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On the structure of the family of Cherry fields on the torus

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Abstract. A class of vector fields on the 2-torus, which includes Cherry fields, is studied. Natural paths through this class are defined and it is shown that the parameters for which the vector field is unstable is the closure of $\{t \mid R_t \circ f$ has irrational rotation number$\}$, where f is a certain map of the circle and R_t is rotation through t. This is shown to be a Cantor set of zero Hausdorff dimension. The Cherry fields are shown to form a family of codimension one submanifolds of the set of vector fields. The natural paths are shown to be stable paths.

1. *Introduction and statement of results*

We are interested in certain flows of class C^∞ on the 2-torus. We will work on its universal cover $\mathbb{R}^2$, so all vector fields X will satisfy $X(x+n, y+m) = X(x, y)$, for all $n, m \in \mathbb{Z}$. All the vector fields considered will satisfy the following:

(A) X has two singularities, a hyperbolic saddle S and a hyperbolic sink P.

(B) X is transverse to the circle $\Sigma = \{(x, y) \mid x = 0\}$.

(C) There exist $a, b \in \Sigma$ such that if $y \in (a, b)$ the positive orbit of X through y goes directly to the sink without re-intersecting Σ, but for $y \notin [a, b]$ the Poincaré map $f: \Sigma \to \Sigma$ is defined and expanding. Furthermore, $f'(y) \to \infty$ as $y \to a^-$ or $y \to b^+$. (See figure 1a.)

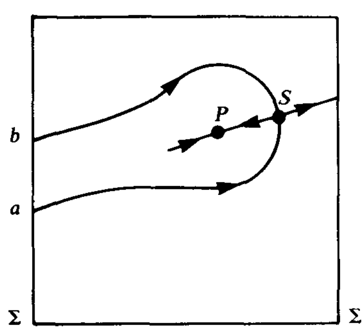

FIGURE 1a

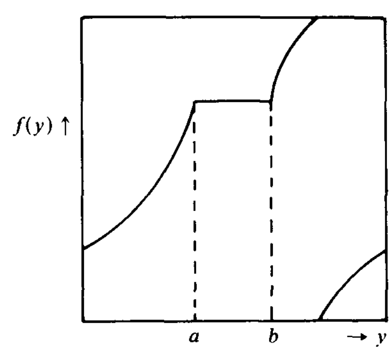

FIGURE 1b

The Poincaré map may be extended to the whole of Σ by making it constant on $[a, b]$ so we have a continuous circle endomorphism $f: \Sigma \to \Sigma$. By condition (C), $f'(y) \geq \lambda > 1$ for all $x \notin [a, b]$. (See figure 1b.)

Since f is monotonic and of degree one it has a rotation number (see e.g. [**8**]). We denote the set of C^∞ vector fields on the 2-torus with the C^∞ topology, by

$\mathfrak{X}^\infty(T^2)$, and the neighbourhood in $\mathfrak{X}^\infty(T^2)$ of all vector fields satisfying (A), (B) and (C), by $\mathcal{N}$.

(1.1) *Definition.* A *Cherry field* is a vector field in $\mathcal{N}$ whose Poincaré map has irrational rotation number.

Cherry fields were first constructed in [3]; see [**8**, pp. 181ff] for a modern construction. The orbit structure of a Cherry field is described by the following:

(1.2) THEOREM ([**8**], p. 186). *Let X be a Cherry field with sink P and saddle S. Then*
(1) *$W^s(P)$ is dense in T^2.*
(2) *P and S are the only minimal sets for X.*
(3) *$\Sigma - W^s(P)$ is a Cantor set.*
(4) *$T^2 - W^s(P)$ is transitive for the flow.*

Vector fields $X \in \mathcal{N}$ whose Poincaré maps have rational rotation number are either Morse–Smale or have a saddle connection, the fields with a saddle connection forming the boundary of the Morse–Smale classes with the same rotation number. The three types of field in $\mathcal{N}$ are determined by what happens to the orbit of the 'free' unstable separatrix of S - that not joined directly to the sink. One of the following must happen:

(i) after intersecting Σ a number of times it intersects (a, b) and goes to the sink. In this case X is a Morse–Smale field;

(ii) after intersecting Σ a number of times it intersects Σ at a or b, so X has a saddle connection;

(iii) it intersects Σ infinitely often without intersecting $[a, b]$. In this case X is a Cherry field.

We investigate paths in $\mathcal{N}$ which change the relative positions of the free separatrix of $W^u(s)$ and $[a, b]$. By measuring how many parameter values correspond to Cherry fields we get an idea of how common they are in $\mathcal{N}$. Let $\phi:[0, 1] \to \mathcal{N}$ be a C^1 path chosen so that

$$f_{\phi(t)}(y) = f_{\phi(0)}(y) + t,$$

where $f_{\phi(t)}$ is the Poincaré map of $\phi(t)$. Such a path may be constructed by making a suitable perturbation to the vector field in a small strip near Σ.

Every number in $[0, 1)$ is represented as the rotation number of $f_{\phi(t)}$ for some t, and it is not difficult to see that the bifurcation set of ϕ is a Cantor set E. The open intervals in the complement of E consist of parameters corresponding to Morse–Smale fields, the boundary points of these intervals correspond to fields with a saddle connection and the remaining points of E correspond to Cherry fields. Our first result reveals that this path contains very few Cherry fields. Let m denote Lebesgue measure.

(1.3) THEOREM 1. *Let $E = \{t \mid \phi_t$ is unstable$\}$. Then $m(E) = 0$ and furthermore E has zero Hausdorff dimension.*

From well-known work of Sotomayor [**10**] it is known that the set of fields in $\mathcal{N}$ with a saddle connection forms an immersed submanifold of $\mathcal{N}$ of class C^∞ and codimension one, or those with a particular rotation number form an embedded submanifold. We are able to show the following for Cherry fields:

(1.4) THEOREM 2. *The set of Cherry fields in $\mathcal{N}$ with a given rotation number forms a codimension one embedded Banach submanifold of $\mathcal{N}$ of class C^1.*

Note that the set of all Cherry fields is not an embedded submanifold, since there would be uncountably many components in any neighbourhood. We have no reason to believe that the submanifold is not, in fact, of class C^∞. Using theorem 2 we prove the following about the path ϕ described above, which shows it is not a particularly special path.

(1.5) THEOREM 3. *The path ϕ is stable in the space of C^1 paths in $\mathcal{N}$, as long as $\phi(0)$ is Morse-Smale.*

ϕ is unusual as a stable path because the Cherry fields are Kupka-Smale but not Morse-Smale (see [**10**, p. 45]).

2. *Proof of theorem* 1

Theorem 1 will be implied by the slightly stronger:

(2.1) THEOREM 1′. *Let f be a continuous monotonic non-decreasing map of the circle of degree one satisfying*

(1) *f is constant on an interval $[a, b]$ and of class C^1 outside $[a, b]$.*

(2) $\inf\{f'(y) | y \notin [a, b]\} = \lambda > 1$.

Let f_t be the map defined by $f_t(y) = f(y) + t$, $t \in [0, 1]$. Let $E = \{t | f_t$ has irrational rotation number$\}$. Then $m(E) = 0$, where m denotes Lebesgue measure, and furthermore E has zero Hausdorff dimension.

Since the Poincaré map of any field in $\mathcal{N}$ satisfies the hypotheses of theorem 1′, it is clear that theorem 1′ implies theorem 1. Note that it does not matter in theorem 1 whether or not we include in E the parameter values corresponding to saddle connections, since there are only countably many of them. Theorem 1′ shows a contrast with the situation for diffeomorphisms of the circle, by comparison with the following:

(2.2) THEOREM (Herman [**5**]). *Let f_t, $t \in [0, 1]$, be a C^1 path in the space of C^r diffeomorphisms of S^1 with the C^r topology, $r \geq 3$. Let $E = \{t | f_t$ is C^{r-2} conjugate to an irrational rotation$\}$. As long as the rotation number changes at all along the path, then $m(E) > 0$.*

