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Milnor K-Groups and Zero-Cycles on Products of Curves over p-Adic Fields

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Abstract. We investigate the Chow groups of zero cycles of products of curves over a p-adic field by means of the Milnor K-groups of theirJacobians as introduced by Somekawa. We prove some finiteness results for $CH_0(X)/m$ for X a product of curves over a p-adic field.

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1. Introduction

Let X be a smooth projective variety over a field k. The structure of the Chow groups $CH^i(X)$ of codimension i-cycles modulo rational equivalence is almost entirely unknown when $i \geqslant 2$. However, when k is a field of arithmetic interest, there are far-reaching conjectures about what these groups should look like. In particular, the conjectures of Bloch [Bl1] and Beilinson [Be] predict that when k is an algebraic number field of finite degree over $\mathbb{Q}$, these groups are finitely generated, and their ranks should be interpreted as the order of vanishing of appropriate L-functions. However, the structure of these groups over complete topological fields is not as well-understood, even conjecturally. The case of a p-adic field (finite extension of $\mathbb{Q}_p$) is intermediate between an algebraically closed field such as $\mathbb{C}$ and an algebraic number field.

Over a p-adic field or $\mathbb{C}$, the torsion subgroup of $CH^i(X)$ is understood to some extent, at least when $i = 2$ or $i = \dim(X)$ (see [CT1] and the references therein). This is accomplished by relating the torsion to an étale cohomology group, using fundamental results in K-theory such as the Merkur'ev–Suslin theorem. One would like to use similar techniques to control the groups $CH^i(X)/n$ in these terms ($n \in \mathbb{N}$). This works well for divisors because the cycle map to étale cohomology:

$$CH^1(X)/n \to H^2(X, \mathbb{Z}/n(1))$$

is injective, as one can easily deduce from the cohomology of the Kummer exact sequence of sheaves for the étale topology on X. For X over a p-adic field, the étale cohomology groups with finite coefficients are known to be finite and, hence, the group $CH^1(X)/n$ is finite (this can also be seen by using a theorem of Mattuck [Mat] and finite generation of the Néron–Severi group). Unfortunately, even when k is algebraically closed, it is not always true that the cycle map

$$CH^i(X)/n \longrightarrow H^{2i}(X, \mathbb{Z}/n(i))$$

is injective for $i \geqslant 2$ (see [BCC], p. 135, [BE] and [To2]; for an example over a p-adic field we refer to [PS]), and the kernel is even less understood when k is not algebraically closed. For this reason, there is very little known in general about the groups $CH^i(X)/n$.

A natural question is whether the group $CH_0(X)/n$ is finite for any smooth projective variety X over a p-adic field k and any $n \in \mathbb{N}$ (here $CH_0(X) = CH^d(X), d = \dim(X)$). This was known to be true for surfaces X such that the Albanese mapping is injective over $\overline{k}$ and for any n (see [CT1], Théorème 8.5 and Remark (4.5.8) below), or for X a surface and n prime to p ([SaSu], Theorem 2.5). However, there were no results that are valid for any n for varieties X with $H^2(X, \mathcal{O}_X) \neq 0$. In this paper, we answer this question affirmatively when X is a product of curves whose Jacobians have a mixture of good ordinary and split multiplicative reduction (we call this *semi-ordinary* below). We are not able to say anything at the moment about the case of, e.g., a self-product of an elliptic curve with supersingular reduction.

Our method is simple to describe, and most of the paper consists of proving the results that are necessary to realize this simple philosophy. To illustrate, let C_1, C_2 be smooth, projective, geometrically connected curves having rational points over a field k, and denote by J_1, J_2 their Jacobians. Let $A_0(C_1 \times C_2)$ denote the group of 0-cycles of degree 0 modulo rational equivalence. We show that the kernel of the Albanese map:

$$A_0(C_1 \times C_2) \longrightarrow J_1(k) \times J_2(k)$$

may be realized as a 'Milnor K-group' $K(k; J_1, J_2)$, which is defined by symbols $\{\alpha_1, \alpha_2\}_{L/k}$, where L ranges over finite extensions of k and $\alpha_i \in J_i(L)$ $(i = 1, 2)$. This approach was started by Bloch (unpublished) and by Somekawa [So] when k is algebraically closed and C_1, C_2 are elliptic curves; we extend Somekawa's methods to general k, C_1, C_2. Kahn [Ka1], [Ka2] has also studied similar groups, but with fewer relations. In many cases, Kahn's groups suffice for our purposes. When k is a p-adic field and n is a positive integer, we try to relate $K(k; J_1, J_2)/n$ with $K_2^M(k)/n$, the Milnor K_2-group of k, modulo n. This works well when the formal groups attached to J_1, J_2 are of multiplicative type. For example, if E_1, E_2 are Tate elliptic curves over k, we show that there is a surjection: $K_2(k)/n \longrightarrow K(k; E_1, E_2)/n$. One easily sees that the group $K_2(k)/n$ is finite, hence that $K(k; E_1, E_2)/n$ is finite.

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Here is a more precise description of our main results: Let $X_1, \dots, X_d$ be smooth, projective, geometrically connected curves over a p-adic field k, and assume that $X_i(k) \neq \emptyset$ for all i. We will say that the Jacobian variety J_i of X_i has *split semi-ordinary reduction* if J_i has semi-Abelian reduction over k, and the special fibre of the Néron model of J_i over the ring of integers of k is an extension of an *ordinary* Abelian variety by a torus that is split over the residue field. Note that this ordinary Abelian variety may be 0, so this includes split multiplicative reduction. Our main result, which is based on work of Kahn [Ka2] and Somekawa [So], is the following (see Corollaries 3.5.1 and 4.5.6 for the precise statements):

THEOREM 1.1. *Assume $X_i(k) \neq \emptyset$ and that the Jacobians $J_1, \dots, J_d$ of $X_1, \dots, X_d$ have split semi-ordinary reduction. Let $A_0(X_1 \times \dots \times X_d)$ be the subgroup of $CH_0(X_1 \times \dots \times X_d)$ generated by zero cycles of degree* 0. *Then the kernel of the Albanese map $A_0(X_1 \times \dots \times X_d) \to J_1(k) \times \dots \times J_d(k)$ is of the form $F \oplus D$ for a finite group F and a divisible group D. In particular, $CH_0(X_1 \times \dots \times X_d)/m$ is finite for every positive integer m.*

