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FAMILIES OF AFFINE RULED SURFACES: EXISTENCE OF CYLINDERS

ADRIEN DUBOULOZ and TAKASHI KISHIMOTO

Abstract. We show that the generic fiber of a family $f:X\to S$ of smooth $\mathbb{A}^1$-ruled affine surfaces always carries an $\mathbb{A}^1$-fibration, possibly after a finite extension of the base S. In the particular case where the general fibers of the family are irrational surfaces, we establish that up to shrinking S, such a family actually factors through an $\mathbb{A}^1$-fibration $\rho:X\to Y$ over a certain S-scheme $Y\to S$ induced by the MRC-fibration of a relative smooth projective model of X over S. For affine threefolds X equipped with a fibration $f:X\to B$ by irrational $\mathbb{A}^1$-ruled surfaces over a smooth curve B, the induced $\mathbb{A}^1$-fibration $\rho:X\to Y$ can also be recovered from a relative minimal model program applied to a smooth projective model of X over B.

Introduction

The general structure of smooth noncomplete surfaces X with negative (logarithmic) Kodaira dimension is not fully understood yet. For say smooth quasi-projective surfaces over an algebraically closed field of characteristic zero, it was established by Keel and McKernan [10] that the negativity of the Kodaira dimension is equivalent to the fact that X is generically covered by images of the affine line $\mathbb{A}^1$ in the sense that the set of points $x\in X$ with the property that there exists a nonconstant morphism $f:\mathbb{A}^1\to X$ such that $x\in f(\mathbb{A}^1)$ is dense in X with respect to the Zariski topology. This property, called $\mathbb{A}^1$-uniruledness is equivalent to the existence of an open embedding $X\hookrightarrow(\overline{X},B)$ into a complete variety $\overline{X}$ covered by proper rational curves meeting the boundary $B=\overline{X}\setminus X$ in at most one point. In the case where X is smooth and affine, an earlier deep result of Miyanishi–Sugie and Fujita [14] asserts the stronger property that X is $\mathbb{A}^1$-ruled: there

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exists a Zariski dense open subset $U \subset X$ of the form $U \simeq Z \times \mathbb{A}^1$ for a suitable smooth curve Z. Equivalently, X admits a surjective flat morphism $\rho : X \to C$ to an open subset C of a smooth projective model $\overline{Z}$ of Z, whose generic fiber is isomorphic to the affine line over the function field of C. Such a morphism $\rho : X \to C$ is called an $\mathbb{A}^1$-fibration, and ρ is said to be of affine type or complete type when the base curve C is affine or complete, respectively.

Smooth $\mathbb{A}^1$-uniruled but not $\mathbb{A}^1$-ruled affine varieties are known to exist in every dimension $\geqslant 3$ [1]. Many examples of $\mathbb{A}^1$-uniruled affine threefolds can be constructed in the form of flat families $f : X \to B$ of smooth $\mathbb{A}^1$-ruled affine surfaces parametrized by a smooth base curve B. For instance, the complement X of a smooth cubic surface $S \subset \mathbb{P}^3_{\mathbb{C}}$ is the total space of a family $f : X \to \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[t])$ of $\mathbb{A}^1$-ruled surfaces induced by the restriction of a pencil $\overline{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ on $\mathbb{P}^3$ generated by S and three times a tangent hyperplane H to S whose intersection with S consists of a cuspidal cubic curve. The general fibers of f have negative Kodaira dimension, carrying $\mathbb{A}^1$-fibrations of complete type only, and the failure of $\mathbb{A}^1$-ruledness is intimately related to the fact that the generic fiber X_η of f, which is a surface defined over the field $K = \mathbb{C}(t)$, does not admit any $\mathbb{A}^1$-fibration defined over $\mathbb{C}(t)$. Nevertheless, it was noticed in [3, Theorem 6.1] that one can infer straight from the construction of $f : X \to \mathbb{A}^1$ the existence of a finite base extension $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ for which the surface $X_\eta \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ carries an $\mathbb{A}^1$-fibration $\rho : X_\eta \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \to \mathbb{P}^1_L$ defined over the field L.

A natural question is then to decide whether this phenomenon holds in general for families $f : X \to B$ of $\mathbb{A}^1$-ruled affine surfaces parameterized by a smooth base curve B, namely, does the existence of $\mathbb{A}^1$-fibrations on the general fibers of f imply the existence of one on the generic fiber of f, possibly after a finite extension of the base B? A partial positive answer is given by Gurjar *et al.* [3, Theorem 3.8] under the additional assumption that the general fibers of f carry $\mathbb{A}^1$-fibrations of affine type. The main result in Gurjar *et al.* [3, Theorem 3.8] is derived from the study of log-deformations of suitable relative normal projective models $\overline{f} : (\overline{X}, D) \to B$ of X over B with appropriate boundaries D. It is established in particular that the structure of the boundary divisor of a well-chosen smooth projective completion of a general closed fiber X_s is stable under small deformations, a property which implies in turn, possibly after a finite extension of the base B, the existence of an $\mathbb{A}^1$-fibration of affine type on the generic fiber of f. This log-deformation theoretic approach is also

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central in the related recent work of Flenner *et al.* [2] on the classification of normal affine surfaces with $\mathbb{A}^1$-fibrations of affine type up to a certain notion of deformation equivalence, defined for families which admit suitable relative projective models satisfying Kamawata's axioms of logarithmic deformations of pairs [8]. The fact that the $\mathbb{A}^1$-fibrations under consideration are of affine type plays again a crucial role and, in contrast with the situation considered in [3], the restrictions imposed on the families imply the existence of $\mathbb{A}^1$-fibrations of affine type on their generic fibers.

Our main result (Theorem 7) consists of a generalization of the results in [3] to families $f : X \to S$ of $\mathbb{A}^1$-ruled surfaces over an arbitrary normal base S, which also includes the case where a general closed fiber X_s of f admits $\mathbb{A}^1$-fibrations of complete type only. In particular, we obtain the following positive answer to [3, Conjecture 6.2]:

THEOREM. *Let $f : X \to S$ be a dominant morphism between normal complex algebraic varieties whose general fibers are smooth $\mathbb{A}^1$-ruled affine surfaces. Then there exist a dense open subset $S_* \subset S$, a finite étale morphism $T \to S_*$ and a normal T-scheme $h : Y \to T$ such that the induced morphism $f_T = \mathrm{pr}_T : X_T = X \times_{S_*} T \to T$ factors as*

$$f_T = h \circ \rho : X_T \xrightarrow{\rho} Y \xrightarrow{h} T,$$

where $\rho : X_T \to Y$ is an $\mathbb{A}^1$-fibration.

In contrast with the log-deformation theoretic strategy used in [3], which involves the study of certain Hilbert schemes of rational curves on well-chosen relative normal projective models $\overline{f} : (\overline{X}, B) \to S$ of X over S, our approach is more elementary, based on the notion of Kodaira dimension [7] adapted to the case of geometrically connected varieties defined over arbitrary base fields of characteristic zero. Indeed, the hypothesis means equivalently that the general fibers of f have negative Kodaira dimension. This property is in turn inherited by the generic fiber of f, which is a smooth affine surface defined over the function field of S, thanks to a standard Lefschetz principle argument. Then we are left with checking that a smooth affine surface X defined over an arbitrary base field k of characteristic zero and with negative Kodaira dimension admits an $\mathbb{A}^1$-fibration, possibly after a suitable finite base extension $\mathrm{Spec}(k_0) \to \mathrm{Spec}(k)$, a fact which ultimately follows from finite type hypotheses and the aforementioned characterization of Miyanishi and Sugie [14].

