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CUTTING AND PASTING Z_p-MANIFOLDS[1]

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ABSTRACT. Let M^n and N^n be n-dimensional closed smooth oriented Z_p-manifolds where p is an odd prime and Z_p is the cyclic group of order p. This paper determines necessary and sufficient conditions under which M^n and N^n are equivalent under a special equivariant cut and past equivalence.

The only invariants are (a) the Euler characteristics of the Z_p-manifolds, (b) the Euler characteristics of the fixed point manifolds in each fixed point dimesnion with specified normal representations, and (c) the oriented Z_p-stratified cobordism class of the Z_p-manifolds.

1. **Introduction.** Dennis Sullivan and W. Neumann independently showed that two non-null closed smooth n-manifolds M^n and N^n are cut and paste equivalent if and only if (i) $\chi(M)=\chi(N)$ (the Euler characteristics are equal) and (ii) M^n and N^n represent the same element in the smooth unoriented cobordism ring $\mathcal{N}_*$.

For a detailed exposition of the proof see E. Y. Miller [8]. Incidentally, the cut and paste relation contained therein is not the same as described by K.K.N.O. [6], Kosniowski [7], and the papers of W. Neumann [9], Hermann and Kreck [5], J. Heithecker [4], *et al.*

Miller was able to show that this cut and paste relation could help detect the local combinatorial invariants of manifolds.

Recently the author was able to show (K. Prevot [11]) that this cutting and pasting theory, called SKV, was in the image of controllable cutting and pasting SKK under an epic map. By means of exact sequences and applications of Heithecker's work [4] on odd order actions, it was shown that SKV and its equivariant versions were precisely the cut and paste operations that allow one to introduce Euler characteristics into cobordism in the sense of Reinhart [12]; and at the same time avoiding the semi-characteristic invariants that appear in SKK.

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This paper will show how to detect SKV invariants for odd prime actions by geometric techniques rather than by exact sequences. In the same vein, we will refer to the special types of cobordism as "Z_p-stratified" since this seems to more adequately describe the geometry in the cobordism. Alternatively, one could use Kosniowski's language of slice types in equivariant cobordism [7]. We will agree throughout the paper that p denotes an odd prime and that $\mathbf{Z}_p$ is the cyclic group of order p.

DEFINITION 1.1. If M^n and N^n are n-dimensional closed smooth oriented Z_p-manifolds then M^n *is* Z_p*-equivariant cut and paste equivalent to* N^n *in one step* if there exists a compact smooth oriented n-dimensional Z_p-manifold P^n and four disjoint equivariant imbeddings i, $\hat{\imath}$, j, $\hat{\jmath}$ of a closed smooth $(n-1)$-dimensional Z_p-manifold T^{-1} into ∂P^n such that

(a) $\partial P^n = i(T^{n-1}) + \hat{\imath}(T^{n-1}) + j(T^{n-1}) + \hat{\jmath}(T^{n-1})$

(b) There are orientation preserving Z_p-equivariant diffeomorphisms

$$M^n \xleftarrow[\sim]{\phi} \begin{cases} P^n \text{ with identifications} \\ i(t) \sim \hat{\imath}(t), \text{ for every } t \in T^{n-1} \\ j(t) \sim \hat{\jmath}(t), \text{ for every } t \in T^{n-1} \end{cases}$$

$$N^n \xleftarrow[\sim]{\psi} \begin{cases} P^n \text{ with identifications} \\ i(t) \sim j(t), \text{ for every } t \in T^{n-1} \\ \hat{\imath}(t) \sim \hat{\jmath}(t), \text{ for every } t \in T^{n-1} \end{cases}$$

REMARK 1.1. The above definition uses + to denote disjoint union, and ∂ to denote the boundary of a given manifold.

DEFINITION 1.2. If M^n and N^n are n-dimensional closed smooth oriented Z_p-manifolds then M^n *is* Z_p*-equivariant cut and paste equivalent to* N^n if there exist n-dimensional closed smooth oriented Z_p-manifolds $V_1^n, V_2^n, \ldots, V_k^n$ with $M^n = V_1^n$, $N^n = V_k^n$, and V_i^n Z_p-equivariant cut and paste equivalent to V_{i+1}^n in one step for $i = 1, 2, \ldots, (k-1)$.

Before the main theorem is stated, a few relevant definitions will be presented on "oriented Z_p-stratified cobordism." It will be assumed that the reader is familiar with the representation theory of the normal bundles of the fixed point sets of a Z_p-manifold. An excellent presentation is found in Conner and Floyd [2].

NOTATION 1.1. If M^n is an n-dimensional compact smooth oriented Z_p-manifold, let

(a) M_m = union of the m-dimensional components of the fixed point set of M^n.

(b) M_{m_i} = union of the m-dimensional components of the fixed point set of M^n with a specified representation i of the normal bundle of $M_{m_i} \to M^n$.

(c) $(M^n - M_n)$ = union of all n-dimensional components of M^n which are not fixed by Z_p. Here $-$ denotes set complement.

DEFINITION 1.3. Let M^n be an n-dimensional closed smooth oriented Z_p-manifold. Then M^n *bounds an oriented Z_p-stratified bordism* if there exists an $(n+1)$-dimensional compact smooth oriented Z_p-manifold W^{n+1} with a Z_p-equivariant orientation preserving diffeomorphism $\psi: M^n \to \partial W^{n+1}$, and $(W_{m+1})_i$ empty if M_{m_i} is empty, for each $m = -1, 0, \ldots, n$ and each representation type i. By convention, each M_{-1_i} is empty.

DEFINITION 1.4. Let M^n and N^n be n-dimensional closed smooth oriented Z_p-manifolds. Then M^n is *oriented Z_p-stratified cobordant* to N^n if

(a) M_{m_i} is empty if and only if N_{m_i} is empty for each $m = 0, 1, \ldots, n$ and each representation type i.

(b) $(M^n - M_n)$ is empty if and only if $(N^n - N_n)$ is empty.

(c) $(M^n + N^n)$ bounds an oriented Z_p-stratified bordism.

(d) $\chi(M_{0_i}) = \chi(N_{0_i})$ for each representation type i.

REMARK 1.2. It is clear that oriented Z_p-stratified cobordism is an equivalence relation on n-dimensional closed smooth oriented Z_p-manifolds.

