial r}(r,\xi) \leq n\frac{C_K}{S_K}, \tag{3}$$

for $\xi \in ST_{x_0}M^n$ and $0 < r < c(\xi)$, which, in view of (1) proves the pointwise inequality. To prove that (2) holds in the distributional sense in $M^n \setminus x_0$, one can adapt Yau's proof of the distributional subharmonicity of the Buseman function ([Y], Appendix), which in turn relies on a construction due to Cheeger and Gromoll ([CG]). Indeed, using the following analogue of Green's theorem: If Ω is a normal domain in M^n with outward normal $\partial/\partial\nu$ and $u \in C^2(\bar{\Omega})$, $v \in C^1(\Omega)$, then

$$-\int_\Omega vLuw\,dV + \int_\Omega\langle\nabla u, \nabla v\rangle = \int_{\partial\Omega} v\frac{\partial u}{\partial\nu}w\,dA,$$

the argument in [Y] can be used to show that

$$-\int_D rL\psi w\,dV \leq -\int_{D\setminus\mathrm{Cut}\{x_0\}} Lr\psi w\,dV,$$

for all relatively compact regions D with smooth boundary ∂D and for all $0 \leq \psi \in C_c^\infty(M^n \setminus \{x_0\})$ with support contained in D. This and the pointwise inequality clearly yield the desired result.

The same argument also yields the following:

COROLLARY 2. *Given $f \in C^\infty(\mathbf{R}^+)$ with $f' \geq 0$, let $\phi(x) = f\big(r(x)\big)$. If $R_w \geq nK$ then*

$$-L\phi \leq f''(r) + f'(r)n\frac{C_K}{S_K}, \tag{4}$$

and the inequality holds pointwise in the complement of the cut locus of x_0 and in the distributional sense in $M^n \setminus x_0$.

2. **Relative volume estimates.** *Given a measurable set $E \subset M^n$ we define its w-volume, denoted* vol_w, *as the measure of E with respect to the measure wdV. For $x \in M^n$ and $r > 0$ we let also $V_w(x,r) = \mathrm{vol}_w\big(B(x,r)\big)$, where B(x,r) is the geodesic ball with radius r centered at x, and define*

$$V_K(r) = \omega_{n-1}\int_0^r S_K^n(s)\,ds,$$

ω_m being the volume of the standard sphere $\mathbf{S}^m \subset \mathbf{R}^{m+1}$, and S_K as in the previous section. Note that, modulo a constant factor, $V_K(r)$ is the volume of the ball of radius r in the $n+1$-dimensional space form with curvature K (if $K > 0$, we restrict to $r \le \pi/\sqrt{K}$).

The comparison theorem used above allows us to generalise to the w-volume the relative volume estimates of [CGT], Proposition 4.1:

LEMMA 3. *Assume that $R_w \ge nK$ and let $x \in M^n$. Then $V_w(x,r)/V_K(r)$ is a monotonically decreasing function of r.*

PROOF. With the notation introduced in §1, in polar coordinates (r,ξ) at x we have

$$\begin{aligned} V_w(r,\xi) &= \int_{\mathbf{S}^{n-1}} d\xi \int_0^{\min\{r,c(\xi)\}} w\sqrt{g}(t,\xi)\,dt \\ &= \int_0^r dt \int_{\{\xi:(t,\xi)\in D_x\}} w\sqrt{g}(t,\xi)\,d\xi. \end{aligned}$$

If we define

$$\phi(r) = \int_{\{\xi:(r,\xi)\in D_x\}} w\sqrt{g}(r,\xi)\,d\xi, \quad \psi(r) = \int_{\mathbf{S}^{n-1}} S_K^n(r)\,d\xi = \omega_{n-1}S_K^n(r),$$

(3) implies that ϕ/ψ is a monotonically decreasing function of r. As in [CGT], p. 42, one concludes that

$$V_w(x,r)/V_K(r) = \int_0^r \phi(s)\,ds \Big/ \int_0^r \psi(s)\,ds$$

is monotonically decreasing.

REMARK. Observe that the analogue statements of ii) iii) iv) in [CGT] Proposition 4.1, follow easily from the lemma. These can then be applied to estimate the number of disjoint unit balls contained in $B(x,R)$, and hence to show that if $R_w \ge -nK$ $(K \ge 0)$ then $V_w(x,R)$ grows at most polynomially if $K = 0$, and at most like $e^{\mathrm{const}\,R}$ if $0 < K < \infty$.

3. **The gradient estimate.** In this section we prove the generalization of Li and Yau's gradient estimate ([LY]) for positive solutions of the heat equation for L. As in the case of the Laplacian the gradient estimate will easily yield a version of the parabolic Harnack inequality. The proof follows closely Davies' proof ([D1]) of Li and Yau's estimate and depends on the following two lemmas.

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LEMMA 4. *Let* $0 < u \in C^\infty(M^n \times [0,T])$ *be a positive solution of* $(L+\partial/\partial t)u = 0$ *in* $M^n \times [0,T]$. *Define* $f(x,t) = \log u(x,t)$ *and* $F(x,t) = t(|\nabla f|^2 - \alpha f_t)$, $\alpha \in \mathbf{R}$. *If* $R_w \geq -nK$, $K \geq 0$, *then*

$$-(L+\partial/\partial t)F \geq t\left\{\frac{2}{n+1}(Lf)^2 - 2nK|\nabla f|^2\right\} - 2\langle \nabla f, \nabla F\rangle - t^{-1}F.$$

PROOF ([D1]). Since $L(\psi \circ g) = (\psi' \circ g)Lg - (\psi'' \circ g)|\nabla g|^2$, we see that

$$Lf = \frac{Lu}{u} + \frac{|\nabla u|^2}{u^2}$$
$$= \frac{-u_t}{u} + \frac{|\nabla u|^2}{u^2} = -f_t + |\nabla f|^2. \tag{5}$$