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The article is organized as follows. The first section contains a review of the structure of smooth affine surfaces of negative Kodaira dimension over arbitrary base fields k of characteristic zero. We show in particular that every such surface X admits an $\mathbb{A}^1$-fibration after a finite extension of the base field k, and we give criteria for the existence of $\mathbb{A}^1$-fibrations defined over k. These results are then applied in the second section to the study of deformations $f : X \to S$ of smooth $\mathbb{A}^1$-ruled affine surfaces: after giving the proof of the main result, Theorem 7, we consider in more detail the particular situation where the general fibers of $f : X \to S$ are irrational. In this case, after shrinking S if necessary, we show that the morphism f actually factors through an $\mathbb{A}^1$-fibration $\rho : X \to Y$ over an S-scheme $h : Y \to S$ which coincides, up to birational equivalence, with the maximally rationally connected quotient of a relative smooth projective model $\overline{f} : \overline{X} \to S$ of X over S. The last section is devoted to the case of affine threefolds equipped with a fibration $f : X \to B$ by irrational $\mathbb{A}^1$-ruled surfaces over a smooth base curve B: we explain in particular how to construct an $\mathbb{A}^1$-fibration $\rho : X \to Y$ factoring f by means of a relative minimal model program applied to a smooth projective model $\overline{f} : \overline{X} \to B$ of X over B.

§1. $\mathbb{A}^1$-ruledness of affine surfaces over nonclosed field

In what follows, the term k-variety refers to a geometrically integral scheme of finite type over a base field k of characteristic zero. A k-variety X is said to be *k-rational* if it is birationally isomorphic over k to the projective space $\mathbb{P}^n_k$, where $n = \dim_k X$. When no particular base field is indicated, we use simply the term *rational* to refer to a geometrically rational variety. We call a variety *irrational* if it is not rational in the previous sense.

1.1 Logarithmic Kodaira dimension

1.1.1. Let X be a smooth algebraic variety defined over a field k of characteristic zero. By virtue of Nagata compactification [15] and Hironaka desingularization [5] theorems, there exists an open immersion $X \hookrightarrow (\overline{X}, B)$ into a smooth complete algebraic variety $\overline{X}$ with reduced SNC boundary divisor $B = \overline{X} \setminus X$. The (logarithmic) Kodaira dimension $\kappa(X)$ of X is then defined as the Iitaka dimension [6] of the pair $(\overline{X}; \omega_{\overline{X}}(\log B))$, where $\omega_{\overline{X}}(\log B) = (\det \Omega^1_{\overline{X}/k}) \otimes \mathcal{O}_{\overline{X}}(B)$. So letting

$$\mathcal{R}(\overline{X}, B) = \bigoplus_{m \geqslant 0} H^0(\overline{X}, \omega_{\overline{X}}(\log B)^{\otimes m}),$$

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we have $\kappa(X) = \mathrm{tr.deg}_k\, \mathcal{R}(\overline{X}, B) - 1$ if $H^0(\overline{X}, \omega_{\overline{X}}(\log B)^{\otimes m}) \neq 0$ for sufficiently large m. Otherwise, if $H^0(\overline{X}, \omega_{\overline{X}}(\log B)^{\otimes m}) = 0$ for every $m \geqslant 1$, we set by convention $\kappa(X) = -\infty$ and we say that $\kappa(X)$ is negative. The so-defined element $\kappa(X) \in \{-\infty\} \cup \{0, \ldots, \dim_k X\}$ is independent of the choice of a smooth complete model $(\overline{X}, B)$ [7].

Furthermore, the Kodaira dimension of X is invariant under arbitrary extensions of the base field k. Indeed, given an extension $k \subset k'$, the pair $(\overline{X}_{k'}, B_{k'})$ obtained by the base change $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is a smooth complete model of $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ with reduced SNC boundary $B_{k'}$. Furthermore letting $\pi : \overline{X}_{k'} \to \overline{X}$ be the corresponding faithfully flat morphism, we have $\omega_{\overline{X}_{k'}}(\log B_{k'}) \simeq \pi^*\omega_X(\log B)$ and so $\mathcal{R}(X_{k'}) \simeq \mathcal{R}(X) \otimes_k k'$ by the flat base change theorem [4, Proposition III.9.3]. Thus $\kappa(X) = \kappa(X_{k'})$.

Example 1. The affine line $\mathbb{A}^1_k$ is the only smooth geometrically connected noncomplete curve C with negative Kodaira dimension. Indeed, let $\overline{C}$ be a smooth projective model of C and let $\overline{C}_{\overline{k}}$ be the curve obtained by the base change to an algebraic closure $\overline{k}$ of k. Since C is noncomplete, $B = \overline{C}_{\overline{k}} \setminus C_{\overline{k}}$ consists of a finite collection of closed points $p_1, \ldots, p_s$, $s \geqslant 1$, on which the Galois group $\mathrm{Gal}(\overline{k}/k)$ acts by k-automorphisms of $\overline{C}_{\overline{k}}$. Clearly, $H^0(\overline{C}_{\overline{k}}, \omega_{\bar{C}_{\overline{k}}}(\log B)^{\otimes m}) \neq 0$ unless $\overline{C}_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}}$ and $s = 1$. Since p_1 is then necessarily $\mathrm{Gal}(\overline{k}/k)$-invariant, $\overline{C} \setminus C$ consists of unique k-rational point, showing that $\overline{C} \simeq \mathbb{P}^1_k$ and $C \simeq \mathbb{A}^1_k$.

1.2 Smooth affine surfaces with negative Kodaira dimension

Recall that by virtue of [14], a smooth affine surface X defined over an algebraically closed field of characteristic zero has negative Kodaira dimension if and only if it is $\mathbb{A}^1$-ruled: there exists a Zariski dense open subset $U \subset X$ of the form $U \simeq Z \times \mathbb{A}^1$ for a suitable smooth curve Z. In fact, the projection $\mathrm{pr}_Z : U \simeq Z \times \mathbb{A}^1 \to Z$ always extends to an $\mathbb{A}^1$-fibration $\rho : X \to C$ over an open subset C of a smooth projective model $\overline{Z}$ of Z. This characterization admits the following straightforward generalization to arbitrary base fields of characteristic zero:

Theorem 2. *Let X be a smooth geometrically connected affine surface defined over a field k of characteristic zero. Then the following are equivalent:*

(a) *The Kodaira dimension $\kappa(X)$ of X is negative.*

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(b) *For some finite extension k_0 of k, the surface X_{k_0} contains an open subset $U \simeq Z \times \mathbb{A}^1_{k_0}$ for some smooth curve Z defined over k_0.*
(c) *There exist a finite extension k_0 of k and an $\mathbb{A}^1$-fibration $\rho : X_{k_0} \to C_0$ over a smooth curve C_0 defined over k_0.*

Proof. Clearly (c) implies (b) and (b) implies (a). To show that (a) implies (c), we observe that letting $\overline{k}$ be an algebraic closure of k, we have $\kappa(X_{\overline{k}}) = \kappa(X) < 0$. It then follows from the aforementioned result of Miyanishi and Sugie [14] that $X_{\overline{k}}$ admits an $\mathbb{A}^1$-fibration $q : X_{\overline{k}} \to C$ over a smooth curve C, with smooth projective model $\overline{C}$. Since $X_{\overline{k}}$ and $\overline{C}$ are of finite type over $\overline{k}$, there exists a finite extension $k \subset k_0$ such that $q : X_{\overline{k}} \to \overline{C}$ is obtained from a morphism $\rho : X_{k_0} \to \overline{C}_0$ to a smooth projective curve $\overline{C}_0$ defined over k_0 by the base extension $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k_0)$. By virtue of Example 1, $\rho : X_{k_0} \to \overline{C}_0$ is an $\mathbb{A}^1$-fibration. □

Examples of smooth affine surfaces X of negative Kodaira dimension without any $\mathbb{A}^1$-fibration defined over the base field but admitting $\mathbb{A}^1$-fibrations of complete type after a finite base extension were already constructed in [1]. The following example illustrates the fact that a similar phenomenon occurs for $\mathbb{A}^1$-fibrations of affine type, providing in particular a negative answer to [3, Problem 3.13].