We are now in a position to state the following.

THEOREM 1.1 *Let M^n and N^n be n-dimensional closed smooth oriented Z_p-manifolds. Then M^n is Z_p-equivariant cut and paste equivalent to N^n if and only if*

(a) *M^n is oriented Z_p-stratified cobordant to N^n,*

(b) *$\chi(M^n) = \chi(N^n)$, and*

(c) *$\chi(M_{m_i}) = \chi(N_{m_i})$ for each representation type i.*

Here χ denotes Euler characteristic.

The proof of Theorem 1.1 will encompass the major portion of this paper.

2. **Proof of Theorem 1.1.** We will need the following special case of Theorem 1.1.

LEMMA 2.1. *Let p be an odd prime and let M^n and N^n be non-null n-dimensional closed smooth oriented free Z_p-manifolds. Then M^n is Z_p-equivariant cut and paste equivalent to N^n as free*[2] *Z_p-manifolds if and only if*

(a) $\chi(M^n) = \chi(N^n)$

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[2] Require that the manifold P^n in Definition 1.1 be free.

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(b) $[M^n]=[N^n]\in\Omega_n(Z_p)$, *where* $\Omega_n(Z_p)$ *is the n-dimensional cobordism group of oriented free* Z_p*-manifolds in the sense of* [2].

Proof. Let BZ_p and BSO denote the classifying spaces of Z_p and SO $(=\varinjlim_r SO(r))$ respectively. Let $\Omega_n(BZ_p)$ denote the group of cobordism classes of oriented n-manifolds mapping into BZ_p.

There is an isomorphism $\Omega_n(Z_p)\to\Omega_n(BZ_p)$, [2]. Moreover, $\Omega_*(BZ_p)$ is the cobordism theory based on the fibration

$$BSO\times BZ_p \xrightarrow{\pi} BSO \xrightarrow{f} BO, \text{ [14].}$$

Since $(BSO\times BZ_p, f\circ\pi)$ is "ordinary" in the sense of [5], the Lemma follows immediately.

REMARK 2.1. Lemma 2.1 also holds for free actions of an arbitrary compact Lie group G with the modification that $\chi(M^n/G)=\chi(N^n/G)$ rather than $\chi(M^n)=\chi(N^n)$ in condition (a).

To begin the formal proof of Theorem 1.1 we first show that if M^n is Z_p-equivariant cut and paste equivalent to N^n, then (a) M^n is oriented Z_p-stratified cobordant to N^n, (b) $\chi(M^n)=\chi(N^n)$, and (c) $\chi(M_{m_i})=\chi(N_{m_i})$ for $m=1,\ldots,n$ and for each representation type i.

In order to achieve (a), assume M^n is Z_p-equivariant cut and paste equivalent to N^n in one step and construct an explicit oriented Z_p-equivariant cobordism W^{n+1} between M^n and N_n. Choose $W^{n+1}=((P^n\times[0,1])+(T^{n-1}\times D^2))/\sim$. Here P^n and T^{n-1} are Z_p-manifolds as in Definition 1.1, and the Z_p-action on $((P^n\times[0,1])+(T^{n-1}\times D^2))$ comes from the given actions on P^n, T^{n-1}, and trivial actions on $[0,1]$ and D^2. The identifications given by $\sim$ are the Z_p-equivariant analogues of those of the cobordism constructed in the proof of the theorem of D. Sullivan and W. Neumann [5]. Also, (b) and (c) follow immediately from the equivariant nature of Z_p-equivariant cutting and pasting.

The real work in the proof of the theorem comes in showing that conditions (a), (b), and (c) are sufficient to achieve Z_p-equivariant cut and paste equivalence between M^n and N^n.

It will be helpful to make the following notational conventions:

1. If M is a Z_p-manifold, and N is not endowed with an action, $N\times M$ is the Z_p-manifold gotten by acting by Z_p on M and trivially on N.

2. (D^K, D_i^l) will denote the K-disk with Z_p-action $D^{K-l}\times D^l\to D^{K-l}\times D^l$ given by $(x,y)\to(i(x),y)$, i.e., act on D^{K-l} by the representation i. Note that the fixed point set is then D^l, and that the corners can be smoothed equivariantly.

3. (S^K, S_i^l) will denote the induced action on the boundary of (D^{K+1}, D_i^{l+1}).

4. If M^n is an oriented manifold, $(Z_p)(M^n)$ will denote the oriented free Z_p manifold $Z_p\times M^n$ with the obvious action.

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5. If M^n is a manifold with boundary ∂M^n, $\mathring{M}^n$ will denote $(M^n - \partial M^n)$.

6. Both $+$ and Σ will denote disjoint union.

7. rM^n will denote r-copies of the manifold M^n where r is a non-negative integer.

8. If M^n and N^n are n-dimensional closed smooth oriented Z_p manifolds, $\{M^n\}=\{N^n\}$ will mean that M^n and N^n are equivalent under Z_p-equivariant cutting and pasting.

9. $M^n = N^n$ will mean that M^n and N^n are equivalent up to an orientation preserving Z_p-equivariant diffeomorphism.

10. $M^n \bigcup_{V^{n-1}} N^n$ means M^n union N^n along V^{n-1}.

The organization of the remaining portion of the proof of Theorem 1.1 goes as follows:

(1) First it is shown how Z_p-equivariant cutting and pasting is related to Z_p-equivariant surgery.

(2) Secondly, after picking an oriented Z_p-stratified cobordism (Z_p, W^{n+1}) between (Z_p, M^n) and (Z_p, N^n), where W^{n+1} is obtained from $M^n \times [0,1]$ by adding handles Z_p-equivariantly, the cobordism W^{n+1} is interpreted as a sequence of Z_p-equivariant surgeries which in turn give rise to Z_p-equivariant cutting and pastings.