To prove Theorem 1.1, we first show, using the simple observation that every closed point is, after a finite base extension, an intersection of divisors supported in the fibres of the projections

$$X_1 \times \dots \times X_d \longrightarrow X_i \quad (i = 1, \dots, d),$$

that the group $CH_0(X_1 \times \dots \times X_d)$ may be described in terms of the Picard groups of $X_1, \dots, X_d$. More precisely, we have a surjection

$$\bigoplus_{E/k \text{ finite}} \operatorname{Pic}((X_1)_E) \otimes \dots \otimes \operatorname{Pic}((X_d)_E) \longrightarrow CH_0(X_1 \times \dots \times X_d) \tag{1.2}$$

induced by the exterior product. By assuming that each X_i has a k-rational point, we can also write (1.2) in terms of the Jacobians:

$$\mathbb{Z} \oplus \bigoplus_{E/k \text{ finite}} \bigoplus_{1 \leqslant \nu \leqslant d} \bigoplus_{1 \leqslant i_1 < \dots < i_\nu \leqslant d} J_{i_1}(E) \otimes \dots \otimes J_{i_\nu}(E) \longrightarrow CH_0(X_1 \times \dots \times X_d). \tag{1.3}$$

In Section 2 we give a precise description of the kernel of this map. We will show that (1.3) induces an isomorphism (Corollary 2.4.1)

$$\mathbb{Z} \oplus \bigoplus_{1 \leqslant \nu \leqslant d} \bigoplus_{1 \leqslant i_1 < \dots < i_\nu \leqslant d} K(k; J_{i_1}, \dots, J_{i_\nu}) \;\cong\; CH_0(X_1 \times \dots \times X_d). \tag{1.4}$$

Here the group $K(k; J_{i_1}, \dots, J_{i_\nu})$ denotes the Milnor K-group attached to $J_{i_1}, \dots, J_{i_\nu}$. This result holds over an arbitrary base field.

In the case where k is a p-adic field and each Jacobian J_i has split semi-ordinary reduction, we can control the groups $K(k; J_{i_1}, \dots, J_{i_\nu})/n$ in terms of $K_\nu^M(k)/n$, and this last group is easily seen to be finite. Theorem 1.1 then follows from some simple Abelian group theory (Lemma 3.4.4). When m is prime to p, the hypothesis of semi-ordinarity of the special fibre is not necessary (Corollary 3.5.1).

For arbitrary semi-Abelian varieties $G_1, \dots, G_n$ over k, Somekawa [So] has defined a Milnor K-group $K(k; G_1, \dots, G_n)$ attached to $G_1, \dots, G_n$, generalizing the usual Milnor K-theory, which corresponds to the case where each G_i is equal to $\mathbb{G}_{m,k}$. He has proved (1.4) in the case of a product of two elliptic curves over an algebraically closed field. The isomorphism (1.4) fits nicely into the picture of the (conjectural) filtrations on Chow groups related to Ext-groups of mixed motives (Remark 2.4.2(b)).

The group $K(k; G_1, \dots, G_n)$ is defined as a quotient of $\bigoplus_{E/k\,\text{finite}} G_1(E) \otimes \dots \otimes G_n(E)$ with respect to two relations; the first one forces $K(k; G_1, \dots, G_n)$ to satisfy a projection formula and the second one forces the existence of a 'reciprocity law' for $K(k; G_1, \dots, G_n)$ (see Definition 2.1.1). Unfortunately, in the case of Abelian varieties $G_1 = A_1, \dots, G_n = A_n$, it is hard to 'calculate' $K(k; A_1, \dots, A_n)$. For the proof of (1.1), we work with a larger quotient of $\bigoplus_{E/k} A_1(E) \otimes \dots \otimes A_n(E)$ denoted by $[A_1 \overset{\mathrm{M}}{\otimes} \dots \overset{\mathrm{M}}{\otimes} A_n](k)$; namely, we factor out only the first relation ($\overset{\mathrm{M}}{\otimes}$ stands for the tensor product in the sense of Mackey functors). These groups were introduced by Kahn, and he uses them to reprove the finiteness of the Chow group of zero cycles of degree zero for products of curves over a finite field [Ka2]. Based on his result, it is easy to deduce finiteness of $CH_0(X_1 \times \dots \times X_d)/m$ for m prime to p in the case where all X_i have good reduction. This is worked out in Section 3. Section 4 is devoted to the study of the group $[A_1 \overset{\mathrm{M}}{\otimes} \dots \overset{\mathrm{M}}{\otimes} A_n](k)/p^m$. In the case where each A_i has semi-ordinary reduction, we can relate $[A_1 \overset{\mathrm{M}}{\otimes} \dots \overset{\mathrm{M}}{\otimes} A_n](k)/p^m$ via flat cohomology and p-adic uniformization to the usual Milnor K-groups of k. Local class field theory allows us then to deduce finiteness of $[A_1 \overset{\mathrm{M}}{\otimes} \dots \overset{\mathrm{M}}{\otimes} A_n](k)/p^m$.

While many of the results in this paper are proved by only using the tensor product of Mackey functors instead of the Milnor K-groups, we have developed the Milnor K-group formalism because we expect it to be necessary when the base field is finitely generated over $\mathbb{Q}$.

NOTATION

$\mathbb{N}$ denotes the set of positive integers. Given an Abelian group A and a nonzero integer m, let ${}_mA$ be the kernel and A/m be the cokernel of multiplication by m on A. For a prime number p, we denote by $A(\text{non-}p)$ the prime to p part of the torsion subgroup of A. For a scheme X and a point x on X, we denote by $k(x)$ the residue field of x. If X is a variety over a field and i is a nonzero integer, we denote by X_i the set of points of dimension i on X and $Z_i(X)$ the group of cycles of dimension i, i.e. $Z_i(X)$ is the free Abelian group with basis X_i. By $CH_i(X)$ (resp. $CH^i(X)$) we denote the Chow group of dimension i (resp. codimension i) cycles on X modulo rational equivalence. For a cycle $z \in Z_i(X)$, we write $[z] \in CH_i(X)$ for the class of z.

For a scheme S and a nonzero integer m, we denote by μ_m the group scheme of mth roots of unity over S. If $\mathcal{A} \to S$ is an Abelian scheme we let $\mathcal{A}[m]$ be the group scheme of m-division points of $\mathcal{A}$. If k is a field we denote by G_k the absolute Galois group of k. By $H^*(k, M)$ we denote the Galois cohomology groups of G_k with values in a discrete G_k-module M with continuous G_k-action. By a p-adic field we will mean a finite extension of $\mathbb{Q}_p$. If k is an arbitrary field, by an extension of k we will always mean a field extension.

2. Milnor *K*-groups of Chow Groups and Zero-cycles on Products of Varieties

2.1. The aim of this section is to give a description of the Chow group of zero cycles of a product of varieties $X_1, \ldots, X_n$ in terms of the Chow groups of zero cycles of the X_i. We begin by defining a Milnor K-group for Chow groups of varieties in the same way as in ([So], sect.1). Let k be a field and let $\mathcal{V}_k$ be the category of smooth projective varieties over k. We will use the notation $X \in \mathcal{V}_k$ as shorthand for writing that X is a smooth projective variety over k. For $X \in \mathcal{V}_k$ and E/k a field extension, let $X_E = X \otimes_k E$. If $j : E \to F$ is a k-morphism of extensions of k, the pull-back $CH_*(X_E) \to CH_*(X_F)$ will be denoted by j^*; if F/E is finite, the push-forward $CH_*(X_F) \to CH_*(X_E)$ will be denoted by j_*, or sometimes $N_{F/E}$. Let $X_1, \ldots, X_n \in \mathcal{V}_k$. We will denote by $\mathcal{CH}_0(X_i)$ the functor on field extensions of k that associates to each field extension E/k, the group $CH_0((X_i)_E)$.