EXAMPLE 3. Let $B \subset \mathbb{P}^2_k = \mathrm{Proj}(k[x, y, z])$ be a smooth conic without k-rational point defined by a quadratic form $q = x^2 + ay^2 + bz^2$, where $a, b \in k^*$, and let $\overline{X} \subset \mathbb{P}^3_k = \mathrm{Proj}(k[x, y, z, t])$ be the smooth quadric surface defined by the equation $q(x, y, z) - t^2 = 0$. The complement $X \subset \overline{X}$ of the hyperplane section $\{t = 0\}$ is a k-rational smooth affine surface with $\kappa(X) < 0$, which does not admit any $\mathbb{A}^1$-fibration $\rho : X \to C$ over a smooth, affine or projective curve C. Indeed, if such a fibration existed then a smooth projective model of C would be isomorphic to $\mathbb{P}^1_k$; since the fiber of ρ over a general k-rational point of C is isomorphic to $\mathbb{A}^1_k$, its closure in $\overline{X}$ would intersect the boundary $\overline{X} \setminus X \simeq B$ in a unique point, necessarily k-rational, in contradiction with the choice of B.

In contrast, for a suitable finite extension $k \subset k'$, the surface $X_{k'}$ becomes isomorphic to the complement of the diagonal in $\overline{X}_{k'} \simeq \mathbb{P}^1_{k'} \times \mathbb{P}^1_{k'}$ and hence, it admits at least two distinct $\mathbb{A}^1$-fibrations over $\mathbb{P}^1_{k'}$, induced by the restriction of the first and second projections from $\overline{X}_{k'}$. Furthermore, since $X_{k'}$ is isomorphic to the smooth affine quadric in $\mathbb{A}^3_{k'} = \mathrm{Spec}(k'[u, v, w])$ with equation $uv - w^2 = 1$, it also admits two distinct $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_{k'}$, induced by the restrictions of the projections pr_u and pr_v.

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1.3 Existence of $\mathbb{A}^1$-fibrations defined over the base field

1.3.1. The previous example illustrates the general fact that if X is a smooth geometrically connected affine surface with $\kappa(X) < 0$ which does not admit any $\mathbb{A}^1$-fibration, then there exists a finite extension k' of k such that $X_{k'}$ admits at least two $\mathbb{A}^1$-fibrations of the same type, either affine or complete, with distinct general fibers. Indeed, by virtue of Theorem 2, there exists a finite extension k_0 of k such that X_{k_0} admits an $\mathbb{A}^1$-fibration $\rho : X_{k_0} \to C$. Let k' be the Galois closure of k_0 in an algebraic closure of k and let $\rho_{k'} : X_{k'} \to C_{k'}$ be the $\mathbb{A}^1$-fibration deduced from ρ. If $\rho_{k'} : X_{k'} \to C_{k'}$ is globally invariant under the action of the Galois group $\mathrm{Gal}(k'/k)$ on $X_{k'}$, in the sense that for every $\Phi \in \mathrm{Gal}(k'/k)$ considered as a Galois automorphism of $X_{k'}$ there exists a commutative diagram

$$\begin{array}{ccc} X_{k'} & \xrightarrow{\Phi} & X_{k'} \\ {\scriptstyle \rho_{k'}}\downarrow & & \downarrow{\scriptstyle \rho_{k'}} \\ C_{k'} & \xrightarrow{\varphi} & C_{k'} \end{array}$$

for a certain k'-automorphism φ of $C_{k'}$, then we would obtain a Galois action of $\mathrm{Gal}(k'/k)$ on $C_{k'}$ for which $\rho_{k'} : X_{k'} \to C_{k'}$ becomes an equivariant morphism. Since $C_{k'}$ is quasi-projective and ρ'_k is affine, it would follow from Galois descent that there exist a curve $\tilde{C}$ defined over k and a morphism $q : X \to \tilde{C}$ defined over k such that $\rho_{k'} : X_{k'} \to C_{k'}$ is obtained from q by the base change $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Since by virtue of Example 1 the affine line does not have any nontrivial form, the generic fiber of q would be isomorphic to the affine line over the field of rational functions of $\tilde{C}$ and so, $q : X \to \tilde{C}$ would be an $\mathbb{A}^1$-fibration defined over k, in contradiction with our hypothesis. So there exists at least an element $\Phi \in \mathrm{Gal}(k'/k)$ considered as a k-automorphism of $X_{k'}$ such that the $\mathbb{A}^1$-fibrations $\rho_{k'} : X_{k'} \to C_{k'}$ and $\rho_{k'} \circ \varphi : X_{k'} \to C_{k'}$ have distinct general fibers.

Arguing backward, we obtain the following criterion:

PROPOSITION 4. *Let X be a smooth geometrically connected affine surface with $\kappa(X) < 0$. If there exists a finite Galois extension k' of k such that $X_{k'}$ admits a unique $\mathbb{A}^1$-fibration $\rho' : X_{k'} \to C_{k'}$ up to composition by automorphisms of $C_{k'}$, then $\rho' : X_{k'} \to C_{k'}$ is obtained by base extension from an $\mathbb{A}^1$-fibration $\rho : X \to C$ defined over k.*

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COROLLARY 5. *A smooth geometrically connected irrational affine surface X has negative Kodaira dimension if and only if it admits an $\mathbb{A}^1$-fibration $\rho : X \to C$ over a smooth irrational curve C defined over the base field k. Furthermore for every extension k' of k, $\rho_{k'} : X_{k'} \to C_{k'}$ is the unique $\mathbb{A}^1$-fibration on $X_{k'}$ up to composition by automorphisms of $C_{k'}$.*

Proof. Uniqueness is clear since otherwise $C_{k'}$ would be dominated by a general fiber of another $\mathbb{A}^1$-fibration on $X_{k'}$, and hence would be rational, implying in turn the rationality of X. By virtue of Theorem 2, there exist a finite Galois extension k' of k and an $\mathbb{A}^1$-fibration $\rho' : X_{k'} \to C'$ over a smooth curve C'. The latter is irrational as X is irrational, which implies that $\rho' : X_{k'} \to C'$ is the unique $\mathbb{A}^1$-fibration on $X_{k'}$. So ρ' descend to an $\mathbb{A}^1$-fibration $\rho : X \to C$ over a smooth irrational curve C defined over k. □

The following example shows that the irrationality hypothesis cannot be weakened to the property that X is geometrically rational but not k-rational.