(2.3) *Proof of theorem* 1′. We will assume, without loss of generality, that $a = 0$, i.e. f is constant on $[0, b]$, $0 < b < 1$, and that $f(0) = 0$ (see figure 2a). Consider the set $A \subset S^1 \times [0, 1]$ defined by

$$A = \{(y, t) | y \in f_t^{-n}[0, b] \quad \text{for some } n \geq 0\}.$$

(See figure 2b.) We write

$$A_y = \{t | (y, t) \in A\}, \qquad A^t = \{y | (y, t) \in A\}.$$

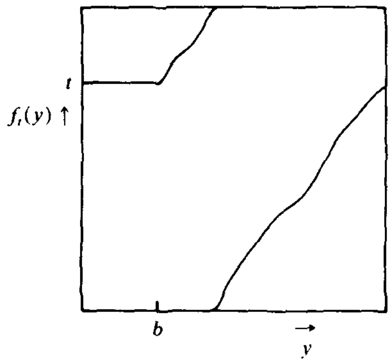

FIGURE 2a

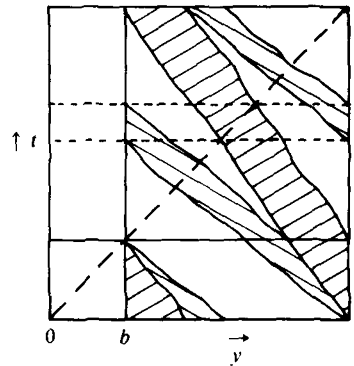

FIGURE 2b

Now

$$\begin{aligned}[0,1]-E &= \{t \mid f_t \text{ has a periodic point}\}\\ &= \{t \mid t \text{ is periodic}\} \qquad \text{since } t \text{ is always in the periodic orbit}\\ &= \{t \mid f_t^n(b)\in[0,b] \quad \text{for some } n\ge 1\} \qquad \text{since } f_t(b)=t\\ &= A'_b\end{aligned}$$

where for $y \in S^1$ we write

$$A'_y = \{t \mid y \in f_t^{-n}[0,b] \quad \text{for some } n \ge 1\}.$$

Note that $A'_y = A_y$ when $y \notin [0, b]$. We will show that in fact $m(A_y) = 1$ for all $y \in S^1$. The first step is

(2.4) LEMMA. $m(A^t) = 1$ *for all* $t \in [0, 1]$.

Proof. Fix $t \in [0, 1]$. We consider two distinct cases.

Case 1. $f_t^{-i}([0, b])$ intersects $f_t^{-j}[0, b]$ for some $i \neq j$. In this case f_t has a periodic point of period $n = |i-j|$. Consider the graph of f_t^n. This has n constant intervals separated by n intervals where $(f_t^n)'(y) > 1$. Hence f_t has exactly one attracting periodic orbit and one repelling periodic orbit. The attracting periodic orbit includes a point in $[0, b]$. Hence all but finitely many points end up in $[0, b]$ and so $\bigcup_{n=0}^{\infty} f_t^{-n}([0, b])$ has measure one; that is $m(A^t) = 1$.

Case 2. $f_t^{-i}[0, b] \cap f_t^{-j}[0, b]$ is empty when $i \neq j$. It follows that

$$|f_t^{-n}([0,b])| \le b/\lambda^n \to 0 \qquad \text{as } n \to \infty,$$

since $\lambda > 1$. Now suppose for a contradiction that $m(\bigcup_{n=0}^{\infty} f_t^{-n}([0, b])) < 1$. Let us write $a_n = \sum_{i=0}^{n} |f_t^{-i}[0, b]|$. Then the monotonic sequence $\{a_n\}_{n=1}^{\infty} \to l$ for some $l < 1$. To each $f_t^{-i}[0, b]$, $i \ge 1$, there corresponds a section of the graph of f of height $|f_t^{-i-1}([0, b])|$ and length $|f_t^{-i}([0, b])|$ (see figure 3). Choose N so large that $(1-a_N)/(1-a_{N+1}) < \lambda$, which is possible since $(1-a_N)/(1-a_{N+1}) \to 1$ as $N \to \infty$. After removing the sections of the graph of f corresponding to $f_t^{-i}([0, b])$, $0 \le i \le N$, there remain at most $N+2$ sections of total length $1-a_{N+1}$ and total height $1-a_N$.

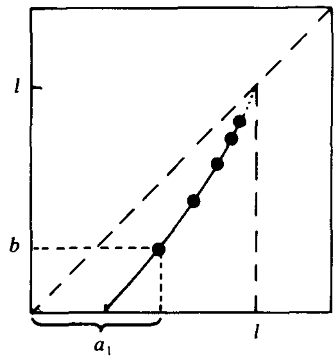

FIGURE 3

Since on each of these sections $f'(y) \geq \lambda$, by the Mean Value Theorem $(1-a_N)/(1-a_{N+1}) \geq \lambda$. This contradiction completes the proof of (2.4). □

(2.5) *Remark.* In particular, the Cantor set $\Sigma - W^s(P)$ mentioned in (1.2) has zero Lebesgue measure. This is because it is a set A^t for some t in case 2. This Cantor set also has zero Hausdorff dimension by arguments similar to those below using (2.7).

(2.6) LEMMA. *$m(A'_y)$ is a continuous function of y.*

Proof. Consider the functions $f_{y,n}(t):[0,1] \to S^1$ defined by $f_{y,n}(t) = f_t^n(y)$, $n \geq 1$. Thus

$$\begin{aligned} f_{y,1}(t) &= f(y)+t, \\ f_{y,2}(t) &= f(f_{y,1}(t))+t, \\ &\vdots \\ f_{y,n}(t) &= f(f_{y,n-1}(t))+t. \end{aligned}$$

Then $A'_y = \{t \mid f_{y,n}(t) \in [0,b]$ for some $n \geq 1\}$. Let us write

$$B_n(y) = \{t \mid f_{y,n}(t) \in [0,b] \quad \text{but } f_{y,m}(t) \notin [0,b] \text{ for } m<n\}.$$

Now $f_{y,n}(t)$ is a map of degree n. Hence B_n consists of at most n intervals each of length not more than $b/(\lambda^n+\lambda^{n-1}+\cdots+\lambda+1)$. Furthermore, since $f_{y,n}(t)$ changes continuously with y, the length of each of these intervals changes continuously with y. Hence $m(B_n(y))$ is a continuous function of y, and

$$m(B_n(y)) \leq mb/(\lambda^n+\cdots+\lambda+1) \qquad \text{for all } y.$$

But $A'_y = \bigcup_{n=1}^{\infty} B_n(y)$ and so $m(\bigcup_{j=1}^{n} B_j(y))$ converges uniformly to $m(A'_y)$. Hence $m(A'_y)$ is continuous as required. □

From (2.4) it follows by Fubini's Theorem (see e.g. [**12**, p. 143]) that $m(A_y)=1$ for almost all y. But for $y \notin [0,b]$, $m(A_y) = m(A'_y)$. Hence by (2.6) it follows that $m(A'_b)=1$. Since $A'_b = [0,1]-E$, as noted above, we have shown $m(E)=0$. To complete the proof we make use of:

(2.7) PROPOSITION (Besicovitch and Taylor [**2**]). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} a_n = 1$. Let $E \subset [0,1]$ be a set whose complement is a union of*

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intervals A_n *with* $m(A_n)=a_n$. *Then*

$$\dim_{\mathscr{H}}(E)\le\inf\left\{\beta\,\middle|\,\sum_{n=1}^{\infty}a_n^{\beta}<\infty\right\}.$$

Here $\dim_{\mathscr{H}}(E)$ denotes the Hausdorff dimension of E. (See [7] for the definition.) For any positive integer n let $\phi(n)$ be the number of positive integers coprime to n and less than n. Then as in the proof of (2.6) we see that the complement of E, A_b', consists of $\phi(n)$ intervals for each n, each of length not more than

$$b/(\lambda^n+\cdots+\lambda+1)=\frac{b(\lambda-1)}{(\lambda^{n+1}-1)}.$$

Now

$$(\lambda-1)\sum_{n=1}^{\infty}\frac{\phi(n)}{(\lambda^{n+1}-1)^{\beta}}<\infty\qquad\text{for all }\beta>0.$$

Hence by (2.7) we have $\dim_{\mathscr{H}}(E)=0$. The proof of theorem 1′ is complete. □

(2.8) *Example.* Consider the 2-parameter family of endomorphisms of the circle defined by

$$f_{a,t}(x)=\begin{cases}t & \text{if } x\le a,\\ \dfrac{x-a}{1-a}+t & \text{if } x\ge a,\end{cases}$$

for $x\in S^1$, $t\in[0,1]$ and $0<a<1$. (See figure 4.) For any fixed a, $f_{a,0}$ satisfies the hypothesis of theorem 1′. In the same way as Arnold and Herman do for diffeomorphisms ([**1**, p. 273], [**5**, p. 280]) we can consider level sets for the rotation number.

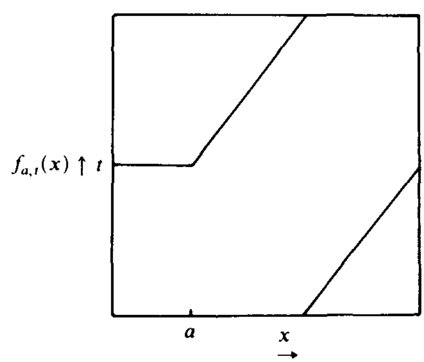

FIGURE 4

To each rational number a 'balloon' is attached, being the level set for that rotation number. (These balloons are analogous to the so-called 'Arnold tongues'.) Even though the width of each balloon tends to zero as a does, theorem 1′ tells us that for each $a_0>0$, the line at height a_0 intersects the balloons in a set of measure one (see figure 5).