(3) Letting $\lambda_{q_{m_i}}$ be the number of $(q+1)$-handles added in the formation of the cobordism $(W_{m+1})_i$ from $M_{m_i} \times [0,1]$, and letting $p\lambda_q$ be the number of $(q+1)$-handles in the formation of the cobordism away from the fixed point sets gives

$$\left\{M^n + \sum_{q,m,i} \lambda_{q_{m_i}}(S^n, S_i^m) + (Z_p)\left(\sum_q \lambda_q S^n\right)\right\}$$
$$= \left\{N^n + \sum_{q,m,i} \lambda_{q_{m_i}}(S^q \times (S^{n-q}, S_i^{m-q})) + (Z_p)\left(\sum_q \lambda_q (S^q \times S^{n-q})\right)\right\}$$

(4) Next, one may cut Z_p-equivariantly to get

$$\{q(S^n, S_i^m) + (S^q \times (S^{n-q}, S_i^{m-q}))\} = \{q(S^n, S_i^m)\}$$

if q is odd, and

$$\{q(S^n, S_i^m) + (S^q \times (S^{n-q}, S_i^{m-q}))\} = \{(q+2)(S^n, S_i^m)\}$$

if q is even.

(5) Combining (3) and (4) one obtains

$$\left[M^n + \sum_{q,m,i} (q+1)\lambda_{q_{m_i}}(S^n, S_i^m) + (Z_p)\left(\sum_q \lambda_q S^n\right)\right\}$$
$$= \left\{N^n + \sum_{m,i}\left(\sum_q q\lambda_{q_{m_i}} + 2\sum_{q\text{ even}} \lambda_{q_{m_i}}\right)(S^n, S_i^m) + (Z_p)\left(\sum_q \lambda_q (S^q \times S^{n-q})\right)\right\}.$$

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(6) Examining the surgery in obtaining $(W_{m+1})_i$ from $M_{m_i}\times[0,1]$ one sees that there are non-negative integers λ_{m_i} such that

$$\Big\{M^n+\sum_{m,i}(\lambda_{m_i}+\chi((W_{m+1})_i)-\chi(M_{m_i}))(S^n,S_i^m)+(Z_p)\Big(\sum_q \lambda_q S^n\Big)\Big\}$$
$$=\Big\{N^n+\sum_{m,i}\lambda_{m_i}(S^n,S_i^m)+(Z_p)\Big(\sum_q \lambda_q(S^q\times S^{n-q})\Big)\Big\}.$$

(7) Next, one shows that there is a non-negative integer d such that

$$\Big\{M^n+\sum_{m,i}(\lambda_{m_i}+\chi((W_{m+1})_i)-\chi(M_{m_i}))(S^n,S_i^m)+(Z_p)(dS^n)\Big\}$$
$$=\Big\{N^n+\sum_{m,i}\lambda_{m_i}(S^n,S_i^m)+(Z_p)(dS^n)\Big\}.$$

(8) If there are non-negative integers h_{m_i} such that

$$\Big\{M^n+\sum_{m,i}h_{m_i}(S^n,S_i^m)+Z_p(dS^n)\Big\}=\Big\{N^n+\sum_{m,i}h_{m_i}(S^n,S_i^m)+(Z_p)(dS^n)\Big\},$$

then

$$\{M^n\}=\{N^n\}.$$

(9) Next, one shows that there are in fact non-negative integers h_{m_i} such that

$$\Big\{M^n+\sum_{m,i}h_{m_i}(S^n,S_i^m)+Z_p(dS^n)\Big\}=\Big\{N^n+\sum_{m,i}h_{m_i}(S^n,S_i^m)+Z_p(dS^n)\Big\}.$$

To carry out the outline of the proof we begin with

LEMMA 2.2. *Let M^n be a closed smooth oriented n-dimensional Z_p-manifold. Assume that M_{m_i} is non-null for some m with $1\le m\le n$ and some normal representation type i. Moreover assume that there is a Z_p-equivariant imbedding of $S^q\times(D^{n-q},D_i^{m-q})$ into a neighborhood of M_{m_i} in M^n which restricts to an imbedding $S^q\times D^{m-q}\to M_{m_i}$ for some q with $0\le q\le m-1$. Then*

$$\{M^n+(S^n,S_i^m)\}$$
$$=\Big\{(M^n-(S^q\times(\mathring{D}^{n-q},\mathring{D}_i^{m-q})))\bigcup_{(S^q\times(S^{n-q-1},S_i^{m-q-1}))}(D^{q+1}\times(S^{n-q-1},S_i^{m-q-1}))+(S^q\times(S^{n-q},S_i^{m-q}))\Big\}.$$

Proof. Note that

$$M^n=(M^n-(S^q\times(\mathring{D}^{n-q},\mathring{D}_i^{m-q})))\bigcup_{S^q\times(S^{n-q-1},S_i^{m-q-1})}(S^q\times(D^{n-q},D_i^{m-q}))$$

and that

$$(S^n,S_i^m)=\partial(D^{n+1},D_i^{m+1})$$
$$=\partial(D^{q+1}\times(D^{n-q},D_i^{m-q}))$$
$$=(D^{q+1}\times(S^{n-q-1},S_i^{m-q-1}))\bigcup_{S^q\times(S^{n-q-1},S_i^{m-q-1})}(S^q\times(D^{n-q},D_i^{m-q})).$$

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Cutting along two copies of $S^q \times (S^{n-q-1}, S_i^{m-q-1})$ gives, after pasting in another way,

$$\{M^n + (S^n, S_i^m)\}$$

$$= \Big\{(M^n - (S^q \times (\mathring{D}^{n-q}, \mathring{D}_i^{m-q}))) \bigcup_{S^q \times (S^{n-q-1}, S_i^{m-q-1})} (D^{q+1} \times (S^{n-q-1}, S_i^{m-q-1}))$$

$$+ (S^q \times (D^{n-q}, D_i^{m-q})) \bigcup_{S^q \times (S^{n-q-1}, S_i^{m-q-1})} (S^q \times (D^{n-q}, D_i^{m-q}))\Big\}.$$

Also,

$$(S^q \times (D^{n-q}, D_i^{m-q})) \bigcup_{S^q \times (S^{n-q-1}, S_i^{m-q-1})} (S^q \times (D^{n-q}, D_i^{m-q}))$$

$$= S^q \times ((D^{n-q}, D_i^{m-q}) \bigcup_{(S^{n-q-1}, S_i^{m-q-1})} (D^{n-q}, D_i^{m-q}))$$

$$= S^q \times (S^{n-q}, S_i^{m-q}).$$

This completes the proof of Lemma 2.2.