EXAMPLE 6. Let $a \in \mathbb{Q}$ be a rational number which is not a cube and let $S = S_a \subset \mathbb{P}^3_{\mathbb{Q}} = \mathrm{Proj}_{\mathbb{Q}}(\mathbb{Q}[x, y, z, t])$ be the smooth cubic surface defined by the equation $x^3 + y^3 + z^3 + at^3 = 0$. All lines on S are defined over the splitting field K of the polynomial $u^3 + a \in \mathbb{Q}[u]$, and one checks by direct computation that no orbit of the action of the Galois group $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathfrak{S}_3$ on S_K consists of a disjoint union of such lines. It follows that the Picard number $\rho(S)$ of S is equal to 1, hence by Segree–Manin Theorem that S is rational but not $\mathbb{Q}$-rational (see e.g., [12, Exercise 2.18 and Theorem 2.1]). Let $H = \{x + y = 0\} \subset \mathbb{P}^3_{\mathbb{Q}}$ be the tangent hyperplane to S at the point $p = [1 : -1 : 0 : 0]$ and let $X = S \setminus (H \cap S)$. So X is a smooth affine surface defined over $\mathbb{Q}$, and since the intersection of $H_{\mathbb{C}}$ with $S_{\mathbb{C}}$ consists of three lines meeting at the Eckardt point p, one checks easily that $\kappa(X) = \kappa(X_{\mathbb{C}}) = -\infty$. Thus $X_{\mathbb{C}}$ admits an $\mathbb{A}^1$-fibration by virtue of [14], but we claim that X does not admit any such fibration defined over $\mathbb{Q}$. Indeed, suppose on the contrary that $\pi : X \to C$ is an $\mathbb{A}^1$-fibration over a smooth curve defined over $\mathbb{Q}$. Since C is geometrically rational and contains a $\mathbb{Q}$-rational point, for instance the image by π of the point $[0 : -1 : 1 : 0] \in X(\mathbb{Q})$, it is $\mathbb{Q}$-rational. But then X whence S would be $\mathbb{Q}$-rational, a contradiction.

§2. Families of $\mathbb{A}^1$-ruled affine surfaces

2.1 Existence of étale $\mathbb{A}^1$-cylinders

This subsection is devoted to the proof of the following:

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THEOREM 7. *Let X and S be normal algebraic varieties defined over a field k of infinite transcendence degree over $\mathbb{Q}$, and let $f : X \to S$ be a dominant affine morphism with the property that for a general closed point $s \in S$, the fiber X_s is a smooth geometrically connected affine surface with negative Kodaira dimension. Then there exist an open subset $S_* \subset S$, a finite étale morphism $T \to S_*$ and a normal T-scheme $h : Y \to T$ such that $f_T = \mathrm{pr}_T : X_T = X \times_{S_*} T \to T$ factors as*

$$f_T = h \circ \rho : X_T \xrightarrow{\rho} Y \xrightarrow{h} T$$

where $\rho : X_T \to Y$ is an $\mathbb{A}^1$-fibration.

Proof. Shrinking S if necessary, we may assume that S is affine, that $f : X \to S$ is smooth and that $\kappa(X_s) < 0$ for every closed point $s \in S$. It is enough to show that the fiber X_η of f over the generic point η of S is geometrically connected, with negative Kodaira dimension. Indeed, if so, then by Theorem 2 above, there exist a finite extension L of $K = \mathrm{Frac}(\Gamma(S, \mathcal{O}_S))$ and an $\mathbb{A}^1$-fibration $\rho : X_\eta \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \to C$ onto a smooth curve C defined over L. Letting T be the normalization of S in L and shrinking T again if necessary, we obtain a finite étale morphism $T \to S$ such that the generic fiber of $\mathrm{pr}_T : X_T \to T$ is isomorphic to the $\mathbb{A}^1$-fibered surface $\rho : X_\eta \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \to C$ and then the assertion follows from Lemma 8 below.

The properties of being geometrically connectedness and having negative Kodaira dimension are invariant under finite algebraic extensions of the base field. So letting $\overline{k}$ be an algebraic closure of k, it is enough to show that the generic fiber of the induced morphism $f_{\overline{k}} : X_{\overline{k}} \to S_{\overline{k}}$ is geometrically connected, of negative Kodaira dimension. We may thus assume from now on that k is algebraically closed. Since X and S are affine and of finite type over k, there exist a subfield k_0 of k of finite transcendence degree over $\mathbb{Q}$, and a smooth morphism $f_0 : X_0 \to S_0$ of k_0-varieties such that $f : X \to S$ is obtained from $f_0 : X_0 \to S_0$ by the base extension $\mathrm{Spec}\,(k) \to \mathrm{Spec}(k_0)$. The field $K_0 = \mathrm{Frac}(\Gamma(S_0, \mathcal{O}_{S_0}))$ has finite transcendence degree over $\mathbb{Q}$ and hence, it admits a k_0-embedding $\xi : K_0 \hookrightarrow k$. Letting $(X_0)_{\eta_0}$ be the fiber of f_0 over the generic point $\eta_0 : \mathrm{Spec}(K_0) \to S_0$ of S_0, the composition $\Gamma(S_0, \mathcal{O}_{S_0}) \hookrightarrow K_0 \hookrightarrow k$ induces a k-homomorphism $\Gamma(S_0, \mathcal{O}_{S_0}) \otimes_{k_0} k \to k$ defining a closed point $s : \mathrm{Spec}(k) \to \mathrm{Spec}(\Gamma(S_0, \mathcal{O}_{S_0}) \otimes_{k_0} k) = S$ of S for which we obtain the following commutative diagram

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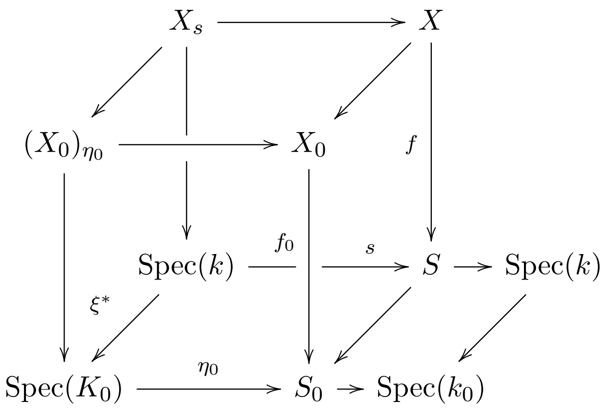

The bottom square of the cube being Cartesian by construction, we deduce that

$$(X_0)_{\eta_0} \times_{\mathrm{Spec}(K_0)} \mathrm{Spec}(k) \simeq X_0 \times_{S_0} \mathrm{Spec}(k) \simeq X \times_S \mathrm{Spec}(k) = X_s.$$

Since by assumption, X_s is geometrically connected with $\kappa(X_s) < 0$, we conclude that $(X_0)_{\eta_0}$ is geometrically connected and has negative Kodaira dimension. This implies in turn that X_η is geometrically connected and that $\kappa(X_\eta) < 0$ as desired. ∎

In the proof of the above theorem, we used the following lemma:

Lemma 8. *Let $f : X \to S$ be a dominant affine morphism between normal varieties defined over a field k of characteristic zero. Then the following are equivalent:*

(a) *The generic fiber X_η of f admits an $\mathbb{A}^1$-fibration $q : X_\eta \to C$ over a smooth curve C defined over the fraction field K of S.*
(b) *There exist an open subset S_* of S and a normal S_*-scheme $h : Y \to S_*$ of relative dimension 1 such that the restriction of f to $V = f^{-1}(S_*)$ factors as $f|_V = h \circ \rho : V \to Y \to S_*$ where $\rho : V \to Y$ is an $\mathbb{A}^1$-fibration.*