3. *Proof of theorem* 2

We turn to the proof of theorem 2. From now on we will write f_Y for the Poincaré map of a vector field $Y\in\mathcal{N}$, and $\rho(f_Y)$ for its rotation number. As in [**10**], the

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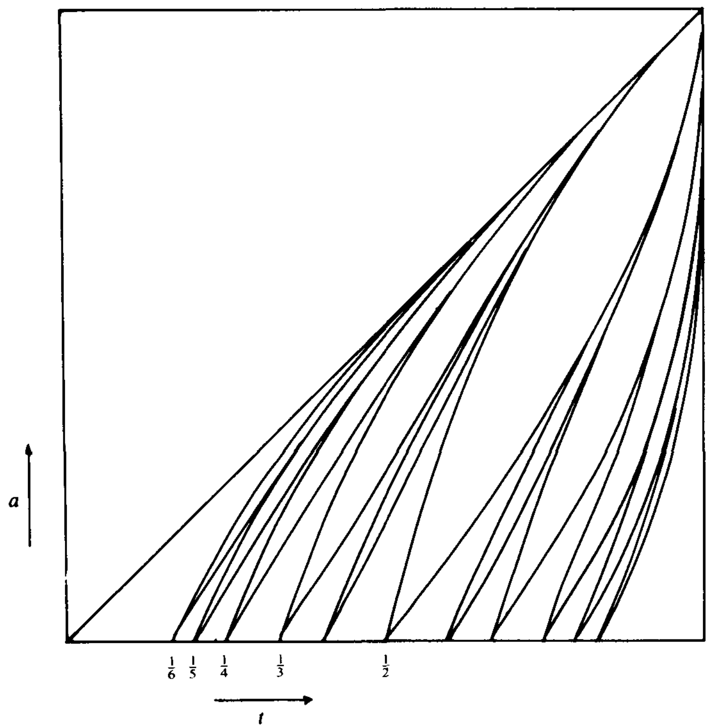

FIGURE 5

procedure is to construct, for each irrational $\alpha \in [0, 1)$, a C^1 function $g_\alpha : \mathcal{N} \to \mathbb{R}$ such that $g_\alpha^{-1}(0) = \{Y \in \mathcal{N} | \rho(f_Y) = \alpha\}$ and $Dg_\alpha(Y) \neq 0$. (Strictly, we will choose g_α to have image S^1.) We first do this for rational rotation numbers m/n, using the Implicit Function Theorem to construct g_n, h_n (for notational convenience we suppress the m's) which take respectively the lower and upper boundaries of the Morse–Smale class $\{Y | \rho(f_Y) = m/n\}$ onto zero (see figure 6). Then g_α is shown to be the C^1 limit of g_{n_i} or h_{n_i} when $(m_j/n_j) \to \alpha$. More precisely we show that for any $Y \in \mathcal{N}$ and a

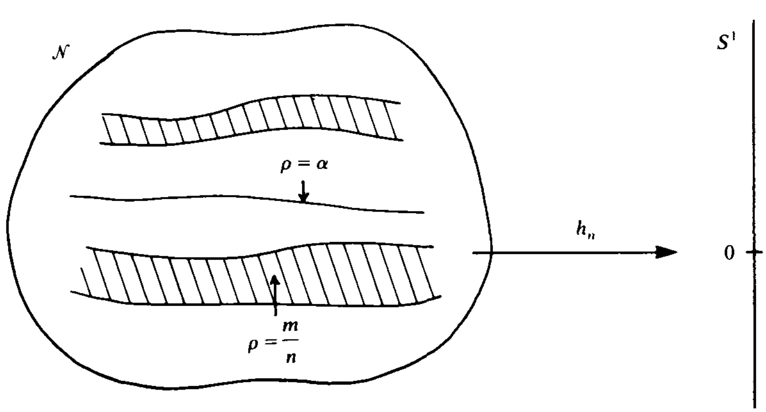

FIGURE 6

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Cauchy sequence $(m_j/n_j)_{j=1}^{\infty}$, the sequence $(Dg_{n_j}(Z))_{j=1}^{\infty}$ is uniformly Cauchy for Z in a neighbourhood of Y. This also shows that any two of these manifolds that are close are in fact C^1-close, which is crucial for theorem 3. It turns out that $g_\alpha(Y)$ will be the solution for t of $\rho(f_Y+t)=\alpha$; that is how far the graph of f_Y must be lifted to have rotation number α.

Let $[a_Y, b_Y]$ be the interval on which f_Y is constant, and let $f(y)=t_Y$ for all $y\in[a_Y, b_Y]$. We may consider a, b and t to be functions of Y, $\mathcal{N}\to S^1$ and by the Stable Manifold Theorem (see e.g. [6]) they are of class C^∞ (consider figure 7). We

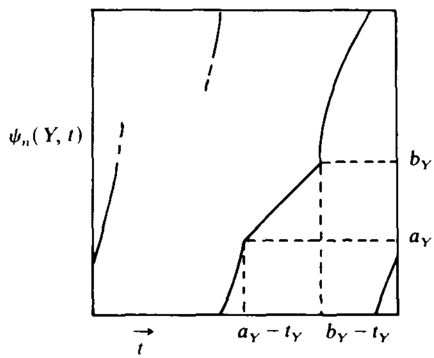

FIGURE 7

will write f_Y+t for the function defined by $(f_Y+t)(x)=f_Y(x)+t$. The f_Y+t has a point of period n exactly when $(f_Y+t)^{n-1}(t_Y+t)\in[a_Y, b_Y]$. Define $\psi_n:\mathcal{N}\times S^1\to S^1$ by

$$\psi_n(Y, t)=(f_Y+t)^{n-1}(t_Y+t).$$

So $\psi_n(Y,\cdot)$ is a circle map of degree n. The boundary points of the Morse-Smale class with rotation number m/n are given by the mth solutions of $\psi_n(Y,t)=a_Y$ or b_Y where the solutions are counted with increasing t, a_Y-t_Y and b_Y-t_Y being the zeroth.

Now in a neighbourhood of such a solution (Y, t_0), ψ_n is of class C^∞. This follows since outside $[a_Y, b_Y]$, f_Y is a 'genuine' Poincaré map and so the map $(Y,t)\mapsto f_Y(t)$ is of class C^∞ (see [**10**, p. 9]). Also it is clear that $D_2\psi_n(Y, t_0)>0$. Hence we may apply the Implicit Function Theorem ([**4**, p. 270]). This tells us that there is a neighbourhood $\mathcal{B}$ of Y in $\mathcal{N}$ and C^∞ functions g_n, $h_n:\mathcal{B}\to S^1$ satisfying

$$\psi_n(Y, g_n(Y))-a(Y)=0,$$
$$\psi_n(Y, h_n(Y))-b(Y)=0,$$

and hence

$$Dg_n(Y)=-\frac{D_1\psi_n(Y, g_n(Y))-Da(Y)}{D_2\psi_n(Y, g_n(Y))}. \tag{3.1}$$

We may write it in this form since $D_2\psi_n(\ ,\ )$ is a real number. Obviously we have the same formula for Dh_n with $Da(Y)$ replaced by $Db(Y)$.

Thus $g_n(Y)-a(Y)$ is a C^∞ function of Y taking the saddle connection fields with rotation number m/n to zero in S^1. Note that $D(g_n(Y)-a(Y))\neq 0$, since, for example, it is non-zero on our particular path ϕ in § 2. This shows that these vector fields with saddle connections form a C^∞ codimension one submanifold of $\mathcal{N}$, as is known from [**10**]. Note also that $g_n(Y)-a(Y)$ is defined for all $Y\in\mathcal{N}$ and C^∞ everywhere.

From now on fix $Y\in\mathcal{N}$ and choose a Cauchy sequence of rationals $\{m_j/n_j\}_{j=1}^\infty$. To prove theorem 2 it is sufficient to show that $\{Dg_{n_j}(Z)\}_{j=1}^\infty$ is uniformly Cauchy for Z in a neighbourhood of Y as $j\to\infty$ (see [**4**, p. 163]). It will be clear from the proof that we could have allowed any Cauchy sequence of saddle connection fields and their corresponding functions h_{n_j} or g_{n_j}. For simplicity we just consider the functions g_{n_j}.