LEMMA 2.3. *Let M^n and N^n be closed smooth oriented n-dimensional Z_p-manifolds, $n > 0$. Assume M^n is oriented Z_p-stratified cobordant to N^n. Then there is an oriented Z_p-stratified cobordism W^{n+1} between M^n and N^n such that*

(1) *$W^{n+1} = K^{n+1} + L^{n+1}$, where K^{n+1} and L^{n+1} are connected,*

(2) *K^{n+1} is fixed by Z_p,*

(3) *Each $(W_{m+1})_i$* in *L^{n+1} is connected when $0 < m < n$, for i a normal representation type.*

(4) *$(W_1)_i$ is a disjoint union of lines in L^{n+1}, for i a normal representation type.*

Proof. Simply make use of equivariant connected sums.

LEMMA 2.4. *Let M^n and N^n be closed smooth oriented n-dimensional Z_p-manifolds which are oriented Z_p-stratified cobordant, and $n > 0$. Then there is an oriented Z_p-stratified cobordism W^{n+1} between M^n and N^n, where W^{n+1} is built from $M^n \times [0, 1]$ by a Z_p-equivariant handle decomposition as follows:*

(1) *The disk bundle of the normal bundle $D(\nu((W_{m+1})_i \to W^{n+1}))$ is built from $D(\nu(M_{m_i} \to M^n))$ by adding $\lambda_{q_{m_i}}$ Z_p-equivariant $(q+1)$-handles $D^{q+1} \times (D^{n-q}, D_i^{m-q})$, where $0 \le q \le m-1$, $1 \le m \le n$, i is a normal representation type, and $\lambda_{q_{m_i}}$ is a non-negative integer depending on q, m, and i.*

(2) *The fixed point free part of the cobordism is then obtained by adding $p\lambda_q$ $(q+1)$-handles $D^{q+1} \times D^{n-q}$ with the prescribed free Z_p-action on the λ_q equivariant handles $(Z_p)(D^{q+1} \times D^{n-q})$. Here $0 \le q \le n-1$ and λ_q is a non-negative integer depending on q.*

Proof. Applying Lemma 2.3, we may assume that there is an oriented Z_p-stratified cobordism W^{n+1} between M^n and N^n with the stated connectivity

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conditions. Thus for $m=0$, there is nothing to do; and the connectivity conditions allow each $\lambda_{q_{m_i}}$ to satisfy $0\le q\le m-1$ and each λ_q to satisfy $0\le q\le n-1$.

Since W^{n+1} is stratified, each $(W_{m+1})_i$ provides a connected cobordism between M_{m_i} and N_{m_i} with maps into the classifying space $B(U(n_1)\times\cdots\times U(n_{(p-1)/2}))$, for $m>0$. See Conner and Floyd [2]. The maps into $B(U(n_1)\times\cdots\times U(n_{(p-1)/2}))$ correspond to a fixed representation type i of the normal bundles $\nu(M_{m_i}\to M^n)$, $\nu(N_{m_i}\to N^n)$, and $\nu((W_{m+1})_i\to W^{n+1})$. See [2].

Let $(W_{m+1})_i \xrightarrow{f} B(U(n_1)\times\cdots\times U(n_{(p-1)/2}))$ be a map into the classifying space corresponding to the normal representation i. Also, let $D^{q+1}\times D^{m-q}$ be a $(q+1)$-handle added to $M_{m_i}\times[0,1]$ in the formation of $(W_{m+1})_i$. The inclusion $D^{q+1}\times D^{m-q}\to(W_{m+1})_i$ induces a trivial $(U(n_1)\times\cdots\times U(n_{(p-1)/2}))$-bundle over $D^{q+1}\times D^{m-q}$. Since $D^{q+1}\times D^{m-q}$ is contractible, the associated disk-bundle of the induced bundle over $D^{q+1}\times D^{m-q}$ is simply $D^{q+1}\times(D^{n-q},D_i^{m-q})$.

Using these facts, we may assume that we have constructed $\tilde{W}^{n+1}\subseteq W^{n+1}$, where

$$\tilde{W}^{n+1}=(M^n\times[0,1] \bigcup_{(\sum_{m_i}D(\nu(M_{m_i}\to M^n)))\times[0,1]} \Big(\sum_{m,i} D(\nu((W_{m+1})_i\to W^{n+1}))\Big)$$

We now need to establish some notation. Let $\mathring{D}(\nu((W_{m+1})_i)\to W^{n+1})$ denote $(D(\nu((W_{m+1})_i\to W^{n+1})))-S(\nu((W_{m+1})_i\to W^{n+1}))$, where D and S are the associated disk and sphere bundles, respectively, of the normal bundle ν. Let

$$\hat{M}^n=(M^n\times 1)-\sum_{m,i}\mathring{D}(\nu((M_{m_i}\times 1)\to(M^n\times 1))),\qquad \hat{N}^n=N-\sum_{m,i}D(\nu(N_{m_i}\to N)),$$

and

$$\hat{W}^{n+1}=\tilde{W}^{n+1}-\Big((M^n\times[0,1)) \bigcup_{\sum_{m_i}(\mathring{D}(\nu(M_{m_i}\to M^n))\times[0,1]]} \sum_{m,i}(\mathring{D}(\nu((W_{m+1})_i\to W^{n+1})))\Big).$$

Note that $\hat{W}^{n+1}\times[0,1]$ may be thought of as a principal oriented Z_p-equivariant cobordism between principal oriented Z_p-manifolds $\hat{M}^n\times 0$ and $\hat{W}^{n+1}\times 1$ with boundary. The cobordism $\hat{W}^{n+1}\times[0,1]$ corresponds to a map $(\hat{W}^{n+1}\times[0,1])/Z_p\to BZ_p$, which is a cobordism of $(\hat{M}^n\times 0)/Z_p\to BZ_p$ and $(\hat{W}^{n+1}\times 1)/Z_p\to BZ_p$ as oriented manifolds with boundary and maps into BZ_p.