Proof. If (b) holds then letting L be the fraction field of Y, we have a commutative diagram

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$$\begin{array}{ccccc}
V_\xi = X_\xi & \longrightarrow & V_\eta = X_\eta & \longrightarrow & V \\
\Big\downarrow{\scriptstyle \rho_\xi} & & \Big\downarrow{\scriptstyle \rho_\eta} & & \Big\downarrow{\scriptstyle \rho} \\
\mathrm{Spec}(L) & \xrightarrow{\ \xi\ } & C = Y_\eta & \longrightarrow & Y \\
 & & \Big\downarrow{\scriptstyle h_\eta} & & \Big\downarrow{\scriptstyle h} \\
 & & \mathrm{Spec}(K) & \xrightarrow{\ \eta\ } & S_*
\end{array}$$

in which each square is Cartesian. It follows that $h_\eta : C \to \mathrm{Spec}(K)$ is a normal whence smooth curve defined over K and that $\rho_\eta : X_\eta \to C$ is an $\mathbb{A}^1$-fibration. Conversely, suppose that X_η admits an $\mathbb{A}^1$-fibration $q : X_\eta \to C$ and let $\overline{C}$ be a smooth projective model of C over K. Then there exist an open subset S_0 of S and a projective S_0-scheme $h : Y \to S_0$ whose generic fiber is isomorphic to $\overline{C}$. After shrinking S_0 if necessary, the rational map $\rho : V \dashrightarrow Y$ of S_0-schemes induced by q becomes a morphism and we obtain a factorization $f\,|_V = h \circ \rho$. By construction, the generic fiber V_ξ of $\rho : V \to Y$ is isomorphic to

$$V \times_Y \mathrm{Spec}(L) \simeq (V \times_Y C) \times_C \mathrm{Spec}(L) \simeq X_\eta \times_C \mathrm{Spec}(L) \simeq \mathbb{A}^1_L$$

since $V \times_Y C \simeq V_\eta \simeq X_\eta$ and $\rho : X_\eta \to C \hookrightarrow \overline{C}$ is an $\mathbb{A}^1$-fibration. So $\rho : V \to Y$ is an $\mathbb{A}^1$-fibration. ∎

EXAMPLE 9. Let $R = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$, $S = \mathrm{Spec}(R)$ and let D be the relatively ample divisor in $\mathbb{P}^2_S = \mathrm{Proj}_R(R[x, y, z])$ defined by the equation $x^2 + sy^2 + tz^2 = 0$. The restriction $h : X = \mathbb{P}^2_S \setminus D \to S$ of the structure morphism defines a family of smooth affine surfaces with the property that for every closed point $s \in S$, X_s is isomorphic to the complement in $\mathbb{P}^2_{\mathbb{C}}$ of the smooth conic D_s. In particular, X_s admits a continuum of pairwise distinct $\mathbb{A}^1$-fibrations $X_s \to \mathbb{A}^1_{\mathbb{C}}$, induced by the restrictions to X_s of the rational pencils on $\mathbb{P}^2_{\mathbb{C}}$ generated by D_s and twice its tangent line at an arbitrary closed point $p_s \in D_s$. On the other hand, the fiber of D over the generic point η of S is a conic without $\mathbb{C}(s, t)$-rational point in $\mathbb{P}^2_{\mathbb{C}(s,t)}$ and hence, we conclude by a similar argument as in Example 3 that X_η does not admit any $\mathbb{A}^1$-fibration defined over $\mathbb{C}(s, t)$. Therefore there is no open subset S_* of S over which h can be factored through an $\mathbb{A}^1$-fibration.

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2.2 Deformations of irrational $\mathbb{A}^1$-ruled affine surfaces

In this subsection, we consider the particular situation of a flat family $f: X \to S$ over a normal variety S whose general fibers are irrational $\mathbb{A}^1$-ruled affine surfaces. A combination of Corollary 5 and Theorem 7 above implies that if $f: X \to S$ is smooth and defined over a field of infinite transcendence degree over $\mathbb{Q}$, then the generic fiber X_η of f is $\mathbb{A}^1$-ruled. Equivalently, there exist an open subset $S_* \subset S$ and a normal S_*-scheme $h: Y \to S_*$ such that the restriction of f to $X_* = X \times_S S_*$ factors through an $\mathbb{A}^1$-fibration $\rho: X_* \to Y$ (see Lemma 8). The restriction of ρ to the fiber of f over a general closed point $s \in S_0$ is an $\mathbb{A}^1$-fibration $\rho_s: X_s \to Y_s$ over the normal, whence smooth, curve Y_s. Since X_s is irrational, Y_s is irrational, and so $\rho_s: X_s \to Y_s$ is the unique $\mathbb{A}^1$-fibration on X_s up to composition by automorphisms of Y_s. So in this case, we can identify $\rho_s: X_s \to Y_s$ with the Maximally Rationally Connected fibration (MRC-fibration) $\varphi: \overline{X}_s \dashrightarrow Y_s$ of a smooth projective model $\overline{X}_s$ of X_s in the sense of [11, IV.5]: recall that φ is unique, characterized by the property that its general fibers are rationally connected and that for a very general point $y \in Y_s$ any rational curve in $\overline{X}_s$ which meets $\overline{X}_y$ is actually contained in $\overline{X}_y$. The $\mathbb{A}^1$-fibration $\rho: X_* \to Y$ can therefore be re-interpreted as being the MRC-fibration of a relative smooth projective model $\overline{X}$ of X over S.

Reversing the argument, general existence and uniqueness results for MRC-fibrations allow actually to get rid of the smoothness hypothesis of a general fiber of $f: X \to S$ and to extend the conclusion of Theorem 7 to arbitrary base fields of characteristic zero. Namely, we obtain the following characterization:

THEOREM 10. *Let X and S be normal varieties defined over a field k of characteristic zero and let $f: X \to S$ be a dominant affine morphism with the property that for a general closed point $s \in S$, the fiber X_s is an irrational $\mathbb{A}^1$-ruled surface. Then there exist a dense open subset S_* of S and a normal S_*-scheme $h: Y \to S_*$ such that the restriction of f to $X_* = X \times_S S_*$ factors as*

$$f|_{X_*} = h \circ \rho: X_* \xrightarrow{\rho} Y \xrightarrow{h} S_*$$

where $\rho: X_* \to Y$ is an $\mathbb{A}^1$-fibration.