We may write f_Y as a map of two variables: $f(Y,t)=f_Y(t)$. We rewrite Dg_n in terms of D_1f and D_2f. For $n\geq 2$, let $\mu(Y,t,s)=f_Y(t)+s$ and $\nu(Y,t)=(Y,\psi_{n-1}(Y,t),t)$. Then $\psi_n=\mu\circ\nu$. Hence, by the chain rule,

$$D\psi_n(Y,t)=(D_1f(\psi_{n-1}(Y,t))+D_2f(\psi_{n-1}(Y,t))\cdot D_1\psi_{n-1}(Y,t),\\ D_2f(\psi_{n-1}(Y,t))\cdot D_2\psi_{n-1}(Y,t)+1).$$

Hence from (3.1) and by induction

(3.2)
$$Dg_n(Y)=-\frac{\alpha_{n,n}+\beta_{n,n}(\alpha_{n,n-1}+\beta_{n,n-1}(\alpha_{n,n-2}\cdots(\alpha_{n,2}+\beta_{n,2}\cdot Dt(Y))\cdots)-Da(Y)}{\beta_{n,n}(\beta_{n,n-1}(\cdots(\beta_{n,2}+1)+1)\cdots)+1}$$

where $\alpha_{n,j}=D_1f(Y,x_{j-1})$ and $\beta_{n,j}=D_2f(Y,x_{j-1})$ and here $x_j=(f_Y+g_n(Y))^{j-1}(t_Y+g_n(Y))$ – that is the $(j-1)$st iterate of $t_Y+g_n(Y)$ which is in the periodic orbit for $f_Y+g_n(Y)$. Since the $\beta_{n,j}$'s are real numbers, (3.2) is the same as

$$\begin{aligned}Dg_n(Y)=&\frac{-\alpha_{n,n}}{\beta_{n,n}(\beta_{n,n-1}(\cdots)+1)+1}+\frac{-\alpha_{n,n-1}}{\beta_{n,n-1}(\cdots)+1+1/\beta_{n,n}}+\cdots\\ &\cdots+\frac{-Dt(Y)}{1+\dfrac{1}{\beta_{n,2}}+\cdots+\dfrac{1}{\beta_{n,2}\cdots\beta_{n,n}}}\\ &+\frac{Da(Y)}{\beta_{n,n}(\beta_{n,n-1}(\cdots)+1)+1}.\end{aligned}\tag{3.3}$$

So to calculate Dg_n we need to know the values of $D_1f(Y,x_j)$ and $D_2f(Y,x_j)$ for x_j in the periodic orbit of $f_Y+g_n(Y)$. To deal with the case when this orbit comes close to a_Y or b_Y we need:

(3.4) PROPOSITION.

$$-\frac{D_1f(Z,a_Z-\delta)}{D_2f(Z,a_Z-\delta)}\to Da(Z)\qquad \textit{as } \delta\to 0^+,$$
$$-\frac{D_1f(Z,b_Z+\delta)}{D_2f(Z,b_Z+\delta)}\to Db(Z)\qquad \textit{as } \delta\to 0^+.$$

(3.5) Corollary. *$\|D_1 f(Z,x)/D_2 f(Z,x)\|$ is uniformly bounded for $x \notin [a_Z, b_Z]$ and Z in a small neighbourhood of Y.*

Since the proof of (3.4) is quite technical, we defer it to the next section.

Fix $\gamma > 0$. We will find η so small that for Z in a small neighbourhood of Y,

$$|g_s(Z) - g_{\bar{s}}(Z)| < \eta \Rightarrow \|Dg_s(Z) - Dg_{\bar{s}}(Z)\| < \gamma,$$

where r/s and $\bar{r}/\bar{s}$ are elements of $\{m_j/n_j\}_{j=1}^{\infty}$. The first step is to show that we can ignore all but finitely many terms in (3.3). We write its jth term

$$T_n^j = \frac{-\alpha_{n,j}}{\beta_{n,j}(\beta_{n,j-1}(\cdots)+1)+1+\dfrac{1}{\beta_{n,j+1}}+\cdots+\dfrac{1}{\beta_{n,j+1}\cdots\beta_{n,n}}}.$$

This is defined for $1 \le j \le n$ by letting $\alpha_{n,1} = Dt(Y)$. Then if M is the uniform bound on $\|D_1 f(Z,x)/D_2 f(Z,x)\|$ for Z in a neighbourhood A of Y, guaranteed by (3.5) we have $\|T_n^j\| \le M/\lambda^{j-2}$. So we may choose n_0 so large that

(3.6)
$$\sum_{j=n_0}^{\infty} \frac{M}{\lambda^{j-2}} < \frac{\gamma}{8},$$

and also

(3.7)
$$\frac{\|Da(Z)\|}{\lambda^{n_0}} < \frac{\gamma}{8} \quad \text{and} \quad \frac{\|Db(Z)\|}{\lambda^{n_0}} < \frac{\gamma}{8},$$

for all $Z \in A$, if A is small enough. Next we choose δ_1 so small that if x_j comes within δ_1 of a_Z or b_Z then $\beta_{n,j}$ is large enough so that we may ignore T_n^i for $i > j$. Precisely, choose $\delta_1 > 0$ so that for all $Z \in A$ and $x \in (a_Z - \delta_1, a_Z)$ or $x \in (b_Z, b_Z + \delta_1)$ the following hold:

(3.8)
$$\frac{1}{D_2 f(Z,x)} \sum_{j=2}^{\infty} \frac{M}{\lambda^{j-2}} < \frac{\gamma}{8}.$$

(3.9)
$$\frac{\|Da(Z)\|}{D_2 f(Z,x)} < \frac{\gamma}{8} \quad \text{and} \quad \frac{\|Db(Z)\|}{D_2 f(Z,x)} < \frac{\gamma}{8}.$$

(3.10)
$$\left\|\frac{D_1 f(Z,x)}{D_2 f(Z,x)} + Da(Z)\right\| < \frac{\gamma}{8} \qquad \text{if } x \in (a_Z - \delta_1, a_Z),$$
$$\left\|\frac{D_1 f(Z,x)}{D_2 f(Z,x)} + Db(Z)\right\| < \frac{\gamma}{8} \qquad \text{if } x \in (b_Z, b_Z + \delta_1).$$

(3.11)
$$\frac{\lambda}{(\lambda-1)} \frac{1}{D_2 f(Z,x)} < \frac{\gamma}{8 \cdot M \cdot n_0}.$$

The inequalities (3.10) are possible by using (3.4). Now choose $0 < \delta_2 < \delta_1$. Let us write

$$x_j = (f_Z + g_s(Z))^{j-1}(t_Z + g_s(Z))$$
$$y_j = (f_Z + g_{\bar{s}}(Z))^{j-1}(t_Z + g_{\bar{s}}(Z)).$$

The idea is to choose η so small that $|g_s(Z) - g_{\bar{s}}(Z)| < \eta$ implies that if x_j is δ_2-close to a_Z or b_Z then y_j is δ_1-close to a_Z or b_Z. We may choose η small enough so that

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if $|g_s(Z)-g_{\bar{s}}(Z)|<\eta$ then the following holds:

(3.12) if x_j and y_j are inside $[b_Z+\delta_2, a_Z-\delta_2]$ then

$$\|T_s^j - T_{\bar{s}}^j\| \le \gamma/4n_0 \qquad \text{for } 1\le j\le n_0.$$

Note that for fixed j, to make $\|T_s^j-T_{\bar{s}}^j\|<\gamma/4n_0$, it is necessary only to make $\beta_{s,k}$ and $\beta_{\bar{s},k}$ close for finitely many k, say $N(j)$. Therefore we may choose n_1 so that if $\beta_{s,k}$ and $\beta_{\bar{s},k}$ are close for $1\le k\le n_1$ then $\|T_s^j-T_{\bar{s}}^j\|<\gamma/4n_0$ for $0\le j\le n_0$, where $n_1=\sup\{N(j)|1\le j\le n_0\}$. To take care of the case $s<n_1$ we also need

(3.13) if $K=\sup_{x\in[b_Z+\delta_2, a_Z-\delta_2], Z\in A}|D_2 f(Z,x)|$ then

$$\delta_2-\delta_1>\eta\cdot K^{n_2-1} \qquad \text{where } n_2=\max(n_0,n_1).$$

This tells us that if $x_j\notin[b_Z+\delta_2, a_Z-\delta_2]$ then $y_j\notin[b_Z+\delta_1, a_Z-\delta_1]$, for $1\le j\le n_2$. So using (3.11), the worst possible case for (3.12) is

$$\begin{aligned}\|T_s^j-T_{\bar{s}}^j\| &\le \left\|\frac{\alpha_{s,j}}{\beta_{s,j}(\beta_{s,j-1}(\cdots)+1)+1}-\frac{\alpha_{\bar{s},j}}{\beta_{\bar{s},j}(\beta_{\bar{s},j-1}(\cdots)+1)+1+\gamma/8Mn_0}\right\|\\ &\le \left\|\frac{(\beta_{\bar{s},j}(\beta_{\bar{s},j-1}(\cdots)+1)\alpha_{s,j}-(\beta_{s,j}(\cdots)+1)\alpha_{\bar{s},j})}{(\beta_{s,j}(\cdots)+1)(\beta_{\bar{s},j}(\cdots)+1+\gamma/8Mn_0)}\right\|\\ &\quad+\left\|\frac{\alpha_{s,j}\gamma/8Mn_0}{(\beta_{s,j}(\cdots)+1)(\beta_{\bar{s},j}(\cdots)+1+\gamma/8Mn_0}\right\|\\ &\le \gamma/4n_0\end{aligned}$$

if $\beta_{s,k}$ and $\beta_{\bar{s},k}$ are close for $1\le k\le j$ and $\alpha_{s,j}$ and $\alpha_{\bar{s},j}$ are close, which is true if $|g_s(Z)-g_{\bar{s}}(Z)|$ is small enough.