DEFINITION 2.1.1. The Milnor K-group $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n))$ attached to $\mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n)$ is the Abelian group

$$\left(\bigoplus_{E/k} CH_0((X_1)_E) \otimes \ldots \otimes CH_0((X_n)_E)\right)/R,$$

where E runs through all finite field extensions of k, and where R is the subgroup generated by the following elements

(2.1.2) If $j\colon E_1 \to E_2$ is a morphism of finite field extensions of k and if $x_{i_0} \in CH_0((X_{i_0})_{E_2})$, $x_i \in CH_0((X_i)_{E_1})$ (for $i \neq i_0$) for some i_0, then

$$(x_1 \otimes \ldots \otimes j_*(x_{i_0}) \otimes \ldots \otimes x_n) - (j^*(x_1) \otimes \ldots \otimes x_{i_0} \otimes \ldots \otimes j^*(x_n)) \in R.$$

(2.1.3) Let K be a function field in one variable over k, i.e. a finitely generated extension of k of transcendence degree one*. Let $f \in K^*$ and $x_i \in CH_0((X_i)_K)$. Then

$$\sum_{v} \mathrm{ord}_v(f)\, s_v(x_1) \otimes \ldots \otimes s_v(x_n) \in R,$$

*We do not assume that k is algebraically closed in K.

where the sum is taken over all places v of K/k, and where $s_v: CH_*(X_K) \to CH_*(X_{k(v)})$ denotes the specialization map (as defined in ([Fu], Chap. 20)).

For a finite field extension E/k and $x_1 \in CH_0((X_1)_E), \ldots, x_n \in CH_0((X_n)_E)$, we denote by $\{x_1, \ldots, x_n\}_{E/k}$ the class of $x_1 \otimes \ldots \otimes x_n$ in $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n))$, and refer to elements of this type as symbols. As in ([So], sect. 1.3), one can define for any field extension k'/k a restriction map

$$\mathrm{res}_{k'/k}: K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n)) \to K(k'; \mathcal{CH}_0((X_1)_{k'}), \ldots, \mathcal{CH}_0((X_n)_{k'})).$$

and, if k'/k is finite, a norm

$$K(k'; \mathcal{CH}_0((X_1)_{k'}), \ldots, \mathcal{CH}_0((X_n)_{k'})) \longrightarrow K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n)).$$

On symbols we have $N_{k'/k}(\{x_1, \ldots, x_n\}_{E/k'}) = \{x_1, \ldots, x_n\}_{E/k}$.

THEOREM 2.2. *There is a canonical isomorphism*

$$K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n)) \xrightarrow{\cong} CH_0(X_1 \times \ldots \times X_n).$$

Proof. First, we define a canonical map

$$\phi: K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n)) \longrightarrow CH_0(X_1 \times \ldots \times X_n).$$

Let $p_i : X_1 \times \ldots \times X_n \to X_i$ be the projection. We also use the same notation for base changes by extensions of k. If K/k is a field extension, let

$$\phi_K: CH_0((X_1)_K) \otimes \ldots \otimes CH_0((X_n)_K) \longrightarrow CH_0((X_1 \times \ldots \times X_n)_K).$$

be the exterior product, i.e. $\phi_K(x_1 \otimes \ldots \otimes x_n) = p_1^*(x_1) \cdot \ldots \cdot p_n^*(x_n)$ (intersection product). Let

$$\widetilde{\phi} = \bigoplus_{E/k} N_{E/k} \circ \phi_E : \bigoplus_{E/k} CH_0((X_1)_E) \otimes \ldots \otimes CH_0((X_n)_E) \longrightarrow CH_0(X_1 \times \ldots \times X_n).$$

We will see that $\widetilde{\phi}$ factors through $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n))$. That $\widetilde{\phi}$ vanishes on elements of the form (2.1.2) follows from the projection formula for the intersection product ([Fu], Proposition 8.3(c)). To see that $\widetilde{\phi}$ vanishes on elements of the form (2.1.3), we need the following simple fact:

(2.2.1) *Let k be a field and K be a function field in one variable over k. Let X be a smooth projective variety over k. Then for every $f \in K^*$ and every $x \in CH_*(X_K)$, we have*

$$\sum_v \mathrm{ord}_v(f)\, N_{k(v)/k}(s_v(x)) = 0$$

in $CH_*(X)$.

Proof. Let C be the regular proper model of K over k. If $z \in Z_*(X_K)$ is a cycle representing x, then we denote by z' its extension to a cycle of $X \times C$ and by

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$x' = [z']$ the corresponding class in $CH_*(X \times C)$. We have

$$\sum_{\upsilon} \mathrm{ord}_{\upsilon}(f)\, N_{k(\upsilon)/k}(s_{\upsilon}(x)) = p^*(\mathrm{div}(f)) \cdot x' = 0,$$

where $p\colon X \times C \to C$ is the projection. This proves (2.2.1).

Now let $K, x_1, \ldots, x_n, f$ be as in (2.1.3). By using (2.2.1) and the compatibility of the specialization map with pull-backs ([Fu], Proposition 20.3(b)), we get:

$$\begin{aligned}
&\widetilde{\phi}\left(\sum_{\upsilon} \mathrm{ord}_{\upsilon}(f)\, s_{\upsilon}(x_1) \otimes \ldots \otimes s_{\upsilon}(x_n)\right) \\
&\quad = \sum_{\upsilon} \mathrm{ord}_{\upsilon}(f)\, N_{k(\upsilon)/k}(\phi_{k(\upsilon)}(s_{\upsilon}(x_1) \otimes \ldots \otimes s_{\upsilon}(x_n))) \\
&\quad = \sum_{\upsilon} \mathrm{ord}_{\upsilon}(f)\, N_{k(\upsilon)/k} \circ s_{\upsilon}(\phi_K(x_1 \otimes \ldots \otimes x_n)) = 0.
\end{aligned}$$

Hence $\widetilde{\phi}$ induces a map $\phi\colon K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n)) \to CH_0(X_1 \times \ldots \times X_n)$. To define the inverse of ϕ, we need some preparation. Let $X \in \mathcal{V}_k$ and let K/k be an extension of fields. A k-morphism $\eta\colon \mathrm{Spec}\, K \to X$ defines a K-rational point $x(\eta)$ of X_K, hence a cycle $z(\eta) \in Z_*(X_K)$. We have (see [Fu], p.399):