Proof. Shrinking S if necessary, we may assume that it is smooth and that for every closed point $s \in S$, X_s is irrational and $\mathbb{A}^1$-ruled, hence carrying a unique $\mathbb{A}^1$-fibration $\pi_s: X_s \to C_s$ over an irrational normal curve C_s. Since $f: X \to S$ is affine, there exist a normal projective S-scheme

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$\overline{X} \to S$ and an open embedding $X \hookrightarrow \overline{X}$ of schemes over S. Letting $W \to \overline{X}$ be a resolution of the singularities of $\overline{X}$, hence in particular of those of X, we may assume up to shrinking S again if necessary that $W \to S$ is a smooth morphism. We let $j : X \dashrightarrow W$ be the birational map of S-schemes induced by the embedding $X \hookrightarrow \overline{X}$. By virtue of [11, Theorem 5.9], there exist an open subset W' of W, an S-scheme $h : Z \to S$ and a proper morphism $\overline{q} : W' \to Z$ such that for every $s \in S$, the induced rational map $\overline{q}_s : W_s \dashrightarrow Z_s$ is the MRC-fibration for W_s. On the other hand, since W_s is a smooth projective model of X_s, the induced rational map $\pi_s : \overline{X}_s \dashrightarrow C_s$ is the MRC-fibration for W_s. Consequently, for a general closed point $z \in Z$ with $h(z) = s$, the fiber W_z of $\overline{q}_s$, which is an irreducible proper rational curve contained in W_s, must coincide with the closure of the image by j of a general closed fiber of π_s. The latter being isomorphic to the affine line $\mathbb{A}^1_\kappa$ over the residue field κ of the corresponding point of C_s, we conclude that there exists an affine open subset U of X on which the composition $\overline{q} \circ j : U \to Z$ is a well-defined morphism with general closed fibers isomorphic to affine lines over the corresponding residue fields. So $\overline{q} \circ j : U \to Z$ is an $\mathbb{A}^1$-fibration by virtue of [9]. The generic fiber of $f : X \to S$ is thus $\mathbb{A}^1$-ruled and the assertion follows from Lemma 8 above. □

EXAMPLE 11. Let $h : Y \to S$ be a smooth family of complex projective curves of genus $g \geqslant 2$ over a normal affine base S and let $\mathcal{T}_{Y/S}$ be the relative tangent sheaf of h. Since by Riemman–Roch $H^0(Y_s, \mathcal{T}_{Y/S,s}) = 0$ and $\dim H^1(Y_s, \mathcal{T}_{Y/S,s}) = 3g - 3$ for every point $s \in S$, $h_*\mathcal{T}_{Y/S,s} = 0$, $R^1h_*\mathcal{T}_{Y/S}$ is locally free of rank $3g - 3$ [4, Corollary III.12.9] and so, $H^1(Y, \mathcal{T}_{Y/S}) \simeq H^0(S, R^1h_*\mathcal{T}_{Y/S})$ by the Leray spectral sequence. Replacing S by an open subset, we may assume that $R^1h_*\mathcal{T}_{Y/S}$ admits a nowhere vanishing global section σ. Via the isomorphism $H^1(Y, \mathcal{T}_{Y/S}) \simeq \mathrm{Ext}^1_Y(\mathcal{O}_Y, \mathcal{T}_{Y/S})$, we may interpret this section as the class of a nontrivial extension $0 \to \mathcal{T}_{Y/S} \to \mathcal{E} \to \mathcal{O}_Y \to 0$ of locally free sheaves over Y. The inclusion $\mathcal{T}_{Y/S} \to \mathcal{E}$ defines a section D of the locally trivial $\mathbb{P}^1$-bundle $\overline{\rho} : \overline{X} = \mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_Y} \mathcal{E}^\vee) \to Y$ and the nonvanishing of σ guarantees that D is the support of an S-ample divisor. Indeed the S-ampleness of D is equivalent to the property that for every $s \in S$ the induced section D_s of the $\mathbb{P}^1$ -bundle $\overline{\rho}_s : \overline{X}_s \to Y_s$ over the smooth projective curve Y_s is ample. Since by construction, $\overline{\rho}_s\mid_{\overline{X}_s \setminus D_s} : \overline{X}_s \setminus D_s \to Y_s$ is a nontrivial torsor under the line bundle $\mathrm{Spec}(\mathrm{Sym}\mathcal{T}^\vee_{Y_s}) \to Y_s$, it follows that D_s intersects positively every section D of $\overline{\rho}_s$ except maybe

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D_s itself. On the other hand, we have $(D_s^2) = -\deg \mathcal{T}_{Y_s} = 2g(Y_s) - 2 > 0$, and so the ampleness of D_s follows from the Nakai–Moishezon criterion and the description of the cone effective cycles on an irrational projective ruled surface given in [4, Propositions 2.20–2.21].

Letting $X = \overline{X} \setminus D$, we obtain a smooth family

$$f = g \circ \overline{\rho}\,|_X : X \xrightarrow{\overline{\rho}|_X} Y \xrightarrow{h} S$$

where $\overline{\rho}\,|_X : X \to Y$ is a nontrivial, locally trivial, $\mathbb{A}^1$-bundle such that for every $s \in S$, X_s is an affine surface with an $\mathbb{A}^1$-fibration $\rho_s : X_s \to Y_s$ of complete type.

In contrast with the previous example, the following proposition shows in particular that if the total space of a family of irrational $\mathbb{A}^1$-ruled affine surfaces $f : X \to S$ has finite divisor class group, then the induced $\mathbb{A}^1$-fibration on a general fiber of $f : X \to S$ is necessarily of affine type.

PROPOSITION 12. *Let X be a geometrically integral normal affine variety with finite divisor class group* $\mathrm{Cl}(X)$ *and let $f : X \to S$ be a dominant affine morphism to a normal variety S with the property that for a general closed point $s \in S$, the fiber X_s is an irrational $\mathbb{A}^1$-ruled surface, say with unique $\mathbb{A}^1$-fibration $\pi_s : X_s \to C_s$. Then there exists an effective action of the additive group scheme $\mathbb{G}_{a,S}$ on X such that for a general closed point $s \in S$, the $\mathbb{A}^1$-fibration $\pi_s : X_s \to C_s$ factors through the algebraic quotient $\rho_s : X_s \to X_s /\!/ \mathbb{G}_{a,s} = \mathrm{Spec}(\Gamma(X_s, \mathcal{O}_{X_s})^{\mathbb{G}_{a,s}})$.*

Proof. Let $f\,|_{X_*} = h \circ \rho : X_* \xrightarrow{\rho} Y \xrightarrow{h} S_*$ be as in Theorem 10. Since ρ is an $\mathbb{A}^1$-fibration, there exists an affine open subset $U \subset Y$ such that $\rho^{-1}(U) \simeq U \times \mathbb{A}^1$ as schemes over U. Since $\rho^{-1}(U)$ is affine, its complement in X is of pure codimension 1, and the finiteness of $\mathrm{Cl}(X)$ implies that it is actually the support of an effective principal divisor $\mathrm{div}_X(a)$ for some $a \in \Gamma(X, \mathcal{O}_X)$. Let ∂_0 be the locally nilpotent derivation of $\Gamma(\rho^{-1}(U), \mathcal{O}_X) \simeq \Gamma(X, \mathcal{O}_X)_a$ corresponding to the $\mathbb{G}_{a,U}$-action by translations on the second factor. Since a is invertible in $\Gamma(\rho^{-1}(U), \mathcal{O}_X)$, it belongs to the kernel of ∂_0, and the finite generation of $\Gamma(X, \mathcal{O}_X)$ guarantees that for a suitably chosen $n \geqslant 0$, $a^n \partial_0$ is a locally nilpotent derivation ∂ of $\Gamma(X, \mathcal{O}_X)$. By construction, the restriction of f to the dense open subset $\rho^{-1}(U)$ of X is invariant under the corresponding $\mathbb{G}_a$-action, and so $f : X \to S$ is $\mathbb{G}_a$-invariant. For a general closed point $s \in S$, the induced $\mathbb{G}_a$-action on X_s is nontrivial, and its algebraic quotient $\rho_s : X_s \to X_s /\!/ \mathbb{G}_a = \mathrm{Spec}(\Gamma(X_s, \mathcal{O}_{X_s})^{\mathbb{G}_a})$ is a surjective

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$\mathbb{A}^1$-fibration onto a normal affine curve $X_s//\mathbb{G}_a$. Since C_s is irrational, the general fibers of ρ_s and π_s must coincide. It follows that π_s is $\mathbb{G}_a$-invariant, whence factors through ρ_s. □

§3. Affine threefolds fibered in irrational $\mathbb{A}^1$-ruled surfaces

In this section, we consider in more detail the case of normal complex affine threefolds X admitting a fibration $f: X \to B$ by irrational $\mathbb{A}^1$-ruled surfaces, over a smooth curve B. We explain how to derive the variety $h: Y \to B$ for which f factors through an $\mathbb{A}^1$-fibration $\rho: X \to Y$ from a relative minimal model program applied to a suitable projective model of X over B. In the case where the divisor class group of X is finite, we provide a complete classification of such fibrations in terms of additive group actions on X.