Therefore, using the generalised BLW formula (*cf.* [BE] Theorem 3 , [S], Theorem 2.1) yields

$$\begin{aligned}-LF &= t\{-L(|\nabla f|^2) + \alpha L f_t\}\\ &= 2t\{|\operatorname{Hess} f|^2 - \langle \nabla f, \nabla(Lf)\rangle + R_w(\nabla f, \nabla f) + w^{-2}\langle \nabla w, \nabla f\rangle^2\}\\ &\quad + \alpha t L f_t \geq 2t\left\{\frac{(Lf)^2}{n+1} + \langle \nabla f, \nabla(f_t - |\nabla f|^2)\rangle - nK|\nabla f|^2\right\} + \alpha t L f_t,\end{aligned}$$

where the last inequality follows from

$$|\operatorname{Hess} f|^2 \geq (\Delta f)^2 = \frac{1}{n}(Lf + w^{-1}\langle \nabla w, \nabla f\rangle)^2 \geq \frac{(Lf)^2}{n+1} - w^{-2}\langle \nabla w, \nabla f\rangle^2$$

which in turn is a consequence of $(a-b)^2 \geq sa^2 - sb^2/(1-s')$, with $s = n/n+1$. Also taking the time derivative of (5) we find

$$\begin{aligned}-F_t &= -t^{-1}F - t\{2\langle \nabla f, \nabla f_t\rangle - \alpha f_{tt}\}\\ &= -t^{-1}F - t\{\alpha L f_t - 2(\alpha - 1)\langle \nabla f, \nabla f_t\rangle\}.\end{aligned}$$

The result now follows easily by combining this with the inequality obtained above.

LEMMA 5. *Assume that* $R_w \geq -nK$, $K \geq 0$. *Then, given* $x_0 \in M^n$ *and* $R > 0$, *there exists* $\phi \in C^{0,1}(M^n) \cap C^\infty\big(M^n \setminus \operatorname{Cut}(x_0)\big)$, $\phi \equiv 1$ *on* $B(0,R)$, $\equiv 0$ *on* $B(0,2R)^c$, *satisfying the following inequalities pointwise in* $M^n \setminus \operatorname{Cut}(x_0)$:

$$\frac{|\nabla \phi|^2}{\phi} \leq \frac{C_1}{R^2}$$
$$-L\phi \geq -\frac{C_2}{R^2} - \frac{C_3}{R}\sqrt{K},$$

with C_i *constants depending only on n.*

PROOF. Define $\phi = \psi\big(r(x)/R\big)$, where $r(x) = d(x_0, x)$ and ψ is a smooth function in $[0,\infty)$ satisfying $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $[0,1]$, $\psi \equiv 0$ in $[2,\infty)$, $\psi' \leq 0$, $(\psi')^2/\psi \leq C_1$ and $\psi'' \geq -C_4$. Using Lemma 1, it is easy to verify that ϕ satisfied the stated inequalities.

THEOREM 6. *Let u be as in the statement of Lemma 4, and assume that $R_w \geq -nK$, $K \geq 0$. Then $\forall \alpha > 1$, and $(x,t) \in M^n \times (0,T)$,*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{(n+1)\alpha^2}{2}\left\{\frac{1}{t} + \frac{nK}{2(\alpha-1)}\right\}. \tag{6}$$

PROOF. We proceed as in the proof of Theorem 1.2 in [LY]. Fix $(x_0,t_0) \in M^n \times (0,T]$ and $R > 0$, and let ϕ be the function constructed in Lemma 5. Let (x_1,t_1) be the point where ϕF attains its maximum over $B(x_0,2R)\times[0,t_0]$. We can assume that $\phi F(x_1,t_1) > 0$, and therefore $t_1 > 0$, for otherwise there is nothing to prove. We consider first the case $x_1 \notin \mathrm{Cut}(x_0)$, so that ϕF is smooth at (x_1,t_1) and

$$\nabla(\phi F)(x_1,t_1) = 0,\ \phi F_t(x_1,t_1) \geq 0,\ \text{and}\ -L(\phi F)(x_1,t_1) \leq 0.$$

Thus at (x_1,t_1) we have

$$\begin{aligned}
0 &\geq -\phi F_t - L(\phi F)\\
&\geq 2\phi t_1\left\{\frac{(Lf)^2}{n+1} - nK|\nabla f|^2\right\} - 2\langle\nabla F,\phi\nabla f\rangle - t_1^{-1}\phi F + \langle\nabla F,\nabla\phi\rangle - FL\phi\\
&\geq 2\phi t_1\left\{\frac{(|\nabla f|^2 - f_t)^2}{n+1} - nK|\nabla f|^2\right\} - 2\sqrt{\phi}F|\nabla f|\frac{C}{R} - t_1^{-1}\phi F\\
&\quad - F\left(\frac{C'}{R^2} + \frac{C''\sqrt{K}}{R}\right)
\end{aligned}$$

where we have used Lemma 4, $\nabla F = -\phi^{-1}\nabla\phi F$, $Lf = |\nabla f|^2 - f_t$ and Lemma 5 in this order. Defining $0 \leq \mu = (F^{-1}|\nabla f|^2)(x_1,t_1)$, so that $f_t(x_1,t_1) = F(x_1,t_1)(t_1\mu - 1)/t_1\alpha$, and substituting above yield

$$\begin{aligned}
0 \geq 2\phi t_1&\left\{\frac{1}{n+1}\left(\mu - \frac{t_1\mu - 1}{t_1\alpha}\right)^2 F^2 - nK\mu F\right\}\\
&- 2(\mu\phi)^{1/2}F^{3/2}\frac{C}{R} - t_1^{-1}\phi F - F\left(\frac{C'}{R^2} + \frac{C''\sqrt{K}}{R}\right).
\end{aligned}$$

Multiplying throughout by t_1F^{-1} and simplifying the last inequality becomes

$$A\lambda^2 - 2B\lambda - D \leq 0$$

where

$$A = \frac{2(t_1\mu(\alpha-1)+1)^2}{(n+1)\alpha^2},\quad B = \frac{C\mu^{1/2}t_1}{R},$$
$$D = 1 + \left(\frac{C'}{R^2} + \frac{C''\sqrt{K}}{R}\right)t_1 + 2\mu nKt_1^2,\quad \lambda = (\phi F)^{1/2}(x_1,t_1).$$