In a similar way we can ensure that if x_j and y_j are both in $(a_Z-\delta_1, a_Z]$ or $[b_Z, b_Z+\delta_1)$ then for $|g_s(Z)-g_{\bar{s}}(Z)|<\eta$ the following holds:

$$\|Da(Z)\|\left(\frac{1}{\beta_{s,j}(\cdots)+1+(1/\beta_{s,j+1})+\cdots+(1/\beta_{s,j+1}\cdots\beta_{s,s})}\right.$$

(3.14)
$$\left.-\frac{1}{\beta_{\bar{s},j}(\cdots)+1+(1/\beta_{\bar{s},j+1})+\cdots+(1/\beta_{\bar{s},j+1}\cdots\beta_{\bar{s},\bar{s}})}\right)<\frac{\gamma}{8},$$

for $1\le j\le n_0$ and $Z\in A$. We now claim that

$$|g_s(Z)-g_{\bar{s}}(Z)|<\eta\Rightarrow\|Dg_s(Z)-Dg_{\bar{s}}(Z)\|<\gamma.$$

We consider three cases:

Case 1. x_j and y_j are in $[b_Z+\delta_2, a_Z-\delta_2]$ for $1\le j\le n_0$. Consider the equation (3.3). The terms T_s^j, $T_{\bar{s}}^j$ are taken care of, for $j>n_0$, by (3.6), and each final term by (3.7). The other terms are dealt with by (3.12). Thus

$$\|Dg_s(Z)-Dg_{\bar{s}}(Z)\|<\frac{\gamma}{4}+\frac{\gamma}{8}+\frac{\gamma}{8}+\frac{\gamma}{8}+\frac{\gamma}{8}<\gamma.$$

If case 1 does not happen let j_0 be the first j where it fails. Suppose $x_{j_0}\notin[b_Z+\delta_2, a_Z-\delta_2]$. Clearly the case $y_{j_0}\notin[b_Z+\delta_2, a_Z-\delta_2]$ is similar.

Case 2. $x_{j_0}=a_Z$. This is the case $s=j_0$, since there is a point of period j_0. If also $y_{j_0}=a_Z$ then Dg_s and $Dg_{\bar{s}}$ differ only in their final terms and this case is clearly all right. Otherwise, by (3.13), $y_{j_0}\in(a_Z-\delta_1, a_Z)$. For $j\le j_0$ the terms T_s^j, $T_{\bar{s}}^j$ are dealt with by (3.12) and for $j \geq j_0+1$, $T^j_{\bar{s}}$ are dealt with by (3.8). Hence

$$\|Dg_s(Z)-Dg_{\bar{s}}(Z)\| < \frac{j_0\cdot\gamma}{4n_0}+\frac{\gamma}{8}+\left\|\frac{Da(Z)}{\beta_{s,s}(\beta_{s,s-1}(\cdots)+1)+1}-T^{j_0+1}_{\bar{s}}\right\| + \frac{\|Da(Z)\|}{\beta_{\bar{s},\bar{s}}(\beta_{\bar{s},\bar{s}-1}(\cdots)+1)+1}.$$

Since $y_{j_0}\in(a_Z-\delta_1, a_Z)$ it follows by (3.10) that

$$\left\|\frac{D_1f(Z,y_{j_0})}{D_2f(Z,y_{j_0})}+Da(z)\right\|<\frac{\gamma}{8} \quad \text{or} \quad \left\|\frac{\alpha_{\bar{s},j_0+1}}{\beta_{\bar{s},j_0+1}}+Da(Z)\right\|<\frac{\gamma}{8}.$$

Thus

$$\left\|T^{j_0+1}_{\bar{s}}-\frac{Da(Z)}{\beta_{\bar{s},j_0}(\cdots)+1+\cdots+(1/\beta_{\bar{s},j_0+1}\cdots\beta_{\bar{s},\bar{s}})}\right\|<\frac{\gamma}{8},$$

and so

$$\begin{aligned}\left\|\frac{Da(Z)}{\beta_{s,s}(\beta_{s,s-1}(\cdots)+1)+1}-T^{j_0+1}_{\bar{s}}\right\| &\leq \frac{\gamma}{8}+\left\|\frac{Da(Z)}{\beta_{s,s}(\beta_{s,s-1}(\cdots)+1)+1}-\frac{Da(Z)}{\beta_{\bar{s},j_0}(\cdots)+1+\cdots+(1/\beta_{\bar{s},j_0+1}\cdots\beta_{\bar{s},\bar{s}})}\right\| \\ &< \frac{\gamma}{8}+\frac{\gamma}{8} \qquad \text{by (3.14).}\end{aligned}$$

Finally, by (3.7)

$$\frac{\|Da(Z)\|}{\beta_{\bar{s},\bar{s}}(\beta_{\bar{s},\bar{s}-1}(\cdots)+1)+1}<\frac{\gamma}{8}.$$

Thus

$$\|Dg_s(Z)-Dg_{\bar{s}}(Z)\|<\frac{\gamma}{4}+\frac{\gamma}{8}+\frac{\gamma}{8}+\frac{\gamma}{8}<\gamma.$$

Case 3. $x_{j_0}\in(a_Z-\delta_2, a_Z)$. Clearly the case $x_{j_0}\in(b_Z, b_Z+\delta_2)$ is similar. By (3.13), $y_{j_0}\in(a_Z-\delta_1, a_Z)$. Using (3.12) and (3.8) as in case 2, we have

$$\begin{aligned}\|Dg_s(Z)-Dg_{\bar{s}}(Z)\| < \frac{j_0\cdot\gamma}{4n_0}+\frac{\gamma}{8}+\frac{\gamma}{8} &+\left\|\frac{\alpha_{s,j_0+1}}{\beta_{s,j_0+1}(\cdots)+1+\cdots+(1/\beta_{s,j_0}\cdots\beta_{s,s})}\right. \\ &\left.-\frac{\alpha_{\bar{s},j_0+1}}{\beta_{\bar{s},j_0+1}(\cdots)+1+\cdots+(1/\beta_{\bar{s},j_0}\cdots\beta_{\bar{s},\bar{s}})}\right\| \\ &+\|Da(Z)\|\left(\frac{1}{\beta_{s,s}(\cdots)+1}-\frac{1}{\beta_{\bar{s},\bar{s}}(\cdots)+1}\right).\end{aligned}$$

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Since both x_{j_0} and y_{j_0} are in $(a_Z-\delta_1, a_Z)$ it follows from (3.10) that

$$\left\| \frac{\alpha_{s,j_0+1}}{\beta_{s,j_0+1}(\cdots)+1+\cdots+(1/\beta_{s,j_0+2}\cdots\beta_{s,s})} - \frac{\alpha_{\bar{s},j_0+1}}{\beta_{\bar{s},j_0+1}(\cdots)+1+\cdots+(1/\beta_{\bar{s},j_0+2}\cdots\beta_{\bar{s},\bar{s}})}\right\|$$
$$< \frac{\gamma}{8}+\frac{\gamma}{8}+\|Da(Z)\|\left(\frac{1}{\beta_{s,j_0}(\cdots)+1+\cdots+(1/\beta_{s,j_0+1}\cdots\beta_{s,s})} - \frac{1}{\beta_{\bar{s},j_0}(\cdots)+1+\cdots+(1/\beta_{\bar{s},j_0+1}\cdots\beta_{\bar{s},\bar{s}})}\right)$$
$$< \frac{\gamma}{4}+\frac{\gamma}{8} \qquad \text{by (3.14).}$$

By applying (3.7) to the final term we now have

$$\|Dg_s(Z)-Dg_{\bar{s}}(Z)\| < \frac{\gamma}{2}+\frac{\gamma}{4}+\frac{\gamma}{8}+\frac{\gamma}{8}=\gamma.$$

Thus theorem 2 is proved. □

4. *Proof of* (3.4) *and* (3.5)