(2.2.2) *Let K/k be a function field in one variable, $\eta : \mathrm{Spec}\, K \to X$ a k-morphism and υ be a place of K/k with valuation ring R. The morphism η extends uniquely to a map $\mathrm{Spec}\, R \to X$ and hence defines a map $s_{\upsilon}(\eta)\colon \mathrm{Spec}\, k(\upsilon) \to \mathrm{Spec}\, R \to X$. Then $s_{\upsilon}([z(\eta)]) = [z(s_{\upsilon}(\eta))]$.*

We need to reformulate (2.2.2) slightly. Let C be the regular proper model of K over k. The morphism $\eta\colon \mathrm{Spec}\, K \to X$ extends uniquely to a morphism $g\colon C \to X$. For a closed point $x \in C$, let υ be the corresponding place of K/k. Then the map $s_{\upsilon}(\eta)\colon \mathrm{Spec}\, k(x) \to X$ is just the composite $g \circ i_x$, where $i_x\colon \mathrm{Spec\ k(x)} \to \mathrm{C}$ is the canonical map. Hence (2.2.2) gives for the cycle $z(x) \stackrel{\mathrm{def}}{=} z(i_x)$:

$$g_*([z(x)]) = g_*([z(i_x)]) = [z(g \circ i_x)] = [z(s_{\upsilon}(\eta))] = s_{\upsilon}([z(\eta)]).$$

Here we have written g_* as shorthand for $(g \otimes_k k(x))_*$.

Now we are going to define the inverse

$$\psi : CH_0(X_1 \times \ldots \times X_n) \to K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n))$$

of ϕ. Let $\widetilde{\psi} : Z_0(X_1 \times \ldots \times X_n) \to K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n))$ be defined by

$$\widetilde{\psi}\left(\sum_x n_x\, x\right) = \sum_x n_x\, \{(p_1)_*[z(x)], \ldots, (p_n)_*[z(x)]\}_{k(x)/k}.$$

To see that $\widetilde{\psi}$ factors through rational equivalence, let $C \subseteq X_1 \times \ldots \times X_n$ be a closed irreducible subset of dimension 1, let K be its function field, and take $f \in K^*$. Let $\widetilde{C}$ be the normalization of C and denote by $\pi\colon \widetilde{C} \to X_1 \times \ldots \times X_n$ the canonical

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morphism. We have to show that $\widetilde{\psi}(\mathrm{div}_C(f)) = 0$. If we let $\eta\colon \mathrm{Spec}\, K \to \widetilde{C} \to X_1 \times \ldots \times X_n$ be the composite and $\eta_i = p_i \circ \eta$, $i = 1, \ldots n$, then we have by (2.2.2) and the remarks following it:

$$\begin{aligned}\widetilde{\psi}(\mathrm{div}(f)) &= \widetilde{\psi}\left(\sum_{x \in \widetilde{C}_0} \mathrm{ord}_x(f)[k(x) : k(\pi(x))]\pi(x)\right)\\ &= \sum_{x\in \widetilde{C}_0} \mathrm{ord}_x(f)\,[k(x) : k(\pi(x))]\,\{(p_1)_*[z(\pi(x))], \ldots, (p_n)_*[z(\pi(x))]\}_{k(\pi(x))/k}\\ &= \sum_{x\in \widetilde{C}_0} \mathrm{ord}_x(f)\,\{(p_1 \circ \pi)_*[z(x)], \ldots, (p_n \circ \pi)_*[z(x)]\}_{k(x)/k}\\ &= \sum_{v} \mathrm{ord}_v(f)\,\{s_v([z(\eta_1)]), \ldots, s_v([z(\eta_n)])\}_{k(v)/k} = 0.\end{aligned}$$

Hence, $\widetilde{\psi}$ induces a map

$$\psi\colon CH_0(X_1 \times \ldots \times X_n) \to K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n)).$$

A simple norm argument shows that $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_n))$ is generated by symbols $\{[x_1], \ldots, [x_n]\}_{E/k}$ where E/k is a finite extension and x_i is an E-rational point of $(X_i)_E$ for $i = 1, \ldots, n$. Moreover it is easy to see that the ϕ's and ψ's commute with norms for finite extensions of k. Therefore, to verify that they are inverse to each other, it is enough to see that $(\psi \circ \phi)(\{[x_1], \ldots, [x_n]\}) = \{[x_1], \ldots, [x_n]\}$ for k-rational points x_i of X_i and $(\phi \circ \psi)([x]) = [x]$ for a k-rational point x of $X_1 \times \ldots \times X_n$. This is clear. □

(2.3) Our aim now is to extend Theorem 2.2 to Chow motives. This can be done in a totally formal manner. Firstly, we recall some notation and facts about correspondences and Chow motives. As general references for what we need here, see ([Fu], Chapter 16), [Man] and [Sou, §1]. For $X, Y \in \mathcal{V}_k$ and an integer r, let $\mathrm{Corr}^r(X, Y)$ be the group of correspondences of degree r between X and Y, i.e. $\mathrm{Corr}^r(X, Y) = \bigoplus_{i=1,\ldots,r} CH^{\dim X_i + r}(X_i \times Y)$ if $X = \bigcup_{i=1,\ldots,r} X_i$ is the decomposition of X into irreducible components. For $X, Y, Z \in \mathcal{V}_k$ we have the composition law

$$\circ\colon \mathrm{Corr}^r(X, Y) \times \mathrm{Corr}^s(Y, Z) \longrightarrow \mathrm{Corr}^{r+s}(X, Z),$$

making $\mathrm{Corr}^0(X, X)$ into a ring with $1 = 1_X = [\Delta_X]$. Let α be a correspondence between X and Y. It induces a map of Chow groups $\alpha_* : CH_*(X) \to CH_*(Y)$, given by $\alpha_*(x) = (p_Y)_*(\alpha \cdot (p_X)^*(x))$. Since flat pull-backs, intersection products and proper push-forwards commute with specialization maps ([Fu], Proposition 20.3), we see that α_* commutes with specialization, too. The category of Chow motives $\mathcal{M}_k$ consists of triples (X, p, m), where X is a variety, $p = p^2 \in \mathrm{Corr}^0(X, X)$ is an idempotent and m is an integer. For two motives (X, p, m), (Y, q, n), the set of morphisms is given

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by

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = q \circ \mathrm{Corr}^{n-m}(X,Y) \circ p.$$

$\mathcal{M}_k$ is a pseudo-Abelian category with tensor products (see, e.g., [Sou], §1), and there is a contravariant faithful functor $h\colon \mathcal{V}_k \to \mathcal{M}_k$ defined on objects by $h(X) = (X, 1_X, 0)$. For $n \in \mathbb{Z}$ we denote by $\mathbb{Z}(n)$ the motive (Spec k, 1, n). The Chow groups $CH_*(M)$ of a motive $M = (X,p,m)$ are defined as $p_*CH_{*+m}(X)$. By the remark above, one can also define specialization maps for Chow groups of motives.