Put

$$\bar{W}^{n+1}=W^{n+1}-\Big((M^n\times[0,1)) \bigcup_{\sum_{m_i}\mathring{D}(\nu(M_{m_i}\to M^n))\times[0,1)} \sum_{m,i}\mathring{D}(\nu(((W_{m+1})_i)\to W^{n+1}))\Big).$$

Applying a free Z_p-equivariant analogue relating cobordism to surgery on manifolds with boundary [16], one may construct $(\bar{W}^{n+1}/Z_p\to BZ_p)$ as an oriented cobordism of $(\hat{M}^n/Z_p\to BZ_p)$ and $(\hat{N}^n/Z_p\to BZ_p)$. One builds $\bar{W}^{n+1}/Z_p\to BZ_p$ by adding λ_q $(q+1)$-handles with maps $D^{q+1}\times D^{n-q}\to BZ_p$ along $(\hat{N}^{n+1}\times 1)/Z_p$. This handle decomposition lifts to give $p\lambda_q$ $(q+1)-Z_p$-equivariant handles $(Z_p)(D^{q+1}\times D^{n-q})$ added to $\hat{W}^{n+1}\times 1$ in forming $\bar{W}^{n+1}$.

The above completes the equivariant handle decomposition of W^{n+1}, and the proof of Lemma 2.4 is complete.

LEMMA 2.5. *With* M^n *and* N^n *as in the statement of Theorem* 1.1 *and the notation from Lemma* 2.4,

$$\left\{M^n+\sum_{q,m,i}\lambda_{q_{m_i}}(S^n,S_i^m)+(Z_p)\left(\sum_q\lambda_qS^n\right)\right\}$$
$$=\left\{N^n+\sum_{q,m,i}\lambda_{q_{m_i}}(S^q\times(S^{n-q},S_i^{m-q}))+(Z_p)\left(\sum_q\lambda_q(S^q\times S^{n-q})\right)\right\}.$$

Proof. The proof follows from Lemma 2.4, repeated application of Lemma 2.2, and repeated application of the oriented free Z_p-equivariant version of Lemma 2.2.

REMARK 2.2. Our goal is to show that M^n and N^n are Z_p-equivariant cut and paste equivalent, i.e., $\{M^n\}=\{N^n\}$. We do this by showing that the "stable effects" of Lemma 2.5 may be reduced from unions of products of spheres with Z_p-actions to unions of spheres with Z_p-actions. We then show that the number of "stabilizing" spheres may be equalized, and then "absorbed" into M^n and N^n.

Lemma 2.7 through 2.13 below are stated without proof. They are Z_p-equivariant analogues of similar results appearing in [5].

LEMMA 2.6. *If* $0\le q\le m-1$, *then*

$$\{2(S^n,S_i^m)\}=\{(S^q\times(S^{n-q},S_i^{m-q}))+(S^{q+1}\times(S^{n-q-1},S_i^{m-q-1}))\}$$

LEMMA 2.7. *If* $q=2r+1$ *and* $0\le q\le m-1$, *then*

$$\left\{(S^n,S_i^m)+\sum_{j=1}^{2r+1}(S^j\times(S^{n-j},S_i^{m-j}))\right\}$$
$$=\{(2r+1)(S^n,S_i^m)+(S^{2r+1}\times(S^{n-2r-1},S_i^{m-2r-1}))\}.$$

LEMMA 2.8. *If* $q=2r+1$ *and* $0\le q\le m-1$, *then*

$$\left\{(S^n,S_i^m)+\sum_{j=1}^{2r+1}(S^j\times(S^{n-j},S_i^{m-j}))\}=\{(2r+1)(S^n,S_i^m)\right\}.$$

LEMMA 2.9. *If* $q=2r+1$ *and* $0\le q\le m-1$, *then*

$$\{(2r+1)(S^n,S_i^m)+(S^{2r+1}\times(S^{n-2r-1},S_i^{m-2r-1}))\}=\{(2r+1)(S^n,S_i^m)\}$$

LEMMA 2.10. *If* $q=2r$ *and* $0\le q\le m-1$, *then*

$$\left\{2(S^n,S_i^m)+\sum_{j=1}^{2r}S^j\times(S^{n-j},S_i^{m-j})\right\}=\{(2r+2)(S^n,S_i^m)\}.$$

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LEMMA 2.11. *If* $q=2r$ *and* $0\le q\le m-1$, *then*

$$\left\{2(S^n, S_i^m)+\sum_{j=1}^{2r} S^j\times(S^{n-j}, S_i^{m-j})\right\}=\left\{2r(S^n, S_i^m)+(S^{2r}\times(S^{n-2r}, S_i^{m-2r}))\right\}.$$

LEMMA 2.12. *If* $q=2r$ *and* $0\le q\le m-1$, *then*

$$\{(S^{2r}\times(S^{n-2r}, S_i^{m-2r}))+2r(S^n, S_i^m)\}=\{(2r+2)(S^n, S_i^m)\}.$$

The following Lemma amalgamates the above results.

LEMMA 2.13. *With the notation of Lemma* 2.5, *one has*

$$\left\{M^n+\sum_{q,m,i}(q+1)\lambda_{q_{m_i}}(S^n, S_i^m)+(Z_p)\left(\sum_q \lambda_q S^n\right)\right\}$$
$$=\left\{N^n+\sum_{m,i}\left(\sum_q q\lambda_{q_{m_i}}+2\sum_{q\text{ even}}\lambda_{q_{m_i}}\right)(S^n, S_i^m)+(Z_p)\left(\sum_q \lambda_q(S^q\times S^{n-q})\right)\right\}.$$

Proof. By Lemma 2.5,

$$\left\{M^n+\sum_{q,m,i}\lambda_{q_{m_i}}(S^n, S_i^m)+(Z_p)\left(\sum_q \lambda_q S^n\right)\right\}$$
$$=\left\{N^n+\sum_{q,m,i}\lambda_{q_{m_i}}(S^q\times(S^{n-q}, S_i^{m-q}))+(Z_p)\left(\sum_q \lambda_q(S^q\times S^{n-q})\right)\right\}.$$

Adding $\sum_{q,m,i} q\lambda_{q_{mi}}(S^n, S_i^m)$ gives

$$\left\{M^n+\sum_{q,m,i}(q+1)\lambda_{q_{m_i}}(S^n, S_i^m)+(Z_p)\left(\sum_q \lambda_q S^n\right)\right\}$$
$$=\left\{N^n+\sum_{q,m,i}(\lambda_{q_{m_i}}((S^q\times(S^{n-q}, S_i^{m-q}))+q(S^n, S_i^m))+(Z_p)\left(\sum_q \lambda_q(S^q\times S^{n-q})\right)\right\}$$
$$=\left\{N^n+\sum_{m,i}\left(\sum_q q\lambda_{q_{m_i}}+2\sum_{q\text{ even}}\lambda_{q_{m_i}}\right)(S^n, S_i^m)+(Z_p)\left(\sum_q (S^q\times S^{n-q})\right)\right\}.$$

This follows from Lemma 2.9 and Lemma 2.12.