3.1 $\mathbb{A}^1$-cylinders via relative minimal model program

Let X be a normal complex affine threefold and let $f: X \to B$ be a flat morphism onto a smooth curve B with the property that a general closed fiber X_b of f is an irreducible irrational $\mathbb{A}^1$-ruled surface. We let $\overline{f}: W \to B$ be a smooth projective model of X over B obtained from an arbitrary normal relative projective completion $X \hookrightarrow \overline{X}$ of X over B by resolving the singularities. We let $j: X \dashrightarrow W$ be the birational map induced by the open immersion $X \hookrightarrow \overline{X}$.

By applying a minimal model program for W over B, we obtain a sequence of birational B-maps

$$W = W_0 \overset{\varphi_1}{\dashrightarrow} W_1 \overset{\varphi_2}{\dashrightarrow} W_2 \dashrightarrow \cdots \dashrightarrow W_{\ell-1} \overset{\varphi_\ell}{\dashrightarrow} W_\ell = W',$$

between B-schemes $\overline{f}_i: W_i \to B$, where $\varphi_i: W_i \dashrightarrow W_{i+1}$ is either a divisorial contraction or a flip, and the rightmost variety W' is the output of a minimal model program over B. The hypotheses imply that W' has the structure of a Mori conic bundle $\overline{\rho}: W' \to Y$ over a projective B-scheme $h: Y \to B$ corresponding to the contraction of an extremal ray of $\overline{\mathrm{NE}}(W'/B)$. Indeed, a general fiber of $\overline{f}$ being a birationally ruled projective surface, the output W' is not a minimal model of W over B. So W' is either a Mori conic bundle over a B-scheme Y of dimension 2 or a del Pezzo fibration over B, the second case being excluded by the fact that the general fibers of $\overline{f}$ are irrational.

Proposition 13. *The induced map $\rho = \overline{\rho}\,|_X: X \dashrightarrow Y$ is a rational $\mathbb{A}^1$-fibration.*

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Proof. Since a general closed fiber X_b is a normal affine surface with an $\mathbb{A}^1$-fibration $\pi_b : X_b \to C_b$ over a certain irrational smooth curve C_b, it follows that there exists a unique maximal affine open subset U_b of C_b such that $\pi_b^{-1}(U_b) \simeq U_b \times \mathbb{A}^1$ and such that the rational map $j_b : \pi_b^{-1}(U_b) \dashrightarrow W_b$ induced by j is regular, inducing an isomorphism between $\pi_b^{-1}(U_b)$ and its image. Each step $\varphi_i : W_i \dashrightarrow W_{i+1}$ consists of either a flip whose flipping and flipped curves are contained in fibers of $\overline{f}_i : W_i \to B$ and $\overline{f}_{i+1} : W_{i+1} \to B$ respectively, or a divisorial contraction whose exceptional divisor is contained in a fiber of $\overline{f}_i : W_i \to B$, or a divisorial contraction whose exceptional divisor intersects a general fiber of $\overline{f}_i : W_i \to B$. Clearly, a general closed fiber of $\overline{f}_i : W_i \to B$ is not affected by the first two types of birational maps. On the other hand, if $\varphi_i : W_i \to W_{i+1}$ is the contraction of a divisor $E_i \subset W_i$ which dominates B, then a general fiber of $\varphi_i \mid_{E_i}$ is a smooth proper rational curve. The intersection of E_i with a general closed fiber $W_{i,b}$ of $\overline{f}_i$ thus consists of proper rational curves, and its intersection with the image of the maximal affine cylinder like open subset $\pi_b^{-1}(U_b)$ of X_b is either empty or composed of affine rational curves. Since U_b is an irrational curve, it follows that each irreducible component of $E_i \cap (\pi_b^{-1}(U_b))$ is contained in a fiber of π_b. This implies that there exists an open subset $U_{b,0}$ of U_b with the property that for every $i = 1, \ldots, \ell$, the restriction of $\varphi_i \circ \cdots \circ \varphi_1 \circ j$ to $\pi_b^{-1}(U_{b,0}) \subset X_b$ is an isomorphism onto its image in $W_{i,b}$. A general fiber of $\overline{\rho} : W' \to Y$ over a closed point $y \in Y$ being a smooth proper rational curve, its intersection with $\pi_{h(y)}^{-1}(U_{h(y),0})$ viewed as an open subset of $W'_{h(y)}$, is thus either empty or equal to a fiber of $\pi_{h(y)}$. So by virtue of [9], there exists an open subset V of X on which $\overline{\rho}$ restricts to an $\mathbb{A}^1$-fibration $\overline{\rho} \mid_V : V \to Y$. ∎

COROLLARY 14. *Let X be a normal complex affine threefold X equipped with a morphism $f : X \to B$ onto a smooth curve B whose general closed fibers are irrational $\mathbb{A}^1$-ruled surfaces. Then X is birationally equivalent to the product of $\mathbb{P}^1$ with a family $h_0 : \mathcal{C}_0 \to B_0$ of smooth projective curves of genus $g \geqslant 1$ over an open subset $B_0 \subset B$.*

Proof. By the previous Proposition, X has the structure of a rational $\mathbb{A}^1$-fibration $\rho : X \dashrightarrow Y$ over a 2-dimensional normal proper B-scheme $h : Y \to B$. In particular, X is birational to $Y \times \mathbb{P}^1$. On the other hand, for a general closed point $b \in B$, the curve Y_b is birational to the base C_b of the unique $\mathbb{A}^1$-fibration $\pi_b : X_b \to C_b$ on the irrational affine surface X_b. Letting $\sigma : \tilde{Y} \to Y$ be a desingularization of Y, there exists an open subset B_0 of

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B over which the composition $h \circ \sigma : \tilde{Y} \to Y$ restricts to a smooth family $h_0 : \mathcal{C}_0 \to B_0$ of projective curves of a certain genus $g \geqslant 1$. By construction, X is birational to $\mathcal{C}_0 \times \mathbb{P}^1$. □

REMARK 15. Example 11 above shows conversely that for every smooth family $h : \mathcal{C} \to B$ of projective curves of genus $g \geqslant 2$, there exists a smooth $\mathbb{A}^1$-ruled affine threefold X birationally equivalent to $\mathcal{C} \times \mathbb{P}^1$. Actually, in the setting of the previous Corollary 14, if we assume further that a general fiber of $f : X \to B$ carries an $\mathbb{A}^1$-fibration $\pi_b : X_b \to C_b$ over a smooth curve C_b whose smooth projective model has genus $g \geqslant 2$, then there exists a uniquely determined family $h : \mathcal{C} \to B$ of proper stable curves of genus g such that X is birationally equivalent to $\mathcal{C} \times \mathbb{P}^1$: indeed, the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geqslant 2$ being proper and separated, the smooth family $h_0 : \mathcal{C}_0 \to B_0$ extends in a unique way to a family $h : \mathcal{C} \to B$ of stable curves of genus g.