By the quadratic formula

$$\lambda^2 = (\phi F)(x_1,t_1) \leq \left\{\frac{B}{A} + \left[\left(\frac{B}{A}\right)^2 + \frac{D}{A}\right]^{1/2}\right\}^2. \tag{7}$$

Assume now that x_1 belongs to the cut locus of x_0. Then one appeals to an idea of Calabi ([Ca]), already used in [CY] and [LY]): Let γ be a minimising geodesic from x_0 to x_1 and let q be a point on γ close to x_0. Denote by $\bar{r}$ the distance function from q and by $\bar{\phi}$ the function defined by

$$\bar{\phi} = \psi\left(\frac{r(q) + \bar{r}(x)}{R}\right).$$

A simple computation shows that $\bar{\phi}$ satisfies the inequalities of Lemma 5 in the complement of the cut locus of q. Moreover, since $r(q) + \bar{r}(x) \geq r(x)$ $\forall x$ with equality at $x = x_1$, and ψ is decreasing, we have

$$\begin{aligned}\bar{\phi}(x_1)F(x_1, t_1) &= \psi\left(\frac{r(x_1)}{R}\right)F(x_1, t_1) \geq \psi\left(\frac{r(x)}{R}\right)F(x, t)\\ &\geq \psi\left(\frac{r(q) + \bar{r}(x)}{R}\right)F(x, t) = \bar{\phi}(x)F(x, t),\end{aligned}$$

so that $\bar{\phi}F$ attains a (local) maximum at (x_1, t_1). Since $\bar{\phi}F$ is smooth in a neighbourhood of (x_1, t_1) the first part of the proof yields

$$(\bar{\phi}F)(x_1, t_1) \leq \left\{\frac{B}{A} + \left[\left(\frac{B}{A}\right)^2 + \frac{D}{A}\right]^{1/2}\right\}^2$$

and (7) follows letting $q \longrightarrow x_0$ along γ.Thus we have

$$\begin{aligned}F(x_0, t_0) &= \phi(x_0)F(x_0, t_0)\\ &\leq \phi(x_1)F(x_1, t_1) \leq \left\{\frac{B}{A} + \left[\left(\frac{B}{A}\right)^2 + \frac{D}{A}\right]^{1/2}\right\}^2.\end{aligned}$$

Now a computation as in [D1], pg. 161, shows that

$$\frac{B}{A} \longrightarrow 0 \text{ and } \frac{D}{A} \longrightarrow \frac{(n+1)\alpha^2}{2}\left\{1 + \frac{nKt_0}{2(\alpha - 1)}\right\}, \text{ as } R \longrightarrow \infty.$$

Therefore (6) follows by letting $R \longrightarrow \infty$ in (7) and recalling the definition of F.

REMARKS. 1) The gradient estimate (6) holds if $u \geq 0$: it suffices to consider $u + \epsilon$ instead of u and then take the limit as $\epsilon \downarrow 0$ in the gradient estimate;

2) For (6) to hold it suffices to assume that $0 \leq u \in C^\infty\big(M^n \times (0, T]\big)$ is a solution of $(L + \partial/\partial t)u = 0$ in $M^n \times (0, T]$: the function $v = v_\epsilon$ defined by $v(x, t) = u(x, t + \epsilon)$, $\epsilon > 0$, is smooth and satisfies the heat equation in $M^n \times [0, T - \epsilon]$. Thus (6) holds for v, $\forall x \in M^n$ and $\forall t \in [0, T - \epsilon)$, and by letting $\epsilon \downarrow 0$ we see that u satisfies (6). In particular the heat kernel $h(x, y, t)$ of L satisfies

$$\text{(8)} \qquad \frac{|\nabla_x h|^2}{h^2} - \alpha\frac{h_t}{h} \leq \frac{(n+1)\alpha^2}{2}\left\{\frac{1}{t} + \frac{nK}{2(\alpha - 1)}\right\},$$

$\forall x, y \in M^n$, $t > 0$, and $\alpha > 1$.

4. **The parabolic Harnack inequality.** Using the gradient estimate of the previous section, it is now a simple matter to extend to L the parabolic Harnack inequality of Li and Yau.

THEOREM 7. *Let $0 \leq u \in C^\infty\big(M^n \times (0,T]\big)$ be a solution of $(L + \partial/\partial t)u = 0$ in $M^n \times (0,T]$. If $R_w \geq -nK$, $K \geq 0$, then $\forall 0 < t \leq t+s < T$, $x, y \in M^n$, and $\alpha > 1$, we have*

$$0 \leq u(x,t) \leq u(y,t+s)\left(\frac{t+s}{t}\right)^{(n+1)\alpha/2} \exp\left\{\frac{\alpha d(x,y)^2}{4s} + \frac{\alpha n(n+1)Ks}{4(\alpha-1)}\right\}. \tag{9}$$

Since the proof of (8) follows from the gradient estimate (6) exactly as in the case of the Laplacian, we refer for the proof to Li and Yau [LY], pp. 166–7, or to Davies [D1], pp. 162–3.

5. **The upper bound.** Using the parabolic Harnack inequality, one could obtain first a diagonal upper bound for the heat kernel of L and then derive a Gaussian upper estimate by adapting Davies's techniques ([D4], §2.). We will follow instead Varopoulos approach ([V], §4), which is more elementary and is suitable to give a unified treatment of the cases $K = 0$ and $K > 0$.