LEMMA 2.14. *With notation from Lemma* 2.13, *there are non-negative integers* λ_{m_i} *such that*

$$\left\{M^n+\sum_{m,i}(\lambda_{m_i}+(\chi((W_{m+1})_i)-\chi(M_{m_i})))(S^n, S_i^m)+(Z_p)\left(\sum_q \lambda_q S^n\right)\right\}$$
$$=\left\{N^n+\sum_{m,i}\lambda_{m_i}(S^n, S_i^m)+(Z_p)\left(\sum_q \lambda_q(S^q\times S^{n-q})\right)\right\}.$$

Proof. Notice that for each m and i,

$$\sum_q (q+1)\lambda_{q_{m_i}}-\left(\sum_q q\lambda_{q_{m_i}}+2\sum_{q\text{ even}}\lambda_{q_{m_i}}\right)=\sum_q \lambda_{q_{mi}}-2\sum_{q\text{ even}}\lambda_{q_{m_i}}$$
$$=\sum_q (-1)^{q+1}\lambda_{q_{m_i}}=\chi((W_{m+1})_i)-\chi(M_{m_i})$$

because $(W_{m+1})_i$ is obtained from $M_{m_i}\times[0,1]$ by adding $\lambda_{q_{mi}}$ handles in dimension $(q+1)$. Take

$$\lambda_{q_{m_i}}=\Big(\sum_q q\lambda_{q_{m_i}}+2\sum_{q\text{ even}}\lambda_{q_{m_i}}\Big).$$

Then the result is immediate.

We now show how to equalize the spheres with free Z_p-action.

LEMMA 2.15. *With the established results, notation, and the additional hypotheses that $\chi(M^n)=\chi(N^n)$ and $\chi(M_{m_i})=\chi(N_{m_i})$ for each m and i, there is a non-negative integer d such that*

$$\Big\{M^n+\sum_{m,i}(\lambda_{m_i}+\chi((W_{m+1})_i)-\chi(M_{m_i}))(S^n,S_i^m)+(Z_p)(dS^n)\Big\}$$
$$=\{N^n+\sum_{m,i}\lambda_{m_i}(S^n,S_i^m)+(Z_p)(dS^n)\Big\}.$$

Proof. Case (1) n even. Since the fixed point manifolds have a complex normal representation, a simple codimension argument shows that m is even. Moreover,

$$\begin{aligned}0&=\chi((W_{m+1})_i)\bigcup_\partial((W_{m+1})_i))=2\chi((W_{m+1})_i)-(\chi(M_{m_i})+\chi(N_{m_i}))\\&=2(\chi((W_{m+1})_i)-\chi(M_{m_i}))\quad\text{since}\quad((W_{m+1})_i)\bigcup_\partial((W_{m+1})_i)\end{aligned}$$

is a closed odd dimensional manifold and $\chi(M_{m_i})=\chi(N_{m_i})$. Thus $(\chi((W_{m+1})_i)-\chi(M_{m_i}))=0$. Also, since $\chi(M^n)=\chi(N^n)$, we deduce that

$$\chi\Big((Z_p)\Big(\sum_q\lambda_qS^n\Big)\Big)=\chi\Big((Z_p)\Big(\sum_q\lambda_q(S^q\times S^{n-q})\Big),$$

and then Lemma 2.1 gives the result.

Case (2) n odd. This case follows immediately from Lemma 2.1.

LEMMA 2.16. *Assume that there are non-negative integers h_{m_i} and d such that*

$$\Big\{M^n+\sum_{m,i}h_{m_i}(S^n,S_i^m)+Z_p(dS^n)\Big\}=\Big\{N^n+\sum_{m,i}h_{m_i}(S^n,S_i^m)+Z_p(dS^n)\Big\}.$$

Then $\{M^n\}=\{N^n\}$.

Proof. Pick $2(d+1)$ disjoint imbeddings of an n-disk D^n into a fixed n-sphere S^n. Label these imbeddings by

$$f_j:D^n\to S^n\quad\text{and}\quad g_j:D^n\to S^n\quad\text{with}\quad j=1,2,\ldots,d,d+1.$$

Let $T_d=S^n-\sum_{j=1}^{d+1}(f_j(\mathring{D}^n)+g_j(\mathring{D}^n))/\sim$ where $\sim$ is the relation given by $f_j(\partial D^n)=g_j(\partial D^n)$, for each $j=1,2,\ldots,d,d+1$.

Using the hypothesis of the Lemma,

$$\Big\{M^n+\sum_{m,i}h_{m_i}(S^n,S_i^m)+(Z_p)(dS^n)+(Z_p)(T_d)\Big\}$$
$$=\Big\{N^n+\sum_{m,i}h_{m_i}(S^n,S_i^m)+(Z_p)(dS^n)+(Z_p)(T_d)\Big\}.$$

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It is not hard to see that

$$\left\{M^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)\right\}=\left\{M^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)+(Z_p)(dS^n)+(Z_p)(T_d)\right\}$$
$$=\left\{N^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)+(Z_p)(dS^n)+(Z_p)(T_d)\right\}$$
$$=\left\{N^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)\right\}.$$

The first equivalence above is gotten by cutting M^n along the boundary of $(Z_p)(D^n)\to M^n$ away from the fixed point sets and cutting along $(Z_p)(dS^{n-1})$ in $(Z_p)(dS^n)$ and cutting along the identifications of $(Z_p)(T_d)$, then one glues everything back into M^n.