Hence, Definition (2.1.1) carries over to motives, i.e. one can define the Milnor K-group $K(k;\mathcal{CH}_0(M_1),\ldots,\mathcal{CH}_0(M_n))$ in exactly the same way as $K(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_0(X_n))$. Note that the map $(M_1,\ldots,M_n) \mapsto K(k;\mathcal{CH}_0(M_1),\ldots,\mathcal{CH}_0(M_n))$ is linear in each component.

COROLLARY 2.3.1. *Let $X_1,\ldots,X_n$ be smooth projective varieties and $p_i \in \mathrm{Corr}^0(X_i,X_i)$ be idempotents. Put $M_i = (X_i,p_i,0)$ for $i = 1,\ldots,n$. Then,*

$$CH_0(M_1 \otimes \ldots \otimes M_n) \cong K(k;\mathcal{CH}_0(M_1),\ldots,\mathcal{CH}_0(M_n)).$$

Proof. By linearity, $K(k;\mathcal{CH}_0(M_1),\ldots,\mathcal{CH}_0(M_n))$ (resp. $CH_0(M_1 \otimes \ldots \otimes M_n)$) is a direct summand of $K(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_0(X_n))$ (resp. $CH_0(X_1 \times \ldots \times X_n)$), and it is easy to see that the isomorphism ϕ maps $K(k;\mathcal{CH}_0(M_1),\ldots,\mathcal{CH}_0(M_n))$ onto $CH_0(M_1 \otimes \ldots \otimes M_n)$. □

(2.4) We apply now 2.3.1 to products of curves. Let $X_1,\ldots,X_d$ be smooth, projective, geometrically connected curves over k. We assume that each X_i has a k-rational point P_i. Then we have a decomposition $h(X_i) = \mathbb{Z}(0) \oplus X_i^+ \oplus \mathbb{Z}(-1)$, where X_i^+ is the motive $(X_i, p_{X_i}, 0)$ with $p_{X_i} = [\Delta_{X_i}] - [P_i \times X_i] - [X_i \times P_i]$ (see, e.g., [Man], §10). Therefore we get

$$\begin{aligned} h(X_1 \times \ldots \times X_d) &= \bigotimes_{i=1}^{d} h(X_i) \\ &= \bigoplus_{\mu+\nu \leqslant \mathrm{d}} \; \bigoplus_{1 \leqslant i_1 < \ldots < i_\nu \leqslant d} \binom{\mathrm{d}-\nu}{\mu} X_{i_1}^+ \otimes \ldots \otimes X_{i_\nu}^+ \otimes \mathbb{Z}(-\mu). \end{aligned}$$

The binomial coefficient comes from the fact that there are $\binom{d-\nu}{\mu}$ ways to get $\mathbb{Z}(-\mu)$ in the tensor product. We now apply the functor CH_0 to this decomposition of motives. Note that CH_0 will be zero on any summand for which $\mu > 0$. Using (2.3.1),

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we then get:

$$\begin{aligned} &CH_0(X_1 \times \ldots \times X_d) \\ &= \mathbb{Z} \oplus \bigoplus_{1 \leqslant v \leqslant d} \bigoplus_{1 \leqslant i_1 < \ldots < i_v \leqslant d} CH_0(X_{i_1}^+ \otimes \ldots \otimes X_{i_v}^+) \\ &= \mathbb{Z} \oplus \bigoplus_{1 \leqslant v \leqslant d} \bigoplus_{1 \leqslant i_1 < \ldots < i_v \leqslant d} K(k; \mathcal{CH}_0(X_{i_1}^+), \ldots, \mathcal{CH}_0(X_{i_v}^+)). \end{aligned}$$

If $J_1, \ldots, J_d$ are the Jacobians of $X_1, \ldots, X_d$, then we have $CH_0(X_i^+) = \mathrm{Ker}([P_i \times X_i]_* + [X_i \times P_i]_* \colon CH_0(X_i) \to CH_0(X_i)) = \mathrm{Ker}(\deg \colon CH_0(X_i) \to \mathbb{Z}) = J_i(k)$.

We explain now why the groups $K(k; \mathcal{CH}_0(X_{i_1}^+), \ldots, \mathcal{CH}_0(X_{i_v}^+))$ coincide with the Milnor K-groups $K(k; J_{i_1}, \ldots, J_{i_v})$ defined by Somekawa in ([So], sect. 1). To simplify notation, assume that $v = d, i_1 = 1, \ldots, i_d = d$. We note first that in the relation (1.2.2) in ([So], p. 107) the choice of signs is not correct; one has to drop $(-1)^{i(\upsilon)}$ in order to get the results of [So]⋆. Clearly it is enough to verify that Somekawa's (corrected) relation (1.2.2) coincides with relation (2.1.3) in the case we consider here. Let K/k be a function field in one variable and let $f \in K^*$ and $x_i \in CH_0((X_i)_K)$. Let $g_i \in J_i(K)$ be the image of x_i under the isomorphism $CH_0((X_i)_K^+) \to J_i(K)$. If v is a place of K with valuation ring $\mathcal{O}_\upsilon$, we have $J_i(\mathcal{O}_\upsilon) = J_i(K)$ since J_i is proper. One can easily verify that $g_i(\upsilon) \in J_i(k(\upsilon))$ corresponds to $s_\upsilon(x_i)$ under $CH_0((X_i)_{k(\upsilon)}^+) \cong J_i(k(\upsilon))$. If we choose for every place v of K/k an index $i(\upsilon) \in \{1, \ldots, d\}$ and denote by $\partial_\upsilon \colon J_{i(\upsilon)}(K) \otimes K^* \to J_{i(\upsilon)}(k(\upsilon))$ the local symbol map defined in ([Se], Chapter III, Section 1), then Somekawa's relation (1.2.2) is

$$\sum_\upsilon \{g_1(\upsilon), \ldots, \partial(g_{i(\upsilon)}, f), \ldots, g_d(\upsilon)\}_{k(\upsilon)/k} = 0.$$

But since $J_i(K) = J_i(\mathcal{O}_\upsilon)$, we have by ([Se], Chapter III, Definition 2)

$$\partial_\upsilon(g_{i(\upsilon)}, f) = \mathrm{ord}_\upsilon(f)\, g_{i(\upsilon)}(\upsilon),$$

so

$$\begin{aligned} &\sum_\upsilon \{g_1(\upsilon), \ldots, \partial_\upsilon(g_{i(\upsilon)}, f), \ldots, g_d(\upsilon)\}_{k(\upsilon)/k} \\ &= \sum_\upsilon \mathrm{ord}_\upsilon(f)\, \{g_1(\upsilon), \ldots, g_d(\upsilon)\}_{k(\upsilon)/k} \\ &= \sum_\upsilon \mathrm{ord}_\upsilon(f)\, \{s_\upsilon(x_1), \ldots, s_\upsilon(x_n)\}_{k(\upsilon)/k}, \end{aligned}$$

which is just the relation (2.1.3).