The idea of the proof of (3.4) is to show that near to the saddle separatrix $f(Z, x)$ behaves like the Poincaré map of a linear saddle. But for a linear vector field L with matrix

$$\begin{pmatrix} \lambda(L) & 0 \\ 0 & -\mu(L) \end{pmatrix}$$

its Poincaré map $p(L, x)$ behaves like $x \mapsto x^{\mu(L)/\lambda(L)}$. Hence $D_1 p(L, x)/D_2 p(L, x)$ behaves like

$$\frac{D(\mu(L)/\lambda(L)) \cdot x^{\mu(L)/\lambda(L)} \log x}{x^{(\mu(L)/\lambda(L))-1}} = D(\mu(L)/\lambda(L)) \cdot x \log x \to 0$$

as $x \to 0$. Because the saddle point of Z moves with Z the term $Da(Z)$ or $Db(Z)$ enters as a correction term. Let us call the saddle point of Z, $S(Z)$. By Sell's Linearization Theorem [**9**] there is a C^1 map $l_Z: U \to \mathbb{R}^2$ from some neighbourhood U of $S(Z)$ conjugating the flow to its linear part $L_Z = DZ_{S(Z)}$:

$$l_Z(\psi_Z(x, t)) = \psi_{L_Z}(l_Z(x), t)$$

if x and $\psi_Z(x, t) \in U$. (Here $\psi(\ \ ,\ \ )$ denotes the flow induced by the vector field X.) Furthermore, Sell shows that the linearization can be chosen to depend in a C^1 way upon Z ([**9**, p. 64]), so $l(Z, t)$ is a C^1 map. Therefore we choose $l(Z, t)$ first for $Z = Y$. Then the neighbourhood referred to in (3.5) in which Z is allowed to lie is the domain of definition of the first component of l. Call this neighbourhood $\mathcal{B}$. For points inside $\bigcup_{t\in\mathbb{R}} \psi_Z(t, U)$ we can extend l_Z, so long as its domain of definition does not overlap itself, by setting

$$l_Z(x) = \psi_{L_Z}(l_Z(\psi_Z(x, t), -t),$$

where t is chosen so that $\psi_Z(x, t) \in U$ and so that the partial orbit joining x to $\psi_Z(x, t)$ has not passed through U. Hence we may extend the domain of l_Z so far

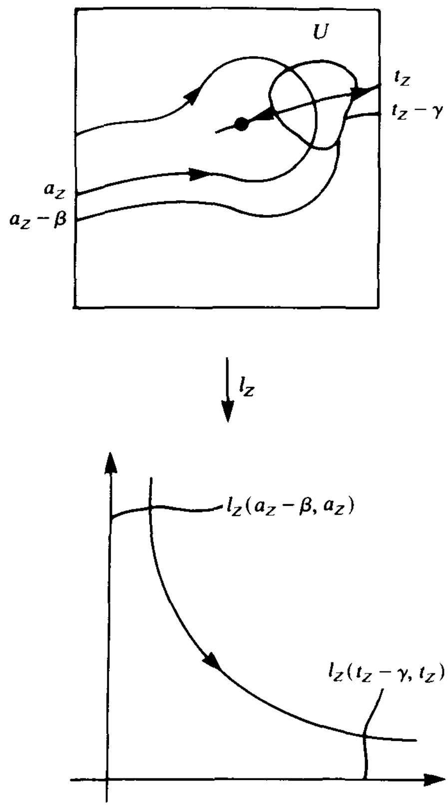

FIGURE 8a

as to include intervals in Σ, $(a_Z-\beta, a_Z)$ and $(t_Z-\gamma, t_Z)$ for small β, γ. (See figure 8a.) Note that $l(Z, t)$ is still a C^1 map on this extended domain. Now for $t\in(a_Z-\beta, a_Z)$ let

$$l(Z,t)=\begin{pmatrix} x(Z,t) \\ \sigma(x(Z,t)) \end{pmatrix},$$

and for $t\in(t_Z-\gamma, t_Z)$ let

$$l(Z,t)=\begin{pmatrix} \tau(y(Z,t)) \\ y(Z,t) \end{pmatrix},$$

where $\sigma:(0,\bar\beta)\to\mathbb{R}$ and $\tau:(0,\bar\gamma)\to\mathbb{R}$ are C^1 functions for some small $\bar\beta$ and $\bar\gamma$. (See figure 8b.) So for $t\in(a_Z-\beta, a_Z), f_Z(t)$ progresses thus:

$$t \underset{l_Z}{\longmapsto} \begin{pmatrix} x(Z,t) \\ \sigma(x(Z,t)) \end{pmatrix} \underset{p_Z}{\longmapsto} \begin{pmatrix} \tau_Z(p_Z(x(Z,t))) \\ p_Z(x(Z,t)) \end{pmatrix} \underset{l_Z^{-1}}{\longmapsto} f_Z(t)$$

where $p_Z:[0,\bar\beta)\to[0,\bar\gamma)$ is the Poincaré map of L_Z from $l_Z(a_Z-\beta, 0)$ to $l_Z(t_Z-\gamma, 0)$.

Let $\Pi_x: l_Z(a_Z-\beta, a_Z]\to\mathbb{R}$ and $\Pi_y: l_Z(t_Z-\gamma, t_Z)\to\mathbb{R}$ be the projections:

$$\Pi_x\begin{pmatrix} x \\ \sigma_Z(x) \end{pmatrix}=x \quad\text{and}\quad \Pi_y\begin{pmatrix} \tau_Z(y) \\ y \end{pmatrix}=y.$$

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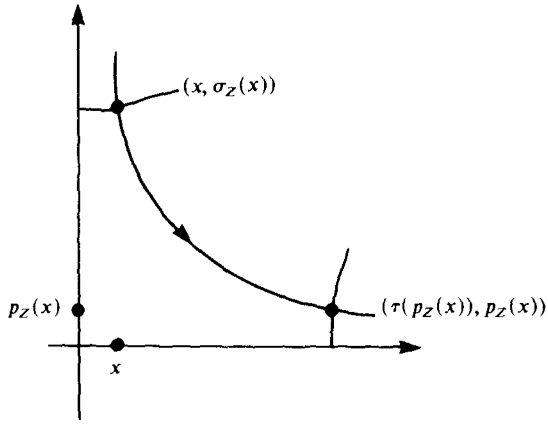

FIGURE 8b

Then if we set

$$j(Z, t) = \Pi_x \circ l(Z, t)$$
$$k(Z, t) = l_Z^{-1} \circ \Pi_y^{-1}(Z, t)$$

we have

(4.1) $$f(Z, t) = k(Z, p_Z(j(Z, t))).$$

Let us write $F_\delta(Z) = f(Z, a_Z - \delta)$. Then

(4.2) $$DF_\delta(Z) = D_1 f(Z, a_Z - \delta) + D_2 f(Z, a_Z - \delta) \cdot Da(Z).$$

On the other hand we also have

$$F_\delta(Z) = k(Z, p_Z(j(Z, a_Z - \delta)))$$

and so

(4.3) $$DF_\delta(Z) = D_1 k + D_2 k \cdot D_1 p + D_2 k \cdot D_2 p \cdot D_1 j + D_2 k \cdot D_2 p \cdot D_2 j \cdot Da(Z).$$

(In order to simplify the equations, from now on we are omitting the points at which Dk, Dj and Dp are evaluated.) From (4.1) we also have

(4.4) $$D_2 f(Z, t) = D_2 k \cdot D_2 p \cdot D_2 j.$$

Together (4.2), (4.3) and (4.4) give us

(4.5) $$\frac{D_1 f(Z, a_Z - \delta)}{D_2 f(Z, a_Z - \delta)} = \frac{D_1 k}{D_2 k \cdot D_2 p \cdot D_2 j} + \frac{D_1 p}{D_2 p \cdot D_2 j} + \frac{D_1 j}{D_2 j}.$$

We consider each term on the right-hand side separately.

$$\lim_{\delta \to 0} \frac{D_1 k}{D_2 k \cdot D_2 p \cdot D_2 j} = \frac{D_1 k(Z, 0)}{D_2 k(Z, 0) D_2 j(Z, 0)} \cdot \lim_{\delta \to 0} \frac{1}{D_2 p}$$
$$= 0,$$

as we will see below. Note that this convergence is uniform in $\mathscr{B}$ since j and k are bounded in $\mathscr{B}$.

$$\lim_{\delta \to 0} \frac{D_1 p(Z, j(Z, a_Z - \delta))}{D_2 p(Z, j(Z, a_Z - \delta))} = \lim_{x \to 0} \frac{D_1 p(Z, x)}{D_2 p(Z, x)} \cdot \frac{1}{D_2 j(Z, a_Z)}.$$

We consider this case below. For the first term let us write $J(Z,\delta)=j(Z,a_Z-\delta)$. Then

$$\frac{D_1J(Z,\delta)}{D_2J(Z,a_Z-\delta)}=\frac{D_1j}{D_2j}+Da(Z).$$

Since $D_1J(Z,0)=0$ and J is C^1, we deduce $\lim_{\delta\to0}D_1j/D_2j=-Da(Z)$. Thus from (4.5) we now have

$$\text{(4.6)}\qquad \lim_{\delta\to0}\frac{D_1f(Z,a_Z-\delta)}{D_2f(Z,a_Z-\delta)}=-Da(Z)+\frac{1}{D_2j(Z,a_Z)}\lim_{x\to0}\frac{D_1p(Z,x)}{D_2p(Z,x)}.$$