THEOREM 8. *Let $h(x,y,t)$ be the heat kernel of L and E the bottom of its $L^2(w\,dV)$-spectrum. Assume that $R_w \geq -nK$.*

1) If $K = 0$, then $\forall 0 < \epsilon < 1$ there exists $C = C(\epsilon, n)$ such that

$$h(x,y,t) \leq C V_w(x,\sqrt{t})^{-1/2} V_w(y,\sqrt{t})^{-1/2} \exp\left\{-\frac{d^2(x,y)}{4(1+\epsilon)t}\right\}.$$

2) If $K > 0$, then $\forall 0 < \epsilon < 1$ there exists $C = C(\epsilon, n, K)$ such that

$$h(x,y,t) \leq C e^{(\epsilon - E)t} V_w(x,\sqrt{t})^{-1/2} V_w(y,\sqrt{t})^{-1/2} \exp\left\{-\frac{d^2(x,y)}{4(1+\epsilon)t}\right\}.$$

PROOF. Using the parabolic Harnack inequality of the previous section the proof follows almost verbatim [V], §4. Let $\phi \in C_c^\infty(M^n)$ be such that $|\nabla\phi| \leq 1$ and define

$$Bf(x) = e^{\lambda\phi(x)} L(e^{-\lambda\phi} f)(x), \quad f \in C_c^\infty(M^n), \ \lambda \in \mathbf{R}.$$

Then $(Bf, f)_{L^2(w\,dV)} = \big(\nabla(e^{\lambda\phi(x)} f), \nabla(e^{-\lambda\phi} f)\big) \geq (E - \lambda^2)\|f\|_2^2$, and the semigroup generated by$B$, $e^{-tB} = e^{\lambda\phi(x)} e^{-tL} e^{-\lambda\phi(\cdot)}$, satisfies

$$\|e^{-tB}\|_{2,2} \leq e^{t(\lambda^2 - E)}$$

(*cf.* [RS], Theorem X.48). Therefore

$$\begin{aligned} &\int_{B(x,\sqrt{t})} w(\xi)\, d\xi \int_{B(y,\sqrt{t})} w(\zeta)\, d\zeta h(\xi,\zeta,t) e^{\lambda\phi(\xi) - \lambda\phi(\zeta)} \\ &\quad = (e^{-tB}\chi_{B(y,\sqrt{t})}, \chi_{B(x,\sqrt{t})}) \leq e^{(\lambda^2 - E)t} V_w(x,\sqrt{t})^{1/2} V_w(y,\sqrt{t})^{1/2}, \end{aligned} \tag{10}$$

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for $x, y \in M^n$, $t > 0$. The parabolic Harnack inequality with $\alpha = 2$ yields

$$h\big(x,y,(1-\epsilon)t\big) \le h(\xi,\zeta,t)(1-\epsilon)^{-(n+1)}\exp\left\{\frac{d^2(x,\xi)+d^2(\zeta,y)}{\epsilon t} + \frac{n(n+1)}{2}K\epsilon t\right\},$$

for $0 < \epsilon < 1$. Integrating over $\xi \in B(x,\sqrt{t})$ and $\zeta \in B(y,\sqrt{t})$, and using (10), $|\phi(x) - \phi(\xi)| \le \|\nabla\phi\|_\infty d(x,\xi) \le \sqrt{t}$, $|\phi(y) - \phi(\zeta)| \le \sqrt{t}$, we obtain

$$h\big(x,y,(1-\epsilon)t\big) \le (1-\epsilon)^{-(n+1)}V_w(x,\sqrt{t})^{-1/2}V_w(y,\sqrt{t})^{-1/2} \times \exp\left\{t\lambda^2 - \lambda\phi(x) + \lambda\phi(y) + 2|\lambda|\sqrt{t} - tE + \frac{2}{\epsilon} + \frac{n(n+1)}{2}K\epsilon t\right\},$$

$\forall x, y \in M^n$, $t > 0$, and $0 < \epsilon < 1$. Putting $\lambda = -d(x,y)/2t$ and $\phi = \phi_i$, where ϕ_i is a sequence in $C_c^\infty(M^n)$ satisfying $\phi_i(x) \to 0$, $\phi_i(y) \to d(x,y)$ as $i \to \infty$, and taking the limit in the last inequality we obtain

$$h\big(x,y,(1-\epsilon)t\big) \le (1-\epsilon)^{-(n+1)}V_w(x,\sqrt{t})^{-1/2}V_w(y,\sqrt{t})^{-1/2} \times \exp\left\{-\frac{d^2(x,y)}{4t} + \frac{d(x,y)}{\sqrt{t}} - tE + \frac{2}{\epsilon} + \frac{n(n+1)}{2}K\epsilon t\right\}.$$

Using

$$-\frac{d^2}{4t} + \frac{d}{\sqrt{t}} = -\left(\frac{d}{2\sqrt{t}} - 1\right)^2 + 1 \le -\gamma\frac{d^2}{4t} + \frac{1}{\gamma - 1}, \quad 0 < \gamma < 1,$$

with $\gamma = (1-2\epsilon)/(1-\epsilon)$, and letting $s = (1-\epsilon)t$, the inequality above becomes

$$h(x,y,s) \le (1-\epsilon)^{-(n+1)}V_w(x,\sqrt{s})^{-1/2}V_w(y,\sqrt{s})^{-1/2} \times \exp\left\{-\frac{d^2(x,y)}{4(1+2\epsilon)s} - sE + \frac{3}{\epsilon} + \frac{n(n+1)\epsilon}{2(1-\epsilon)}Ks\right\}.$$

Observing that $E = 0$ if $K = 0$ (because in this case the w-volume of balls grows subexponentially, *cf* [S], Proposition 3.1), by redefining ϵ, 1) and 2) follow with $C = c_1 e^{c_2/\epsilon}$, where $c_1 = c_1(n)$, and $c_2 = c_2(n,K)$.

6. **The lower bound.** This section is devoted to obtaining a lower bound for $h(x,y,t)$ comparable with the upper estimate of the previous section. The idea of the proof is exactly as in Davies [D1], §6, and therefore we will only sketch the proofs, briefly indicating what changes must be made. We start with a lemma.

LEMMA 9. *Assume that $R_w \ge -nK$, $K \ge 0$. Then for every $T > 0$ there exists a constant $a = a(n,K,T)$ such that*

$$\int_{B(x,a\sqrt{t})} h(x,y,t)w(y)\,dV(y) \ge \frac{1}{2}, \tag{11}$$

$\forall 0 < t < T$. *Moreover if* $K = 0$, *a is independent of* T *and (11) holds for* $0 < t < \infty$.