Summarizing our discussion in this section, we have

⋆For example, the calculation $\sum_\upsilon (-1)^{i(\upsilon)} N_{k(\upsilon)/k}(\{g_1(\upsilon), \ldots, \partial_\upsilon(g_{i(\upsilon)}, h), \ldots, g_d(\upsilon)\}_{k(\upsilon)/k}) = (N \circ \partial)(\{h, g_1, \ldots, g_r\})$ is not correct, in general; one finds rather $\sum_\upsilon N_{k(\upsilon)/k}(\{g_1(\upsilon), \ldots, \partial_\upsilon(g_{i(\upsilon)}, h), \ldots, g_d(\upsilon)\}_{k(\upsilon)/k}) = (N \circ \partial)(\{g_1, \ldots, g_r, h\})$. Therefore ([So], Theorem 1.4) holds only if we drop the signs $(-1)^{i(\upsilon)}$ in his relation (1.2.2).

COROLLARY 2.4.1. *Let $X_1, \ldots, X_d$ be smooth projective geometrically connected curves over k with Jacobians $J_1, \ldots, J_d$ such that $X_i(k) \neq \emptyset$ for each i. Then,*

$$CH_0(X_1 \times \ldots \times X_d) = \mathbb{Z} \oplus \bigoplus_{1 \leqslant \nu \leqslant d} \bigoplus_{1 \leqslant i_1 < \ldots < i_\nu \leqslant d} K(k; J_{i_1}, \ldots, J_{i_\nu}).$$

Remarks 2.4.2. (a) If k is an algebraically closed field, $d = 2$ and X_1, X_2 are elliptic curves, this was proved by Somekawa ([So], Theorem (2.4)).

(b) Let X be a smooth projective variety over k. A conjecture of Beilinson predicts the existence of a filtration on Chow groups arising from a spectral sequence

$$E_2^{\nu,\mu} = \mathrm{Ext}^{\nu}_{\mathcal{MM}}(1, h^{\mu}(X)(m)) \Longrightarrow CH^m(X, 2m - (\nu + \mu)),$$

where $\mathcal{MM}$ is the (conjectural) category of mixed motives over k with integral coefficients and $CH^*(X, *)$ are Bloch's higher Chow groups (see, e.g., [Ja]). If $X = X_1 \times \ldots \times X_d$ is a product of curves with $X(k) \neq \emptyset$, the spectral sequence should degenerate (for general X, it is only expected to degenerate after tensoring with $\mathbb{Q}$). Let again $J_1, \ldots, J_d$ be the Jacobians. Somekawa has given a conjectural interpretation of the groups $K(k; J_{i_1}, \ldots, J_{i_\nu})$ in terms of Ext-groups of mixed motives. By using this description, one can 'compute' formally

$$\mathrm{Ext}^{\nu}_{\mathcal{MM}}(1, h^{2d-\nu}(X)(d)) = \bigoplus_{1 \leqslant i_1 < \ldots < i_\nu \leqslant d} K(k; J_{i_1}, \ldots, J_{i_\nu})$$

Hence, Corollary 2.4.1 conjecturally describes the filtration on $CH_0(X_1 \times \ldots \times X_d)$.

(c) One can extend Theorem 2.2 and Corollary 2.3.1 to higher Chow groups as well. Let $X_1, \ldots, X_n$ be in $\mathcal{V}_k$ and let $\nu_1, \ldots, \nu_n, \mu_1, \ldots, \mu_n$ be nonnegative integers. Set $d = d_1 + \ldots + d_n$ and $\nu = \nu_1 + \ldots + \nu_n$. The Milnor K-groups $K(k; \mathcal{CH}^{\nu_1}(X_1, \mu_1), \ldots, \mathcal{CH}^{\nu_n}(X_n, \mu_n))$ can be defined as in (2.1.1) except that the relation (2.1.3) has to be replaced by the following (since there are no canonical specialization maps in general for higher Chow groups*):

(2.1.3′) Let K be a function field in one variable over k. Let $f \in K^* = CH^1(K, 1)$ and $x_i \in CH^{\nu_i}((X_i)_K, \mu_i)$ for $i = 1, \ldots, n$. Then

$$\sum_{\upsilon} N_{k(\upsilon)/k} \circ \partial_{\upsilon}(f \times x_1 \times \ldots \times x_n) = 0$$

*More precisely the specialization maps depend on the choice of uniformizing parameter in general; see Remark 2.4.2(e) below for more on this.

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Here $f \times x_1 \times \ldots \times x_n$ is the image of $f \otimes x_1 \otimes \ldots \otimes x_n$ under the exterior product

$$CH^1(K,1) \otimes CH^{\nu_1}((X_1)_K, \mu_1) \otimes \ldots \otimes CH^{\nu_n}((X_n)_K, \mu_n) \longrightarrow CH^{1+\nu} \times ((X_1 \times \ldots \times X_n)_K, 1+\mu)$$

and

$$\partial_\upsilon : CH^{1+\nu}((X_1 \times \ldots \times X_n)_K, 1+\mu) \to CH^{\nu}((X_1 \times \ldots \times X_n)_{k(\upsilon)}, \mu)$$

is the connecting map in the localization sequence (see [Bl2], Theorem (3.1) and also [Bl3]). Again the definition works also for Chow motives. We have:

(2.4.4) *Let $X_1, \ldots, X_n$ be smooth projective varieties of dimensions $d_1, \ldots, d_n$, let $\nu_1, \ldots, \nu_n$ be integers, and let $p_i \in \mathrm{Corr}^0(X_i, X_i)$ be idempotents. Put $M_i = (X_i, p_i, 0)$ for $i = 1, \ldots, n$, $d = d_1 + \ldots d_n$ and $\nu = \nu_1 + \ldots \nu_n$. Then,*

$$CH^{\nu}(M_1 \otimes \ldots \otimes M_n, d-\nu) \cong K(k; \mathcal{CH}^{\nu_1}(M_1, d_1 - \nu_1), \ldots, \mathcal{CH}^{\nu_n}(M_n, d_n - \nu_n)).$$

The proof is a straightforward generalization of the proof of (2.2) if one uses cubes (as in [To], p. 179) instead of simplices in the definition of higher Chow groups (note that we are dealing with 0-cycles in (2.4.4)). Let us consider the special case where $X_1 = \ldots = X_n = \mathrm{Spec}\, k$ and $\nu_1 = \ldots = \nu_n = 1$. According to ([So], Theorem (1.4)), we have $K(k; \mathcal{CH}^1(k,1), \ldots, \mathcal{CH}^1(k,1)) \cong K_n^M(k)$, the usual Milnor K-group of k. Hence (2.4.4) yields $CH^n(k,n) \cong K_n^M(k)$, a result due to Nesterenko and Suslin [NS] and Totaro [To].