Hence it now suffices to examine $p(Z,x)$, the Poincaré map of L_Z. We may assume L_Z has the form

$$\begin{pmatrix}\lambda(Z) & 0\\ 0 & -\mu(Z)\end{pmatrix}.$$

Let us write $\alpha(Z)=\mu(Z)/\lambda(Z)$. Then we may integrate the vector field L_Z to find

$$\text{(4.7)}\qquad p(Z,x)=\sigma_Z(x)\left(\frac{x}{\tau_Z(p_Z(x))}\right)^{\alpha(Z)}.$$

Differentiating we get

$$D_1p(Z,x)=x^{\alpha(Z)}\left[\frac{D_1\sigma\cdot\tau+\sigma\cdot Da[\log x-\log\tau]\tau-\sigma\cdot\alpha\cdot D_1\tau}{\tau^{\alpha(Z)+1}+D_2\tau\cdot\sigma\cdot x^{\alpha(Z)}}\right],$$

$$D_2p(Z,x)=x^{\alpha(Z)-1}\left[\frac{D_2\sigma\cdot\tau\cdot x+\sigma\cdot\tau\cdot\alpha}{\tau^{\alpha(Z)+1}+\sigma\cdot\alpha\cdot x^{\alpha(Z)}\cdot D_2\tau}\right],$$

and hence that

$$\lim_{x\to0}\frac{D_1p(Z,x)}{D_2p(Z,x)}=\lim_{x\to0}\frac{x[(D_1\sigma\cdot\tau+\sigma\cdot Da[\log x-\log\tau]\tau-\sigma\cdot\alpha\cdot D_1\tau)\cdot\tau^{\alpha+1}]}{\tau^{\alpha+1}\cdot\sigma\cdot\tau\cdot\alpha}$$
$$=0,$$

since $\lim_{x\to0}x\log x=0$. Notice that since σ, τ and α are bounded in $\mathcal{B}$ this convergence is uniform in $\mathcal{B}$. By (4.6) the proof of (3.4) is complete. □

Proof of (3.5). Consider (4.5). We have already noted that the first two terms on the right hand side are bounded uniformly in $\mathcal{B}$. But the final term is defined and continuous for $\delta=0$ so is also bounded in $\mathcal{B}$. Thus (3.5) is proved. □

5. *Proof of theorem* 3

Let $C^1(I,\mathfrak{X})$ be the space of C^1 paths in $\mathfrak{X}^\infty(T^2)$ with the C^1 topology. We recall what it means for a path to be stable (see e.g. [**11**]).

(5.1) *Definition.* Two paths $X, Y\in C^1(I,\mathfrak{X})$ are (*mildly*) *equivalent* if there exists a reparametrising homeomorphism $h: I\to I$ such that $X(t)$ and $Y(h(t))$ are topologically equivalent vector fields. X and Y are *strongly equivalent* if in addition the topological equivalence between $X(t)$ and $Y(h(t))$ can be chosen to change continuously with t. X is a *stable path* if there is a neighbourhood U of X in $C^1(I,\mathfrak{X})$ such that all paths Y in U are strongly equivalent to X.

Before showing that our chosen path ϕ is stable we note that all Cherry fields with the same rotation number are topologically equivalent. Similarly all Morse–Smale fields in the neighbourhood $\mathcal{N}$ with the same rotation number are topologically equivalent, as are any fields in the corresponding boundaries – those with saddle connections (only we must distinguish between 'lower' and 'upper' saddle connections). The topological equivalence can be constructed in essentially the same way in all cases, as follows.

Choose X, $Y \in \mathcal{N}$ of the same type and with the same rotation number. We first restrict to the transverse circle Σ. Call the successive inverse intersections of the stable manifolds of the sinks of X, Y with Σ, I_j, $\tilde{I}_j$ respectively. If the fields X and Y are Morse-Smale, I_j and $\tilde{I}_j$ will have two components for large enough j. The restriction of the topological equivalence to Σ is defined by taking I_j to $\tilde{I}_j$ affinely according to the ratio of their lengths (naturally we deal with the two components of I_j and $\tilde{I}_j$ separately, if necessary). Since the stable manifold of each sink is dense in Σ (cf. (1.2)) the map can be extended uniquely to the whole of Σ. Because X and Y have the same rotation number the I_j's, $\tilde{I}_j$'s intersect Σ in the same order, and so the map is indeed continuous. The map may now be extended to the whole of Σ, but care must be taken near the saddle separatrices.

For points x, y in the same orbit we let $l(xy)$ denote the arc length of the orbit between x and y, using the metric induced by the Euclidean metric. Then put

$$l(a_X S_X) = \alpha \qquad l(a_Y S_Y) = \hat{\alpha}$$
$$l(S_X t_X) = \beta \qquad l(S_Y t_Y) = \hat{\beta}$$
$$l(S_X P_X) = \gamma \qquad l(S_Y P_Y) = \hat{\gamma}$$
$$l(b_X S_X) = \delta \qquad l(b_Y S_Y) = \hat{\delta}$$

(see figure 9). Let $x \in \Sigma$ be close to, and below, a_X. Then the arc of orbit $xf_X(x)$ is

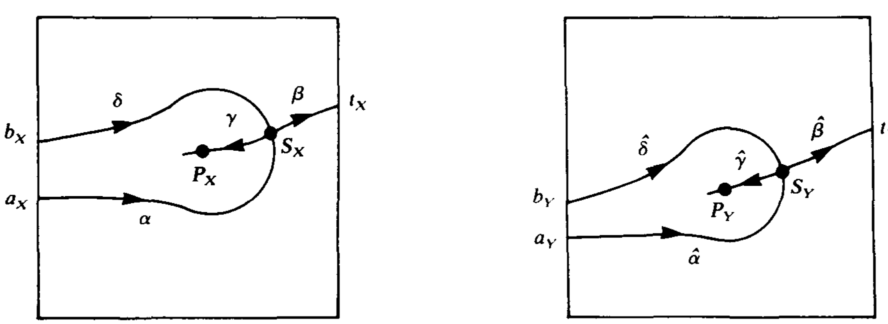

FIGURE 9

mapped onto an arc $yf_Y(y)$ determined by the map on Σ. Let $l(xf_X(x)) = r$ and $l(yf_Y(y)) = s$. We split $xf_X(x)$ into two parts of length $r\alpha/(\alpha+\beta)$ and $r\beta/(\alpha+\beta)$ and $yf_Y(y)$ into two parts of length $s\hat{\alpha}/(\hat{\alpha}+\hat{\beta})$ and $s\hat{\beta}/(\hat{\alpha}+\hat{\beta})$. The first and second parts of $xf_X(x)$ are then mapped onto the corresponding parts of $yf_Y(y)$ according to ratio of arc length. For points close to, and above, a_X we do a similar procedure, this time splitting into parts of ratio $\alpha/(\alpha+\gamma)$ and $\gamma/(\alpha+\gamma)$ for X and of ratio $\hat{\alpha}/(\hat{\alpha}+\hat{\gamma})$ and $\hat{\gamma}/(\hat{\alpha}+\hat{\gamma})$ for Y. We then do a similar procedure for points close to b_X. Finally for points away from a_X and b_X we map arcs of trajectories according to their whole lengths between intersections of Σ and smooth these separate maps together by a partition of unity subordinate to a suitable cover of Σ.

Recall now the path $\phi \in C^1(I, \mathcal{N})$ examined in § 2 and defined so that $f_{\phi(t)} = f_{\phi(0)} + t$. ϕ intersects each submanifold of Cherry fields or fields with a saddle connection at exactly one point. By the above remarks, to show that ϕ is mildly stable it suffices to show that a nearby path also meets every submanifold exactly once. This is true because ϕ intersects every submanifold transversely and furthermore when these submanifolds are close, they are C^1-close. Precisely, recall the maps $g_\alpha : \mathcal{N} \to S^1$ defined in § 2 for any $\alpha \in [0, 1)$. g_α takes the submanifold with rotation number α onto 0 (for α rational, g_α is really two maps). These maps tell us how much to add to the Poincaré map of the field to get the right rotation number α. Then it follows that $g_\alpha \circ \phi = -\mathrm{id} + K_\alpha$ where $K_\alpha = \phi^{-1}(g_\alpha^{-1}(0))$. So in particular this shows ϕ is transverse to the submanifold $g_\alpha^{-1}(0)$. Furthermore, as already noted if g_α and g_β are close in C^0 sense, they are also close in the C^1 sense. Hence if $t_0 = (g_\alpha \circ \phi)^{-1}(0)$ there are neighbourhoods V_α of ϕ in $C^1(I, \mathcal{N})$ and U_α of t_0 in I such that if $\tilde{\phi} \in V_\alpha$ then $\tilde{\phi}$ intersects exactly once the same submanifolds as ϕ does, for $t \in U_\alpha$. We may find such a U_α for every $t \in E$, the bifurcation set of ϕ and then take a finite cover of E by U_α's, $\{U_1, \dots, U_n\}$. Then $V = \bigcap_{i=1}^n V_i$ is a neighbourhood of ϕ such that $\tilde{\phi} \in V$ crosses every submanifold exactly once. Thus ϕ is mildly stable.