Now, to show that the spheres with fixed points may be absorbed in an analogous argument. Let $f_j^{m,i}$, $g_j^{m,i}:(D^n, D_i^m)\to(S^n, S_i^n)$ with $j=1,2,\ldots,(h_{m_i}+1)$ be $2(h_{m_i}+1)$ disjoint equivariant imbeddings for each m and i. Let

$$T_{h_{m_i}}=(S^n, S_i^m)-\sum_{j=1}^{(h_{m_i}+1)}(f_j^{m,i}(\mathring{D}^n, \mathring{D}_i^m)+g_j^{m,i}(D^n, D_i^m))/\sim$$

where $\sim$ is given by the identifications $f_j^{m,i}(S^{n-1}, S_i^{m-1})=g_j^{m,i}(S^{n-1}, S_i^{m-1})$ on the boundary. Thus,

$$\left\{M^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)+\sum_{m,i} T_{h_{m_i}}\right\}=\left\{N^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)+\sum_{m,i} T_{h_{m_i}}\right\}.$$

Cutting M^n along $\partial(D^n, D_i^m)\to D(\nu(M_{m_i}\to M^n))$, cutting $\sum_{m,i} h_{m_i}(S^n, S_i^m)$ along $\sum_{m,i} h_{m_i}\partial(D^n, D_i^m)$, and cutting $\sum_{m,i} T_{h_{m_i}}$ along the identifications shows that

$$\{M^n\}=\left\{M^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)+\sum_{m,i} T_{h_{m_i}}\right\}=\left\{N^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)+\sum_{m,i} T_{h_{m_i}}\right\}=\{N^n\}.$$

REMARK 2.3. Lemma 2.16 has shown that if the "stabilizing" spheres may be equalized, then they may be absorbed. Lemma 2.15 has shown that the spheres without fixed points may be equalized. Thus it remains to equalize the spheres with fixed points.

LEMMA 2.17. *Taking into account Lemma* 2.15 *and Lemma* 2.16, *there are non-negative integers* h_{m_i} *such that*

$$\left\{M^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)\}=\{N^n+\sum_{m,i} h_{m_i}(S^n, S_i^m)\right\}.$$

Proof. Case (1) n even. The result follows immediately from the proof of Lemma 2.15. In fact, one may take $h_{m_i}=\lambda_{m_i}$.

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Case (2) n odd. This case is the difficult one, which breaks down into several cases. Before Case 2 is finished, we should indicate the following results.

LEMMA 2.18. *Let $[j]$ denote the action of Z_p on the complex numbers $\mathbf{C}$ via $(\mathbf{C}\to\mathbf{C}, z\to\rho^j z)$ where $\rho=e^{2\pi i/p}$ and here $i=\sqrt{-1}$. If $j\not\equiv 0(p)$, then there is a Z_p-action on the Riemann surface S_j of genus $(pg\times p(p-3)/2+1)$ for $g\geq 0$, with p fixed points each of normal type $[j]$ such that (S_j/Z_p) is a Riemann surface of genus g.*

COROLLARY 2.1. *By taking the product $S_{j_1}\times S_{j_2}\times\cdots\times S_{j_r}$ we get a Z_p-manifold with diagonal Z_p-action and P^r fixed points each of normal type $[j_1]+[j_2]+\cdots+[j_r]$.*

COROLLARY 2.2. *With trivial action on the complex projective space $\mathbf{C}p^{2K}$, $\mathbf{C}p^{2K}\times S_{j_1}\times\cdots\times S_{j_r}$ has fixed points p^r copies of $\mathbf{C}p^{2K}$ with normal type $[j_1]+[j_2]+\cdots+[j_r]$ for each component of the fixed point set. Moreover, χ(Fixed Points)$=(2K+1)p^r=1 \bmod 2$.*

We will omit the proofs of Lemma 2.18, Corollary 2.1, and Corollary 2.2, and refer the interested reader to [10]. Now, continuing the proof of Lemma 2.17,

Case (2) n odd. We first show that if a is any integer, then there exists a positive integer μ such that $\mu+a>0$ and $\{(\mu+a)(S^n, S_i^m)\}=\{\mu(S^n, S_i^m)\}$.

Case (2A) n odd, $m=4r-1$. There exists an $(n+1)$-dimensional closed smooth oriented Z_p-manifold $(X^{n+1}, (X_{4r})_i)$ such that $((X_{4r})_i)=a+2$. Here $(X^{n+1}, (X_{4r})_i)$ means that the fixed point set of X^{n+1} occurs in dimension $4r$ with representation type i. The existence of $(X^{n+1}, (X_{4r})_i)$ follows from Lemma (2.1.2) [6] and Corollary 2.2. Removing two imbedded Z_p-equivariant disks $2(D^{n+1}, D_i^{4r})$ from $(X^{n+1}, (X_{4r})_i)$ gives rise to an oriented Z_p-stratified cobordism between two copies of (S^n, S_i^{4r-1}). The cobordism is $(Y^{n+1}, (Y_{4r})_i)=(X^{n+1}, (X_{4r})_i)-2(\mathring{D}^{n+1}, \mathring{D}_i^{4r})$. Also, $\chi((Y_{4r})_i)=\chi(X_{4r}-2\mathring{D}^{4r})-\chi(S^{4r-1})=a$. Using Lemma 2.15 and Lemma 2.16, $\{(S^n, S_i^{4r-1})+(\lambda_i+a)(S^n, S_i^{4r-1})\}=\{(S^n, S_i^{4r-1})+\lambda_i(S^n, S_i^{4r-1})\}$ for some λ_i. Thus $\{((\lambda_i+1)+a)(S^n, S_i^{4r-1})\}=\{(\lambda_i+1)(S^n, S_i^{4r-1})\}$. In this case, take $\mu=\lambda_i+1$.

Case (2B) n odd, $m=4r+1$, $r>0$. Construct $(X^{n+1}, (X_{4r})_i)$ as in Case (2A) but with $\chi((X_{4r}))=a$. Let $D^2\times(X^{n+1}, (X_{4r})_i)$ be the Z_p-manifold gotten by acting trivially on D^2. Pick a Z_p-equivariant imbedding of (D^{n+3}, D_i^{4r+2}) into the interior of $D^2\times(X^{n+1}, (X_{4r})_i)$. Then $((D^2\times(X^{n+1}, (X_{4r})_i))-(\mathring{D}^{n+3}, \mathring{D}_i^{4r+2}))$ is an oriented Z_p-stratified cobordism between (S^{n+2}, S_i^{4r+1}) and $S^1\times(X^{n+1}, (X_{4r})_i)$. Also, $\chi((D^2\times X_{4r})-\mathring{D}^{4r+2})-\chi(S^{4r+1})=a-1$.