PROOF. The argument is as in Varopoulos [V], §4. Let $\psi \in C^\infty(\mathbf{R}^+)$ satisfy $0 \le \psi \le 1$, with $\psi \equiv 1$ in $[0, 1/2]$, $\psi \equiv 0$ in $[1, \infty)$, $-C_1 \le \psi' \le 0$, $|\psi''| \le C_2$, and define

$$\phi(z,t) = \psi\Big(\frac{\alpha r^2}{t}\Big), \; r(z) = d(x,z), \quad 0 < \alpha < 1.$$

Since $rC_K/S_K(r) \le C(1+\sqrt{K}r)$, and $\psi'(\alpha r^2/t), \psi''(\alpha r^2/t) = 0$ unless $r^2 \le t/\alpha \le T/\alpha$, using Corollary 2 we see that

$$-L\phi(\cdot,t) \ge -\frac{\sqrt{\alpha}}{t} C(\sqrt{\alpha} + \sqrt{KT})$$

holds in the sense of distributions, with C constant depending only on n (and on the choice of ψ). Proceeding as in [V], p. 266, we find

$$\int_{M^n} \phi(y,t)h(x,y,t)w(y)\,dV(y) - \int_{M^n} \phi(y,t)h(x,y,\epsilon)w(y)\,dV(y) \ge -C\sqrt{\alpha}(\sqrt{\alpha} + \sqrt{KT}),$$

Since, by definition of h, the second integral converges to $\phi(x,t) = 1$ as $\epsilon \downarrow 0$, one can choose ϵ and $\alpha = \alpha(K,T)$ small enough so that

$$\int_{M^n} \phi(y,t)h(x,y,t)w(y)\,dV(y) \ge 1/2, \quad x \in M^n, \; 0 < t < T,$$

whence, recalling the definition of ϕ, (11) follows with $a = \alpha^{-1/2}$. As for the second statement it suffices to observe that if $K = 0$, then α and therefore a can be chosen independently of T so that (11) holds for $0 < t < \infty$.

LEMMA 10. *Assume that* $R_w \ge -nK$, $K \ge 0$. *Then for every* $T > 0$ *there exists a constant* $C_1 = C_1(n,K,T)$ *such that*

$$h(x,x,t) \ge C_1 V_w(x,\sqrt{t})^{-1}, \quad x \in M^n, \; 0 < t < T. \tag{12}$$

If $K = 0$, C_1 *depends only upon n and (12) holds for* $0 < t < \infty$.

PROOF ([D1], LEMMA 5.6.2). Given $T > 0$ let a be such that (11) holds. The parabolic Harnack inequality with $\alpha = 2$ yields

$$h(x,y,t/2) \le h(x,x,t)2^{n+1} \exp\Big\{\frac{d^2(x,y)}{t} + \frac{n(n+1)}{4}KT\Big\},$$

so that integrating over $B(x, a\sqrt{t/2})$ gives

$$1/2 \le h(x,x,t)V_w(x, a\sqrt{t/2})2^{n+1} \exp\Big\{a^2 + \frac{n(n+1)}{4}KT\Big\}.$$

By the relative w-volume estimate of Lemma 1,

$$V_w(x, a\sqrt{t/2}) \le bV_w(x,\sqrt{t}), \quad 0 < t < T,$$

where

$$b = b(a,T) = \begin{cases} 1 & \text{if } a/\sqrt{2} \le 1 \\ \sup_{0<t\le T} \frac{V_K(a\sqrt{t/2})}{V_K(\sqrt{t})} & \text{if } a/\sqrt{2} > 1, \end{cases}$$

and (12) follows with $C_1 = 2^{-n-2}b^{-1}\exp\{-a^2 - n(n+1)KT/4\}$.

If $K = 0$, then $a = a(n)$ is independent of T and $V_0(a\sqrt{t/2})/V_0(\sqrt{t}) = (a/\sqrt{2})^{n+1}$, so that b is also independent of T. It is then clear that (12) holds for $0 < t < \infty$ with $C_1 = 2^{-n-2}\exp\{-a^2\}b^{-1}$.

THEOREM 11. *Assume that $R_w \ge -nK$, $K \ge 0$. Then for every $T > 0$ and $0 < \epsilon < 1$ there exists a constant $C_2 = C_2(n, K, T, \epsilon)$ such that*

$$h(x,y,t) \ge C_2 V_w(x,\sqrt{t})^{-1/2} V_w(y,\sqrt{t})^{-1/2} \exp\left\{-\frac{d^2(x,y)}{4(1-\epsilon)t}\right\}, \tag{13}$$

$\forall x, y \in M^n$ and $0 < t < T$. Moreover if $K = 0$ then C_2 does not depend on T and (13) holds for $0 < t < \infty$.

PROOF. One argues as in Davies [D1], Theorem 5.6.3. Let $C_1 = C_1(n, K, T)$ be such that (12) holds. Given $0 < \epsilon < 1$, the parabolic Harnack inequality with $\alpha = (1-\epsilon/2)/(1-\epsilon)$ gives

$$h(x,x,\epsilon t/2) \le h(x,y,t)(2/\epsilon)^{n+1} \exp\left\{\frac{d^2(x,y)}{4(1-\epsilon)t} + \frac{n(n+1)}{2\epsilon}KT\right\},$$

for $x, y \in M^n$ and $0 < t < T$. By the Lemma above

$$h(x,y,t) \ge C_1(\epsilon/2)^{n+1} \exp\left\{-\frac{n(n+1)}{2\epsilon}KT\right\} V_w(x,\sqrt{\epsilon t/2})^{-1} \exp\left\{-\frac{d^2(x,y)}{4(1-\epsilon)t}\right\}.$$

By Lemma 3,

$$V_w(x,\sqrt{\epsilon t/2}) \ge b_2 V_w(x,\sqrt{t})$$

with $b_2 = b_2(\epsilon, T) = \inf_{0\le t\le T} V_K(\sqrt{\epsilon t/2})/V_K(\sqrt{t})$. Substituting this into the inequality above, and using the symmetry of $h(x,y,t)$, (13) follows with

$$C_2 = C_1(\epsilon/2)^{n+1} b_2 \exp\left\{-\frac{n(n+1)}{2\epsilon}KT\right\}.$$

From this expression it is clear that if $K = 0$, C_2 does not depend on T and consequently (13) holds for $0 < t < \infty$.