As another special case, one can recover the following formula of Somekawa ([So], Theorem (2.1)) for a curve X which has a rational point over k (J=Jacobian of X):

$$K(k; \mathbb{G}_m, J) \cong V(X) = \mathrm{Ker}(\mathrm{Norm} : SK_1(X) \to k^*).$$

(d) The following remark was communicated to us by B. Kahn. Let k be a field and let $DM(k)$ denote Voevodsky's derived category of geometrical effective motives over k as constructed in [V]. It is an additive triangulated tensor category. Given a smooth variety X over k (not necessarily projective) or a Chow motive, there is an associated motive $M(X)$ in $DM(k)$ (in fact the category of Chow motives embeds as a full subcategory of $DM(k)$). For arbitrary Chow motives $M_1, \ldots, M_n$ it should be possible to see directly from the definition of the tensor products in $DM(k)$ that

$$K(k; \mathcal{CH}_0(M_1), \ldots, \mathcal{CH}_0(M_n)) \cong \mathrm{Hom}_{DM}(\mathbb{Z}, M_1 \otimes \ldots \otimes M_n).$$

This suggests a more general formalism than that presented above.

(e) Let X be a variety over a field k, and let C be a smooth projective, irreducible curve over k with function field K. Let ν be a place of K with valuation ring A. Let A^h be the henselization of A and K^h its quotient field. Then there is an exact localization sequence:

$$\dots \longrightarrow CH^i(X \times_k A^h, j) \longrightarrow CH^i(X_{K^h}, j) \longrightarrow CH^{i-1}(X_{k(\upsilon)}, j-1) \longrightarrow \dots$$

(see [Bl2], Theorem 3.1 and [Bl3] for corrections in the proof of the moving lemma). The boundary map can be split by the choice of a uniformizing parameter π of A^h, which gives a class in $CH^1(X_{K^h}, 1)$. In this way, we can define a specialization map:

$$CH^i(X_K, j) \longrightarrow CH^i(X_{K^h}, j) \longrightarrow CH^i(X_{k(\upsilon)}, j).$$

However, it seems difficult to make a good choice of uniformizing parameter for all places of K so that the analogue of the reciprocity law (2.2.1) will hold.

3. The Group $CH_0(X_1 \times \dots \times X_d)/m$ for Curves $X_1, \dots, X_d$ over a p-adic Field and m Prime to p

(3.1) In this section we investigate the groups $CH_0(X_1 \times \dots \times X_d)/m$, for $X_1, \dots, X_d$ smooth, projective geometrically connected curves over a p-adic field k and m prime to p, by using the results of the last section. However we only use the fact that there is a surjection, due to Kahn and valid over any perfect* field k:

$$\mathbb{Z} \oplus \bigoplus_{1 \leqslant \nu \leqslant d} \bigoplus_{1 \leqslant i_1 < \dots < i_\nu \leqslant d} [J_{i_1} \overset{M}{\otimes} \dots \overset{M}{\otimes} J_{i_\nu}](k) \longrightarrow CH_0(X_1 \times \dots \times X_d), \tag{3.1.1}$$

where $\overset{M}{\otimes}$ denotes the tensor product of Mackey functors (see [Ka2]). This follows from Corollary 2.4.1 above, because the tensor product of Mackey functors only involves the relation (2.1.2) that was used to define our Milnor K-groups. While Section 2 describes the kernel of this surjection, we do not need it in this section.

Before stating and proving the main results of this section, we need to recall some facts about Mackey functors and the Néron model.

(3.2) Let k be a perfect field. A *Mackey functor* A is a co- and contravariant functor from the category of étale k-schemes to the category of Abelian groups (ie for every finite k-morphism $f: X \to Y$ of étale k-schemes, we have maps $f_*: A(X) \to A(Y)$ and $f^*: A(Y) \to A(X)$) such that

$$A(X_1 \amalg X_2) = A(X_1) \oplus A(X_2). \tag{3.2.1}$$

*The restriction to perfect fields here is not necessary if one defines Mackey functors as functors on all finite extensions rather than finite separable extensions. However, in this case, (3.2.2) has to be replaced by a slightly more complicated axiom. Since we are dealing in the following with p-adic fields only, we choose to work with the simpler but more restricted definition.

(3.2.2) If

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f'\downarrow & & f\downarrow \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a Cartesian square, then

$$\begin{array}{ccc} A(X') & \xrightarrow{g'_*} & A(X) \\ f'^*\uparrow & & f^*\uparrow \\ A(Y') & \xrightarrow{g'_*} & A(Y) \end{array}$$

commutes.

By (3.2.1) we see that A is uniquely determined by its value $A(K)\stackrel{\text{def}}{=}A(\text{Spec } K)$ on finite separable extensions K/k. It is explained in [Ne] (where Mackey functors are called G-modulations) how one can express (3.2.1), (3.2.2) in terms of fields only. A G_k-module A gives rise to the Mackey functor $K \mapsto A^{G_K}$ with the natural norm and restriction maps. If $L/K/k$ are finite separable extensions and j : Spec $L \to$ Spec K is the natural map, we put $\text{res}_{L/K} = j^*$ and $N_{L/K} = j_*$. The Mackey functors form an Abelian category with a tensor product (introduced by Kahn [Ka1]), which is defined as follows: For Mackey functors $A_1 \ldots, A_n, n \geqslant 2$ and for a finite separable extension K/k, $[A_1 \stackrel{M}{\otimes} \ldots \stackrel{M}{\otimes} A_n](K)$ is the quotient of $\bigoplus_{E/K \text{ finite separable}} A_1(E) \otimes \ldots \otimes A_n(E)$ modulo the subgroup generated by elements of the form (2.1.2). The image of a tensor

$$a_1 \otimes \ldots \otimes a_n \in A_1(E) \otimes \ldots \otimes A_n(E)$$

in $[A_1 \stackrel{M}{\otimes} \ldots \stackrel{M}{\otimes} A_n](K)$ will be denoted by $(a_1, \ldots, a_n)_{E/K}$ and will be referred to as a symbol. One can easily verify that $\ldots \stackrel{M}{\otimes} A$ is right-exact for any Mackey functor A.