Applying Lemmas 2.15 and 2.16 gives

$$\begin{aligned}\{(a+\lambda_i)(S^{n+2}, S_i^{4r+1})\}&=\{(S^{n+2}, S_i^{4r+1})+((a-1)+\lambda_i)(S^{n+2}, S_i^{4r+1})\}\\&=\{S^1\times(X^{n+1}, (X_{4r})_i)+\lambda_i(S^{n+2}, S_i^{4r+1})\}\text{ for some }\lambda_i.\end{aligned}$$

Hence,

$$\begin{aligned}
\{(2a+3\lambda_i)(S^{n+2}, S_i^{4r+1})\}\\
&=\{(a+\lambda_i)(S^{n+2}, S_i^{4r+1})+(a+\lambda_i)(S^{n+2}, S_i^{4r+1})+\lambda_i(S^{n+2}, S_i^{4r+1})\}\\
&=\{2(S^1\times(X^{n+1}, (X_{4r})_i)+\lambda_i(S^{n+2}, S_i^{4r+1}))+\lambda_i(S^{n+2}, S_i^{4r+1})\}\\
&=\{2(S^1\times(X^{n+1}, (X_{4r})_i)+3\lambda_i(S^{n+2}, S_i^{4r+1})\}\\
&=\{(S^1\times(X^{n+1}, (X_{4r})_i))+3\lambda_i(S^{n+2}, S_i^{4r+1})\}\\
&=\{(a+\lambda_i)(S^{n+2}, S_i^{4r+1})+2\lambda_i(S^{n+2}, S_i^{4r+1})\}\\
&=\{(a+3\lambda_i)(S^{n+2}, S_i^{4r+1})\}.
\end{aligned}$$

Here take $\mu=a+3\lambda_i$.

Case (2C) n odd and $m=1$. Let $V^{n+1}=S_{j_1}\times\cdots\times S_{j_r}$ as in Corollary 2.1, and let i = representation type $[j_1]+\cdots+[j_r]$. Let $S^1\times V^{n-1}=\partial(D^2\times V^{n-1})$ with trivial action on D^2, and $(S^n, S_i^1)=\partial(D^{n+1}, D_i^2)$. Let $(W^{n+1}, (W_2)_i)=(D^2\times V^{n-1})+(D^{n+1}, D_i^2)$. Then

$$\begin{aligned}
\chi(W_2)&=\chi(D^2\times(p^r \text{ points}))+\chi(D^2)\\
&=1(\text{mod } 2)+1(\text{mod } 2)\\
&=0(\text{mod } 2).
\end{aligned}$$

Moreover, $(W^{n+1}, (W_2)_i)$ is an oriented Z_p-stratified cobordism between (S^n, S_i^1) and $(S^1\times V^{n-1})$. Applying Lemma 2.15 and Lemma 2.16 gives

$$\{(S^1\times V^{n-1})+(\lambda_i+\chi((W_2)_i))(S^n, S_i^1)\}=\{(S^n, S_i^1)+\lambda_i(S^n, S_i^1)\}$$

for some $\lambda_i>0$ with $\lambda_i=0$ mod 2.

Let $\beta_i=(\lambda_i+\chi((W_2)_i))/2$. Then,

$$\{(\lambda_i+\chi((W_2)_i))(S^n, S_i^1)\}=\{\beta_i(S^1\times(S^{n-1}, S_i^0))+\beta_i(S^2\times S^{n-2})\},$$

where S^2 has trivial action and S^{n-2} has the induced action of $S^{n-2}\to\partial D^{n-1}$, with D^{n-1} being acted upon by representation type i.

Cutting along S^1 in the first factor and applying Lemma 2.1 to the second factor shows

$$\begin{aligned}
\{\beta_i(S^1\times(S^{n-1}, S_i^0))+\beta_i(S^2\times S^{n-2})\}&=\{(\lambda_i)/2(S^1\times(S^{n-1}, S_i^0))+\{(\lambda_i)/2(S^2\times S^{n-2})\}\\
&=\{\lambda_i(S^n, S_i^1)\}.
\end{aligned}$$

Hence, $\{(\lambda_i+\chi((W_2)_i))(S^n, S_i^1)\}=\{\lambda_i(S^n, S_i^1)\}$.

Plugging into the above gives

$$\{(S^1\times V^{n-1})+\lambda_i(S^n, S_i^1)\}=\{(S^n, S_i^1)+\lambda_i(S^n, S_i^1)\},$$

and Lemma 2.16 shows $\{(S^1\times V^{n-1})\}=\{(S^n, S_i^1)\}$. Similar reasoning shows $\{2(S^1\times V^{n-1})\}=\{2(S^n, S_i^1)\}$, so, $\{(S^n, S_i^1)\}=\{(S^1\times V^{n-1})\}=\{2(S^1\times V^{n-1})\}$ (cutting along S^1) $=\{2(S^n, S_i^1)\}$.

Thus, given a in Case (2C), we may take $\mu = |a| + 1$. Putting the results from Cases (2A), (2B), and (2C) together gives: there exists positive integers μ_{m_i} such that, if n is odd,

$$\{(\mu_{m_i} + (-\chi((W_{m+1})_i) + \chi(M_{m_i}))(S^n, S_i^m)\} = \{\mu_{m_i}(S^n, S_i^m)\}.$$

Using Lemma 2.15 yields

$$\begin{aligned}\Big\{M^n + \sum_{m,i} (\mu_{m_i} + \lambda_{m_i})(S^n, S_i^m) + (Z_p)(dS^n)\Big\} \\ = \Big\{M_n + \sum_{m,i} (\mu_{m_i} + (-\chi((W_{m+1})_i) + \chi(M_{m_i})))(S^n, S_i^m) \\ + \sum_{m,i} (\lambda_{m_i} + \chi((W_{m+1})_i) - \chi(M_{m_i}))(S^n, S_i^m) + (Z_p)(dS^n)\Big\} \\ = \Big\{N^n + \sum_{m,i} (\mu_{m_i} + \lambda_{m_i})(S^n, S_i^m) + (Z_p)(dS^n)\Big\}.\end{aligned}$$

This shows that the spheres may be equalized. Thus the proof of Theorem 1.1 is complete.

After writing the paper "Modifications of Controllable Cutting and Pasting," I noticed that Theorem 1.1 in this paper could be extended to include *any* odd order group G acting as a group of orientation preserving diffeomorphisms, provided that one takes into account the fixed data of subgroups H of G and their slice representations. The interested reader is referred to [11].

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