7. **An example.** Let $(M^n, g) = (\mathbf{R}, \text{can})$, and consider the weight function $w = e^{-x^2}$, so that

$$L = -\frac{d^2}{dx^2} - 2x\frac{d}{dx}.$$

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The spectrum of L is given by $\{2k\}_{k=0}^{\infty}$ and the eigenfunction belonging to $2k$ is the Hermite polynomial H_k. Using the generating function of the product of Hermite polynomials (*cf.* [Le], p. 61) one shows that the heat kernel of L is given by Mehler's formula:

$$h(x,y,t) = \{\pi(1-e^{-4t})\}^{-1/2}\exp\Big\{2xy\frac{e^{-2t}}{1+e^{-2t}} - (x-y)^2\frac{e^{-4t}}{1-e^{-4t}}\Big\},$$

for $x, y \in \mathbf{R}$, $t > 0$. It is then easy to see that neither a gradient estimate of the form

$$(14) \qquad \frac{|\nabla_x h(x,y,t)|^2}{h^2(x,y,t)} - \alpha\frac{h_t(x,y,t)}{h(x,y,t)} \le C_0\Big(\frac{1}{t}+1\Big), \quad x,y \in \mathbf{R},\ t>0,\ \alpha>0,$$

nor a parabolic Harnack inequality

$$(15) \quad h(x,y,t) \le h(x,z,t+s)\Big(\frac{t+s}{t}\Big)^C \exp\Big\{C_1\frac{(y-z)^2}{s} + C_2 s\Big\}, \quad x,y,z \in \mathbf{R},\ t,s>0,$$

hold for h. Indeed the left hand side of (14) equals

$$4\Big\{\frac{e^{-4t}}{1-e^{-4t}}(x-y) - \frac{e^{-2t}}{1+e^{-2t}}y\Big\}^2 \\ -\alpha\Big\{4(x-y)^2\frac{e^{-4t}}{(1-e^{-4t})^2} - 4xy\frac{e^{-2t}}{(1+e^{-2t})^2} - 2\frac{e^{-4t}}{1-e^{-4t}}\Big\}.$$

For $x = y$ this reduces to

$$4x^2\frac{e^{-2t}}{(1+e^{-2t})^2}\{e^{-2t}+\alpha\} + 2\alpha\frac{e^{-4t}}{1-e^{-4t}},$$

which, for t fixed, is not bounded independently of x. As for a parabolic Harnack inequality, assuming $x = y$ and $0 < z \le x$, then

$$\frac{h(x,x,t)}{h(x,z,t+s)} \ge \Big\{\frac{1-e^{-4(t+s)}}{1-e^{-4t}}\Big\}^{-1/2} \exp\{(x-z)^2\frac{e^{-4(t+s)}}{1-e^{-4(t+s)}} + 2x\frac{e^{-2t}}{1+e^{-2t}}(x-z).$$

If we further assume that $c_0^{-1} \le x - z \le c_0$ with $c_0 > 0$, then

$$\frac{h(x,x,t)}{h(x,z,t+s)} \ge c_1 e^{c_2 x},\ c_1 = c_1(t,s),\ c_2 = c_2(t) > 0,$$

which is unbounded as $x \longrightarrow \infty$.

Since in this case $R_w = -w''/w = -2(2x^2-1)$ is not bounded from below, we see that some assumption on R_w is necessary for the kind of bounds obtained here.

In this connection observe that Bakry's tensor $R = \mathrm{Ric} - \mathrm{Hess}(\log w)$, (*cf.* [Bk1], [Bk2], [BE]) in this case is identically equal to 2, showing that a control on R does not imply the results described here. From this point of view R_w rather than R seems to be a more useful generalization of the Ricci tensor. On the other hand, it was shown in [DS] that $R \ge k > 0$ implies that the spectral gap of L is bounded below by k, and the example above shows that this bound is sharp.

8. α**-Dimensional measures.** In this section we define various notions of α-dimensionality which generalise Strichartz's locally uniform α-dimensionality, and use the results of the previous sections to extend some of his results relating α-dimensionality of μ to L^p bounds for $e^{-tL}\mu$. The notation is unchanged and we continue to assume that $R_w \geq -nK$ $(K \geq 0)$.

A Borel measure μ on M^n is locally w-uniformly α-dimensional $(0 \leq \alpha \leq n)$ if there exists a constant C_0 such that

$$\sup_{0<r\leq 1} r^{n-\alpha} \left\| \frac{\mu(B(x,r))}{\mathrm{vol}_w(B(x,r))} \right\|_\infty \leq C_0. \tag{17}$$

Note that if $w \equiv 1$ and M^n has bounded geometry (17) reduces to Strichartz's α-dimensionality condition

$$\mu\big(B(x,r)\big) \leq Cr^\alpha, \quad 0 < r \leq 1.$$

PROPOSITION 12. *The measure μ is locally w-uniformly α-dimensional if and only if*

$$\int_{M^n} h(x,y,t)\,d\mu(y) \leq Ct^{(\alpha-n)/2}, \quad 0 < t \leq 1, \tag{18}$$

where the constant C depends only on n, K, and on the constant C_0 in (17).

PROOF. The proof follows that of [St3], Theorem 2.4, almost verbatim: Given $0 < t \leq 1$, let $\{M_j\}$ be a paving of size $\sqrt{t}$, defined (*cf.* [St3], §2) as a disjoint decomposition of M^n into Borel sets satisfying $B(x_j,\sqrt{t}) \subset M_j \subset B(x_j, 2\sqrt{t})$. If μ satisfies (17)

$$\int_{M_j} h(x,y,t)\,d\mu(y) \leq \mathrm{vol}_w B(x_j, 2\sqrt{t})(2\sqrt{t})^{\alpha-n} \sup_{z\in M_j} h(x,z,t).$$

Integrating the parabolic Harnack over M_j yields

$$\sup_{z\in M_j} h(x,z,t) \leq c(n,K)\,\mathrm{vol}_w B(x_j,\sqrt{t})^{-1} \int_{M_j} h(x,y,t)\,d\mu(y).$$