3.2 Factorial affine threefolds

PROPOSITION 16. *Let X be a normal affine threefold with finite divisor class group $\mathrm{Cl}(X)$ and let $f : X \to B$ be a morphism onto a smooth curve B whose general closed fibers are irrational $\mathbb{A}^1$-ruled surfaces. Then there exists a factorization $f = h \circ \rho : X \to Y \to B$ where $\rho : X \to Y$ is the algebraic quotient morphism of an effective $\mathbb{G}_{a,B}$-action on X. In particular, a general fiber of f admits an $\mathbb{A}^1$-fibration of affine type.*

Proof. By virtue of Proposition 12, there exists an effective $\mathbb{G}_{a,B}$-action on X such that for a general closed point $b \in B$, the $\mathbb{A}^1$-fibration $\pi_b : X_b \to C_b$ on X_b factors through the algebraic quotient

$$\rho_b : X_b \to X_b /\!/ \mathbb{G}_{a,b} = \mathrm{Spec}(\Gamma(X_b, \mathcal{O}_{X_b})^{\mathbb{G}_{a,b}}).$$

Since X is a threefold, the ring of invariants $\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_{a,B}}$ is finitely generated [16]. The quotient morphism $\rho : X \to Y = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_{a,B}})$ is an $\mathbb{A}^1$-fibration, and since Y is a categorical quotient in the category of algebraic varieties, the invariant morphism $f : X \to B$ factors through ρ. □

COROLLARY 17. *Let $f : \mathbb{A}^3 \to B$ be a morphism onto a smooth curve B with irrational $\mathbb{A}^1$-ruled general fibers. Then B is isomorphic to either $\mathbb{P}^1$ or $\mathbb{A}^1$ and there exists a factorization $f = h \circ \rho : \mathbb{A}^3 \to \mathbb{A}^2 \to B$, where $\rho : \mathbb{A}^3 \to \mathbb{A}^2$ is the quotient morphism of an effective $\mathbb{G}_{a,B}$-action on $\mathbb{A}^3$.*

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Proof. Since B is dominated by a general line in $\mathbb{A}^3$, it is necessarily isomorphic to $\mathbb{P}^1$ or $\mathbb{A}^1$. The second assertion follows from Proposition 16 and the fact that the algebraic quotient of every nontrivial $\mathbb{G}_a$-action on $\mathbb{A}^3$ is isomorphic to $\mathbb{A}^2$ [13]. □

EXAMPLE 18. In Corollary 17 above, the base curve B need not be affine. For instance, the morphism

$$f : \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z]) \longrightarrow \mathbb{P}^1, (x, y, z) \mapsto [(xz - y^2)x^2 + 1 : (xz - y^2)^3]$$

defines a family whose general member is isomorphic to the product $C_\lambda \times \mathbb{A}^1$ where $C_\lambda \subset \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[xz - y^2, x])$ is the affine elliptic curve with equation $(xz - y^2)^3 + \lambda((xz - y^2)x^2 + 1) = 0$. The subring $\mathbb{C}[xz - y^2, x]$ of $\mathbb{C}[x, y, z]$ coincides with the ring of invariants of the $\mathbb{G}_a$-action associated with the locally nilpotent $\mathbb{C}[x]$-derivation $x\partial_y + 2y\partial_z$ and f is the composition of the quotient morphism $\rho : \mathbb{A}^3 \to \mathbb{A}^2 = \mathbb{A}^3//\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[u, v])$ defined by $(x, y, z) \mapsto (xz - y^2, x)$ and of the morphism $h : \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u, v]) \to \mathbb{P}^1$ defined by $(u, v) \mapsto [uv^2 + 1 : u^3]$.

Corollary 17 above implies in particular that if a general fiber of a regular function $f : \mathbb{A}^3 \to \mathbb{A}^1$ is irrational and admits an $\mathbb{A}^1$-fibration, then the latter is necessarily of affine type. In contrast, regular functions $f : \mathbb{A}^3 \to \mathbb{A}^1$ whose general fibers are rational and equipped with $\mathbb{A}^1$-fibrations of complete type only do exist, as illustrated by the following example.

EXAMPLE 19. Let $f = x^3 - y^3 + z(z + 1) \in \mathbb{C}[x, y, z]$ and let $f : \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z]) \to \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[\lambda])$ be the corresponding morphism. The closure $\overline{S}_\lambda$ in $\mathbb{P}^3 = \mathrm{Proj}(\mathbb{C}[x, y, z, t])$ of a general fiber $S_\lambda = f^*(\lambda)$ of f is a smooth cubic surface which intersects the hyperplane $H_\infty = \{t = 0\}$ along the union B_λ of three lines meeting at the Eckardt point $p = [0 : 0 : 1 : 0]$. Thus S_λ is rational and a direct computation reveals that $\kappa(S_\lambda) = -\infty$. So by virtue of [14], S_λ admits an $\mathbb{A}^1$-fibration $\pi_\lambda : S_\lambda \to C_\lambda$ over a smooth rational curve C_λ. If C_λ was affine, then there would exist a nontrivial $\mathbb{G}_a$-action on S_λ having the general fibers of π_λ as general orbits. But it is straightforward to check that every automorphism of S_λ considered as a birational self-map of $\overline{S}_\lambda$ is in fact a biregular automorphism of $\overline{S}_\lambda$ preserving the boundary B_λ. So the automorphism group of S_λ injects into the group $\mathrm{Aut}(\overline{S}_\lambda, B_\lambda)$ of automorphisms of the pair $(\overline{S}_\lambda, B_\lambda)$. The latter being a finite group, we conclude that no such $\mathbb{G}_a$-action exists, and hence that S_λ only admits $\mathbb{A}^1$-fibrations of complete type. An $\mathbb{A}^1$-fibration $\pi_\lambda : S_\lambda \to \mathbb{P}^1$ can be obtained as follows: letting $B_\lambda = L_1 \cup L_2 \cup L_3$,

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L_1 is a member of a 6-tuple of pairwise disjoint lines whose simultaneous contraction realizes $\overline{S}_\lambda$ as a blow-up $\sigma : \overline{S}_\lambda \to \mathbb{P}^2$ of $\mathbb{P}^2$ in such a way that $\sigma(L_2)$ and $\sigma(L_3)$ are respectively a smooth conic and its tangent line at the point $p = \sigma(L_1)$. The birational transform $\overline{\pi}_\lambda : \overline{S}_\lambda \dashrightarrow \mathbb{P}^1$ on $\overline{S}_\lambda$ of the pencil generated by $\sigma(L_2)$ and $2\sigma(L_3)$ restricts to an $\mathbb{A}^1$-fibration $\pi_\lambda : S_\lambda \to \mathbb{P}^1$ with two degenerate fibers: an irreducible one, of multiplicity two, consisting of the intersection with S_λ of the unique exceptional divisor of σ whose center is supported on $\sigma(L_3) \setminus \{p\}$, and a smooth one consisting of the intersection with S_λ of the four exceptional divisors of σ with centers supported on $\sigma(L_2) \setminus \{p\}$.

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Adrien Dubouloz
IMB UMR5584
CNRS
Univ. Bourgogne Franche-Comté
F-21000 Dijon
France
adrien.dubouloz@u-bourgogne.fr

Takashi Kishimoto
Department of Mathematics
Faculty of Science
Saitama University
Saitama 338-8570
Japan
tkishimo@rimath.saitama-u.ac.jp

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