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FREE PRODUCTS OF LOCALLY INDICABLE GROUPS WITH A SINGLE RELATOR

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The method of proof of Magnus introduced in 1930 is adapted to prove the following theorem of Howie. If A and B are groups for which every finitely generated subgroup has an infinite cyclic image, and if one adds an additional relation (with obvious exceptions), then in the resultant group both A and B appear isomorphically.

We recall that a group is said to be locally indicable if every finitely generated subgroup has an infinite cyclic homomorphic image. For convenience, if R is an element of a group G, G/R will denote G/N where N is the normal closure of R in G.

THEOREM 1. *Let* A *and* B *be locally indicable groups and let* $G = (A*B)/R$, *where* R *is a cyclically reduced word of length at least* 2. *Then the canonical maps* $A \to G$, $B \to G$ *are injective.*

This theorem, which is due to Howie [2] (see also references in his paper to the work of Brodskii and Short), is a generalisation of the Freiheitsatz of [3]. It is the object of this note to prove Theorem 1 using combinatorial methods like those of Magnus [3], in the style of [1].

Proof of Theorem 1. We argue by induction on the length λ of R. Let A_0 (respectively B_0) denote the subgroup of A (respectively B) generated by those elements which appear in R. Then R regarded as a word in A_0*B_0 is cyclically reduced of length λ. Let $G_0 = A_0*B_0/R$.

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If we can show that the canonical maps of A_0 and B_0 into G_0 are injective, it will follow that

$$G \cong A \underset{A_0}{*} G_0 \underset{B_0}{*} B_0$$

so that the canonical maps of A and B into G will also be injective. Hence we may assume, without loss of generality, that $A = A_0$ and $B = B_0$. In particular, if $\lambda = 2$ (the initial case of the induction), then A and B are infinite cyclic groups and the result holds. Since $A = A_0$ and $B = B_0$, both are finitely generated, and hence each has an infinite cyclic image. By symmetry, it suffices to show that the canonical map of A into G is injective.

Case 1. $B = \langle b \rangle$ is infinite cyclic.

Let $D = \langle d \rangle$ be infinite cyclic. Then there is an epimorphism of A onto D with kernel N say and with $A/N = \langle aN \rangle$. This epimorphism induces a homomorphism of $A*B$ onto $D*B$, and suppose that under this homomorphism R is mapped to R_1.

Subcase A. Suppose that d occurs with zero exponent sum in R_1. Thus R lies in the normal closure in $A*B$ of $N \cup B$. Put $b_i = a^{-i}ba^i$ and write R in terms of the b_i and the elements of N. At least two of the b_i must be involved in expressing R, since R is of length at least 4 and some $a^r n$ (r a non-zero integer and $n \in N$) must occur in R by our assumption that $A = A_0$.

Let s be the least suffix of a b_i appearing in R and let t be largest. Let

$$K = N*\langle b_s \rangle * \langle b_{s+1} \rangle * \dots * \langle b_t \rangle, \quad K_0 = N*\langle b_s \rangle * \langle b_{s+1} \rangle * \dots * \langle b_{t-1} \rangle$$
$$\text{and} \quad K_1 = N*\langle b_{s+1} \rangle * \langle b_{s+2} \rangle * \dots * \langle b_t \rangle .$$

R is of length smaller than λ as a word in the free product $K_0 * \langle b_t \rangle$. Hence by the induction hypothesis, K_0 is embedded in $\bar{K} = K/R$. Similarly, R as a word in the free product $\langle b_s \rangle * K_1$ is of length less than λ and so K_1 is embedded in $\bar{K}$. Thus G is the HNN extension of $\bar{K}$ with free element a and associated subgroups K_0 and K_1, with a taking b_i to b_{i+1} $(s \leqslant i < t)$, and acting on the elements of N by conjugation in A. Thus N and hence A is embedded in G.

Subcase B. Suppose d appears with non-zero exponent sum δ in R_1 and b appears with non-zero exponent sum β.

Adjoin a β-th root $\hat{a}$ of a to obtain the group $\hat{A}$ and a β-th root $\hat{d}$ of d to obtain the infinite cyclic group $\hat{D}$. Then $\hat{A}$ is again locally indicable (as can be shown using the subgroup theorem for a free product with an amalgamation) and there is an obvious homomorphism $\alpha : \hat{A} \to \hat{D}$ which takes $\hat{a}$ to $\hat{d}$ and a to d. Let $\hat{b} = \hat{b}a^{\delta}$ and $B = \langle \hat{b} \rangle$. Then it is easy to check that $\hat{A} * B = \hat{A} * \hat{B}$ by using the homomorphism definition of a free product. It is also clear that in the image of R under the homomorphism induced by α from $\hat{A} * \hat{B}$ into $\hat{D} * \hat{B}$, that $\hat{d}$ appears with zero exponent sum. We can then apply subcase A.

Case 2. B arbitrary.

Then there exists an epimorphism ϕ of B onto an infinite cyclic group $C = \langle c \rangle$. This induces a homomorphism of $A * B$ onto $A * C$. Let R_2 be the image of R under this map. If c appears with non-zero exponent sum in R_2, then if the length of R_2 is 1, then $A * C/R_2$ is the free product of A and a finite cyclic group and hence A is embedded in it, which implies the required result.

Otherwise if R_2 is of length at least 2, then we can use Case 1 to establish that A is embedded in $A * C/R_2$, and hence the result.

So we are left with the case that c appears with zero exponent sum in R_2 which is itself of length at least two. This is the case dealt with in the last five paragraphs of page 173 of [2]; we repeat this argument for completeness.

Let K be the kernel of ϕ and let b be a pre-image of c under ϕ. Then R lies in the normal closure of A and K which has the form $L = (*A_i | i$ in the integers$) * K$, where $A_i = b^{-i}Ab^i$.

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Since $B = B_0$ it follows that at least two of the groups A_i are involved in R. Suppose s is smallest index such that A_s is involved in R, and t is the largest such index.

Define $K_0 = A_{s+1} * \ldots * A_t * K$ and $K_1 = A_s * \ldots * A_{t-1} * K$. Then R belongs to $L_0 = A_s * K_0 = A_t * K_1$, and all the groups A_s, A_t, K_0, K_1 are locally indicable. Let $G_2 = L_0/R$.

Let λ_2 denote the length of some cyclically reduced conjugate of R regarded as a word in $A_s * K_0$. Then $\lambda_2 \leqslant \lambda_1 \leqslant \lambda$. If $\lambda_2 = \lambda$, then $s = t = 0$, and R belongs to $A*K$ which contradicts $B = B_0$. If $\lambda_2 \leqslant 1$, then R is conjugate in $A_s * K_0$ to an element of A_s or K_0. The first contradicts the fact that $\lambda \geqslant 2$, the second the choice of s. Hence $2 \leqslant \lambda_2 < \lambda$.

By the inductive hypothesis, the canonical maps of A_s and K_0 into G_2 are injective. The result now follows since G is the HNN extension of G_2 with b conjugating K_1 onto K_0.

References

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