Substituting this above, using the relative w-volume estimate, and summing over j, (18) follows with

$$C = 2^{\alpha-n} C_0 c(n,K) \sup_{x,t} \frac{\mathrm{vol}_w B(x,2\sqrt{t})}{\mathrm{vol}_w B(x,\sqrt{t})}.$$

Conversely, by the proof of Theorem 11

$$\inf_{z\in B(x,\sqrt{t})} h(x,z,t) \geq C_1 \,\mathrm{vol}_w B(x,\sqrt{t})^{-1},$$

with C_1 independent of x and $0 < t \leq 1$. Hence

$$\int_{B(x,\sqrt{t})} h(x,y,t)\,d\mu(y) \geq C\,\mathrm{vol}_w B(x,\sqrt{t})^{-1}\mu(B(x,\sqrt{t})), \tag{19}$$

and (18) implies (17)

As in [St3], Theorem 3.1, interpolating between $p = 1$ and $p = \infty$, we obtain

THEOREM 13. *Let μ be a locally w-uniformly α-dimensional measure. Then there is a constant C which depends on n, K, and on the constant C_0 in (17), such that $\forall 1 \leq p \leq \infty$, and $\forall f \in L^p(d\mu)$*

$$\sup_{0<t\leq 1} t^{(n-\alpha)/2p'} \|e^{-tL}(f\,d\mu)\|_{L^p(w\,dV)} \leq C\|f\|_{L^p(d\mu)}, \quad p^{-1}+p'^{-1}=1. \tag{20}$$

Here $e^{-tL}(f\mu)$ is defined by

$$\int_{M^n} h(x,y,t)f(y)\,d\mu(y)$$

if $f \in L^p(d\mu)\cap L^\infty(d\mu)$ and extended to all of $L^p(d\mu)$ by density.

We can define different notions of α-dimensionality by using L^p norms instead of sup norms in (17): We shall say that a locally finite (complex) measure ν is L^p weakly α-dimensional if

$$\sup_{0<r\leq 1} r^{(n-\alpha)/p'} \left\| \frac{|\nu|(B(x,r))}{\mathrm{vol}_w(B(x,r))} \right\|_{L^p(w\,dV)} \leq C_0, \quad p^{-1}+p'^{-1}=1. \tag{21}$$

(21) is related to a condition considered by Lau ([La]) in the Euclidean setting. For $p=2$ it is a generalisation of Strichartz's condition of weak α-dimensionality ([St3], §5). The following lemma shows that (21) is essentially equivalent to a discrete condition.

LEMMA 14. *For every $1 \leq p < \infty$, there is a constant $C = C(n,K,p)$ such that for every $0 < r \leq 1$ and for every paving $\{M_j\}$ of size r,*

$$\frac{1}{C}\left\| \frac{|\nu|(B(x,r))}{\mathrm{vol}_w(B(x,r))} \right\| \leq \left\{ \sum_j \frac{|\nu|(M_j)^p}{\mathrm{vol}_w(M_j)^{(p-1)}} \right\}^{1/p} \leq C\left\| \frac{|\nu|(B(x,4r))}{\mathrm{vol}_w(B(x,4r))} \right\|,$$

where the norms are taken in $L^p(w\,dV)$.

PROOF. Let $\{M_j\}$ be a paving of size r. By elementary geometry $x \in M_j \subset B(x_j,2r)$ implies $M_j \subset B(x,4r)$. By the relative w-volume estimate

$$\frac{\mathrm{vol}_w(B(x,4r))}{\mathrm{vol}_w(M_j)} \leq \frac{\mathrm{vol}_w(B(x_j,6r))}{\mathrm{vol}_w(B(x_j,r)))} \leq C_1, \quad 0<r\leq 1,$$

and therefore

$$\begin{aligned}\frac{|\nu|(M_j)^p}{\mathrm{vol}_w(M_j)^{p-1}} &\leq \int_{M_j} \frac{|\nu|(B(x,4r))^p}{\mathrm{vol}_w(M_j)^p} w(x)\,dV(x)\\ &\leq C_1 \int_{M_j} \frac{|\nu|(B(x,4r))^p}{\mathrm{vol}_w(B(x,4r)))^p} w(x)\,dV(x),\end{aligned}$$

whence the second inequality in the statement follows summing over j. To prove the first inequality, let again M_j be a paving of size r. If $x \in M_j$,

$$\frac{1}{\mathrm{vol}_w(B(x,r))} \leq \frac{\mathrm{vol}_w(B(x,4r))}{\mathrm{vol}_w(B(x,r))}\frac{1}{\mathrm{vol}_w M_j} \leq C_2\frac{1}{\mathrm{vol}_w M_j},$$

and

$$|\nu|\big(B(x,r)\big) \le \sum\{|\nu|(M_k) : M_k \cap B(x_j, 3r) \ne \emptyset\}.$$

Since $M_k \cap B(x_j, 3r) \ne \emptyset$ implies $d(x_j, x_k) \le 5r$, we have

$$\frac{\mathrm{vol}_w(M_k)}{\mathrm{vol}_w(M_j)} \le \frac{\mathrm{vol}_w(B(x_j, 7r))}{\mathrm{vol}_w(B(x_j, r))} \le C_3, \text{ and } \frac{\mathrm{vol}_w(B(x_j, r))}{\mathrm{vol}_w(B(x_k, r))} \le C_3.$$

This and the fact that the M_k's intersecting $B(x_j, 3r)$ are contained in $B(x_j, 7r)$ yield

$$\mathrm{card}\{k : M_k \cap B(x_j, 3r) \ne \emptyset\} \le C_3 \frac{\mathrm{vol}_w(B(x_j, 7r))}{\mathrm{vol}_w(B(x_j, r))} \le C_3^2.$$

Therefore we conclude that

$$\int_{M_j} \frac{|\nu|(B(x,r))^p}{\mathrm{vol}_w(B(x,r)))^p} w(x)\, dV(x) \le C_4 \sum \left\{ \frac{|\nu|(M_k)^p}{\mathrm{vol}_w(M_k)^{(p-1)}} : M_k \cap B(x_j, 3r) \ne \emptyset \right\}.$$

Summing over j and arguing as before to estimate the number of $B(x_j, 3r)$ that intersect a given M_k we conclude that the first inequality of the lemma holds.