We now show that the topological equivalence changes continuously with t. First note that it is sufficient to do this on the restriction to Σ. So choose $\tilde{\phi} \in V$. We reparametrize $\tilde{\phi}$ by mapping the parameter linearly between corresponding bifurcation points. Let q_t be the topological equivalence between $\phi(t)$ and $\tilde{\phi}(h(t))$, where h is this reparametrizing homeomorphism. Fix $t_0 \in I$. Let $I_1, I_2, \dots,$ and $\tilde{I}_1, \tilde{I}_2, \dots$ be the successive inverse intersections with Σ of the stable manifolds of $\phi(t_0)$ and $\tilde{\phi}(h(t_0))$ respectively. Again, these 'intervals' may have two components if $\phi(t_0)$ is Morse–Smale. Let $\varepsilon > 0$. We consider the three cases:

(a) $\phi(t_0)$ and $\tilde{\phi}(h(t_0))$ are Cherry fields. Choose N so large that $I_1, \dots, I_N$ and $\tilde{I}_1, \dots, \tilde{I}_N$ both have total lengths at least $1 - \varepsilon/4$. Then choose δ so small that if $|t - t_0| < \delta$, the corresponding intervals for $\phi(t)$, $I_{1,t}, \dots, I_{n,t}$, remain disjoint and let

$$M = \sup_{\substack{|t-t_0|<\delta \\ 1 \le j \le N}} \frac{|\tilde{I}_{t,j}|}{|I_{t,j}|}.$$

Then choose $\eta < \delta$ so small that the boundary points of $I_{t,1}, \dots, I_{t,N}$ and $\tilde{I}_{t,1}, \dots, \tilde{I}_{t,N}$ do not change by more than $\varepsilon/4M$ while $|t - t_0| < \eta$. Then $|t - t_0| < \eta \Rightarrow \|q_{t_0} - q_t\| < \varepsilon$ (see below).

(b) $\phi(t_0)$ and $\tilde{\phi}(h(t_0))$ are Morse–Smale. Choose N as in case (a). Choose δ so small that if $|t - t_0| < \delta$ then $\phi(t)$ is in the same Morse–Smale class. Put

$$M = \sup_{\substack{|t-t_0|<\delta/2 \\ 1 \le j \le N}} \frac{|\tilde{I}_{t,j}|}{|I_{t,j}|}$$

and choose $\eta < \delta/2$ so small that for $|t - t_0| < \eta$ the endpoints of $I_{t,1}, \dots, I_{t,N}$ and $\tilde{I}_{t,1}, \dots, \tilde{I}_{t,N}$ do not change by more than $\varepsilon/4M$. Then $|t - t_0| < \eta \Rightarrow \|q_{t_0} - q_t\| < \varepsilon$ (see below).

(c) $\phi(t_0)$ and $\phi(h(t_0))$ have saddle connections. Choose N and M as in case (a). $\phi(t_0)$ is on the boundary of a certain Morse-Smale class. For $t<t_0$ suppose $\phi(t)$ fails to fall into this class. Then we may choose δ so small that for $|t-t_0|<\delta$, $I_{t,1}, I_{t,2}, \ldots, I_{t,N}$ and $\tilde{I}_{t,1}, \ldots, \tilde{I}_{t,N}$ remain disjoint and their endpoints move by no more than $\varepsilon/4M$. On the other hand, for $t>t_0$, $\phi(t)$ is in the Morse-Smale class. Choose γ so that for $|t-t_0|<\gamma$ the large components of $I_{t,1}, \ldots, I_{t,N}$ and $\tilde{I}_{t,1}, \ldots, \tilde{I}_{t,N}$ still have total length not less than $1-\varepsilon/2$, and their endpoints move by no more than $\varepsilon/4M$. Put $\eta=\min(\delta, \gamma)$. Then $|t-t_0|<\eta \Rightarrow \|q_{t_0}-q_t\|<\varepsilon$.

To see this final step in each case, consider any $x\in\Sigma$. It must satisfy one of the following:

(i) x remains outside any interval. Then $g_t(x)$ is outside any interval so is constrained to move by not more than $\varepsilon/4+\varepsilon/4M$ in case (a) or (b), or $\varepsilon/2+\varepsilon/4M$ in case (c).

(ii) x remains inside a single interval. Consider figure 10. The graph of q_t for $|t-t_0|<\eta$ connects two points in the boxes. It is not hard to see that for the worst possible x we have

$$|q_t(x)-q_{t_0}(x)|<2\cdot\varepsilon/4M+M\cdot\varepsilon/4M<\varepsilon.$$

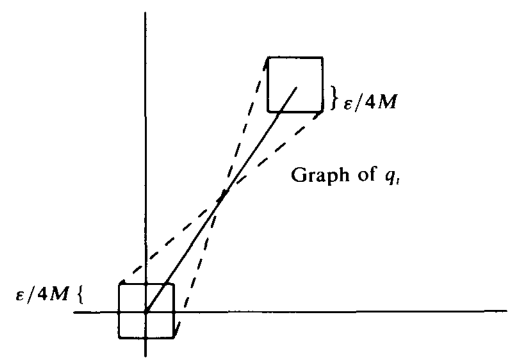

FIGURE 10

(iii) The endpoint of an interval crosses x. Call e_t the endpoint directly below x and suppose it was above x for $t=t_0$. Then $d(x, e_{t_0})<\varepsilon/4M$ and $d(x, e_t)<\varepsilon/4M$. In cases (a) or (b) we have

$$d(q_{t_0}(x), \tilde{e}_{t_0})<M\cdot\varepsilon/4M+\varepsilon/4 \quad \text{and} \quad d(q_t(x), \tilde{e}_t)<\varepsilon/4,$$

where $\tilde{e}_t$ is the corresponding endpoint for $\tilde{\phi}(t)$. Since we also have $d(e_{t_0}, e_t)<\varepsilon/4M$ it follows that

$$d(q_{t_0}(x), q_t(x))<\varepsilon/2+\varepsilon/4+\varepsilon/4M\leq\varepsilon$$

(see figure 11). In case (c) the same argument holds, except we may need to replace $\varepsilon/4$ by $\varepsilon/2$ in one place.

Thus we have shown that q_t is continuous in t and so ϕ is a stable path as claimed.

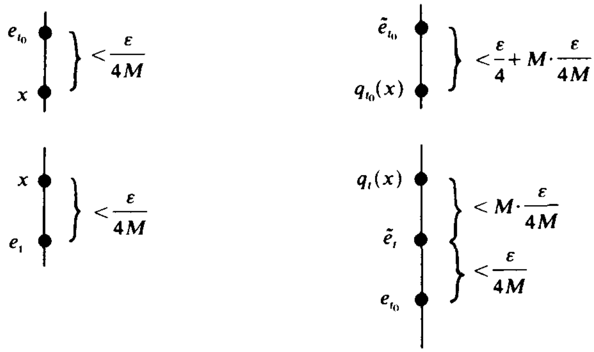

FIGURE 11

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REFERENCES

[1] V. I. Arnold. Small denominators, I. Mappings of the circumference onto itself. *AMS Translations, Series* 2, **46**, 213-284.

[2] A. S. Besicovitch & S. J. Taylor. On the complementary intervals of a linear closed set of zero Lebesgue measure. *J. London Math. Soc.* **29** (1954), 449-459.

[3] T. M. Cherry. Analytic quasi-periodic curves of discontinuous type on a torus. *Proc. London Math. Soc.* **44** (1938), 175-215.

[4] J. Dieudonné. *Foundations of Modern Analysis.* Academic Press, 1969.

[5] M. Herman. Mesure de Lebesgue et nombre de rotation. In: *Geometry and Topology Rio de Janeiro* 1976. Lecture Notes in Mathematics vol. 597, Springer-Verlag, 1977.

[6] M. W. Hirsch, C. C. Pugh & M. Shub. *Invariant Manifolds.* Lecture Notes in Mathematics vol. 583. Springer-Verlag, 1977.

[7] W. Hurewicz & H. Wallman. *Dimension Theory.* Princeton, 1941.

[8] J. Palis Jr. & W. de Melo. *Geometric Theory of Dynamical Systems.* Springer-Verlag, 1982.

[9] G. R. Sell. Smooth linearization near a fixed point. Preprint, Minnesota, 1983.

[10] J. Sotomayor. Generic one parameter families of vector fields on two-dimensional manifolds. *Pub. Math. Inst. Hautes Études Scientifiques,* **43** (1973), 6-46.

[11] S. J. van Strien. One parameter families of vector fields. Thesis, Utrecht, 1982.

[12] S. J. Taylor. *Introduction to Measure and Integration.* Cambridge University Press, 1966.