As a corollary of Lemma 14 it is easy to see that if μ is a locally w-uniformly α-dimensional measure and $f \in L^p(d\mu)$, $1 \le p \le \infty$ then the measure $\nu = f\, d\mu$ is L^p weakly α-dimensional: Indeed given a paving $\{M_j\}$ of size r we use Hölder inequality and $\mu(M_j) \le \mathrm{vol}_w(M_j) r^{\alpha-n}$to estimate $|\nu|(M_j)^p$. Summing over j we conclude that

$$\left\{ \sum_j \frac{|\nu|(M_j)^p}{\mathrm{vol}_w(M_j)^{(p-1)}} \right\}^{1/p} \le C \|f\|_{L^p} r^{(\alpha-n)/p'}, \quad p^{-1} + p'^{-1} = 1,$$

with C depending on μ only through the constant of locally w-uniform α-dimensionality.

THEOREM 15. *Let ν be a locally finite (complex) measure. If ν is L^p weakly α-dimensional, $1 \le p \le \infty$, then there is a constant C_1 that depends on ν only through the constant C_0 in (21) such that*

$$\sup_{0<t\le 1} t^{(n-\alpha)/2p'} \|e^{-tL}\nu\|_{L^p(w\,dV)} \le C_1. \tag{22}$$

The converse holds if ν is a positive measure.

PROOF. We only need consider $p < \infty$. Assuming first that ν is L^p weakly α-dimensional we proceed as in [St3], Theorem 5.2: Given $0 < t \le 1/8$, let $\{M_j\}$ be a paving of size $r = \sqrt{t}$. By Lemma 14

$$\left\{ \sum_j \frac{|\nu|(M_j)^p}{\mathrm{vol}_w(M_j)^{(p-1)}} \right\}^{1/p} \le C_0' r^{(\alpha-n)/p'}.$$

Define a measure μ_r by

$$\mu_r(A) = \frac{|\nu|(A)}{|\nu|(M_j)} \mathrm{vol}_w(M_j), \text{ if } A \subset M_j,$$

so that $\nu = f_r \mu_r$ with $\|f_r\|_{L^p(d\mu_r)} = C_0' r^{(\alpha-n)/p'}$. Arguing as in the proof of Proposition 12, one shows that there exists a constant $C = C(n, K)$ such that

$$\int_{M^n} h(x, y, t)\, d\mu_r(y) \leq C,$$

whence, interpolating between L^1 and L^∞,

$$\|e^{-tL}(f\, d\mu_r)\|_{L^p(w\,dV)} \leq C^{1/p'} \|f\|_{L^p(d\mu_r)}.$$

Taking $f = f_r$ we conclude that (21) holds with $C_1 = C_0' C^{1/p'}$ for $0 < t \leq 1/8$, and, since e^{-tL} is a contraction semigroup on $L^p(w\,dV)$, for $0 < t \leq 1$. Conversely, if ν is a positive measure, again as in the proof of Proposition 12 we have

$$e^{-tL}\nu \geq \int_{B(x,\sqrt{t})} h(x, y, t)\, d\nu(y) \geq C_2 \frac{\nu(B(x, \sqrt{t}))}{\mathrm{vol}_w(x, \sqrt{t})}, \quad 0 < t \leq 1,$$

and, by taking L^p norms (22) implies that ν is L^p weakly α-dimensional.

Remarks and Further Results. Referring back to Theorem 13, we note that, for locally uniformly α-dimensional measures on a manifold M^n with bounded geometry and for $p = 2$, Strichartz ([St3], Corollary 3.7) has proven an estimate analogous to (20) but with $\sup_{0<t\leq 1}$ replaced by $\limsup_{t\downarrow 0}$:

$$\limsup_{t\downarrow 0} t^{(n-\alpha)/4} \|e^{t\Delta}(f\, d\mu)\|_{L^2(dV)} \leq C \int_{M^n} |f|^2 \phi\, d\mu_\alpha, \tag{23}$$

where μ_α is α-dimensional Hausdorff measure on M^n and $\phi \in L^1_{\mathrm{loc}}(d\mu_\alpha)$ is the function that appears in the decomposition $\mu = \phi\, d\mu_\alpha + \nu$ proven by Strichartz ([St2], Theorem 3.1) as a generalisation of the Radon-Nykodim theorem for non σ-finite measure. Strichartz also proves an extension of Wiener's Theorem for 0-dimensional measures ([St3], Theorem 3.2).

The corresponding results for locally w-uniformly α-dimensional measures do not seem to hold only under the assumption that R_w is bounded from below, mainly because this does not give enough control on $\mathrm{vol}_w\big(B(x, r)\big)$. If we are willing to impose additional conditions on M^n, namely that it is of bounded geometry and that $|\nabla(\log w)|$ is bounded above, then

$$\mathrm{vol}_w\big(B(x, r)\big) \asymp w(x) r^n, \quad 0 \leq r \leq 1,$$

and Strichartz's method of proof can be applied to show that the obvious generalisation of (23) to locally w-uniformly α-dimensional measures holds. Moreover by a direct application of (1.2) in Kannai ([Ka]), or by adapting the argument in McKean Singer ([MkS]), pp. 44–46, one verifies that, under minimal assumptions,

$$\lim_{t\downarrow 0} t^{n/2} h(x, y, t) = \begin{cases} 0 & \text{if } x \neq y \\ (4\pi)^{-n/2} w(x)^{-1} & \text{if } x = y \end{cases}$$

and, with the additional hypotheses on M^n imposed above, the proof of Theorem (3.2) in [St3] can be carried through to show that if μ is locally w-uniformly α-dimensional and

$$\mu = \sum c_j\delta_j + \mu_c$$

is its decomposition in discrete and continuous parts, then $\forall f \in L^2(d\mu)$

$$\lim_{t\downarrow 0} t^{n/2}\|e^{-tL}(d\mu)\|^2_{L^2(w\,dV)} = (8\pi)^{-n/2}\sum_j |f(a_j)|^2 c_j^2 w(a_j)^{-1},$$

where the right hand side is bounded by const $\cdot \|f\|^2_{L^2(d\mu)}$, and therefore finite.

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