(3.3) We recall some facts about the Néron model of an Abelian variety A over a complete discretely valued field k with valuation ring $\mathcal{O}$ and residue field $\mathcal{F}$. The basic references are [N] and [BLR]. The Néron model $\mathcal{A}$ of A is a smooth group scheme over $\mathcal{O}$ with $\mathcal{A} \times_{\mathcal{O}} k \cong A$, and which represents the functor

$$Y \mapsto \text{Hom}_k(Y \times_{\mathcal{O}} k, A)$$

on the category of schemes smooth over $\mathcal{O}$ ([BLR], Ch. 1, §§2,3). We have that A/k has good reduction if and only if the Néron model is proper over $\mathcal{O}$. Let $\mathcal{A}^0$ be the neutral component of $\mathcal{A}$. The special fibre $\mathcal{A}_s^0$ of $\mathcal{A}^0$ is an extension of an Abelian variety over F by a linear algebraic group, which itself is an extension of a torus by a unipotent group. We say that A has *semi-Abelian reduction* if this linear algebraic group is a torus, and *split semi-Abelian reduction* if this torus is split over the residue field F. After a finite extension of the base field, we can always achieve split semi-Abelian reduction (see e.g. [BLR], Chapter 7, §4, Theorem 1).

Now assume that A is an Abelian variety over a p-adic field k with semi-Abelian reduction. The Mackey functor $K \mapsto A(K)$ on finite extensions of k has a natural filtration $A^1 \subseteq A^0 \subseteq A$ which we are going to recall now. First, we introduce some notation. For a finite extension K/k, let $\mathcal{O}_K$ be the ring of integers of K and let κ_K be the residue field. The absolute Galois group of κ_K will be denoted by $\mathfrak{g}_K$. We set $\mathcal{O} = \mathcal{O}_k$, $\kappa = \kappa_k$ and $\mathfrak{g} = \mathfrak{g}_k$. Then A^0 is given by $A^0(K) = \mathcal{A}^0(\mathcal{O}_K)$ and $A^1(K)$ is the kernel of the specialization $\mathcal{A}^0(\mathcal{O}_K) \to \mathcal{A}^0_s(\kappa_K)$. This group can be also described as the $\mathcal{O}_K$-valued points of the formal group attached to A, hence it is a $\mathbb{Z}_p$-module.

The quotient A^0/A^1 has the following description: $[A^0/A^1](K) \cong \mathcal{A}^0_s(\kappa_K)$. If L/K is a finite extension, then $\mathrm{res}_{L/K} : [A^0/A^1](K) \to [A^0/A^1](L)$ and $N_{L/K} : [A^0/A^1](L) \to [A^0/A^1](K)$ can be identified with, respectively, $\mathrm{res}_{\kappa_L/\kappa_K} : \mathcal{A}^0_s(\kappa_K) \to \mathcal{A}^0_s(\kappa_L)$ and $e_{L/K} N_{\kappa_L/\kappa_K} : \mathcal{A}^0_s(\kappa_L) \to \mathcal{A}^0_s(\kappa_K)$. Finally, we consider the quotient A/A^0. Let Γ be a $\mathfrak{g}$-lattice, i.e. a discrete $\mathfrak{g}$-module which is — as Abelian group — free of finite rank. Let $\tilde{\Gamma} \subseteq \Gamma$ be a $\mathfrak{g}$-invariant sublattice of finite index. We denote by $\Phi = \Phi_{(\Gamma, \tilde{\Gamma})}$ the following Mackey functor ($e_K \overset{\mathrm{def}}{=} e_{K/k}$):

$$K \mapsto \Phi(K) = (\Gamma / e_K \tilde{\Gamma})^{\mathfrak{g}_K}.$$

If L/K is a finite extension, define

$$\mathrm{res}_{L/K} = e_{L/K} \mathrm{res}_{\kappa_L/\kappa_K} : (\Gamma/e_K\tilde{\Gamma})^{\mathfrak{g}_K} \longrightarrow (\Gamma/e_L\tilde{\Gamma})^{\mathfrak{g}_L}$$

and

$$N_{L/K} = N_{\kappa_L/\kappa_K} : (\Gamma/e_L\tilde{\Gamma})^{\mathfrak{g}_L} \longrightarrow (\Gamma/e_K\tilde{\Gamma})^{\mathfrak{g}_K}.$$

According to ([SGA7], Exposé IX, Théorème 12.5), we have $A/A^0 = \Phi_{(\Gamma, \tilde{\Gamma})}$, where Γ (resp. $\tilde{\Gamma}$) is the character group of the maximal torus in $\mathcal{A}^0_s$ (resp. of the maximal torus in the special fiber of the Néron model of the dual Abelian variety A^t).

(3.4) Now we investigate the tensor products $[A_1/A_1^0] \overset{M}{\otimes} \dots \overset{M}{\otimes} [A_n/A_n^0]$, $[A_1/A_1^0] \overset{M}{\otimes} [A_2^0/A_2^1] \overset{M}{\otimes} \dots$ etc. for Abelian varieties $A_1, \dots, A_n/k$ with semi-Abelian reduction.

LEMMA 3.4.1. *Let* $\Phi_i = \Phi_{(\Gamma_i, \tilde{\Gamma}_i)}$, $i = 1, \dots, n$, $n \geqslant 2$ *be Mackey functors attached to pairs* $(\Gamma_i, \tilde{\Gamma}_i)$ *of* $\mathfrak{g}$-*lattices. Then the torsion group* $[\Phi_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \Phi_n](k)$ *is of the form*

$$[\Phi_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \Phi_n](k) \cong F \oplus D,$$

where F *is finite and* D *is* m-*divisible for all integers* m *prime to* p.

Proof. We consider here only the case where the Γ_i's have trivial $\mathfrak{g}$-action, and we show that $[\Phi_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \Phi_n](k)$ is divisible. The general case will be postponed to the appendix since it is not of great importance for the following and rather involved.

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We take $x_i \in \Gamma_i$ for $i = 1, \ldots, n$, K/k a finite extension and m a nonzero integer. We choose a totally ramified extension L/K of degree m. Then for the symbol

$$(x_1 + e_K\widetilde{\Gamma}_1, \ldots, x_n + e_K\widetilde{\Gamma}_n)_{K/k} \in [\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](k),$$

we have

$$\begin{aligned}(x_1 &+ e_K\widetilde{\Gamma}_1, \ldots, x_n + e_K\widetilde{\Gamma}_n)_{K/k} \\ &= (x_1 + e_K\widetilde{\Gamma}_1, \ldots, x_{n-1} + e_K\widetilde{\Gamma}_{n-1}, N_{L/K}(x_n + e_L\widetilde{\Gamma}_n))_{K/k} \\ &= m(x_1 + e_L\widetilde{\Gamma}_1, \ldots, x_n + e_L\widetilde{\Gamma}_n)_{L/k}.\end{aligned}$$

Thus $[\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](k)$ is divisible. □

LEMMA 3.4.2. *Let A/k be an Abelian variety with semi-Abelian reduction, and let $\Phi_i = \Phi_{(\Gamma_i, \widetilde{\Gamma}_i)}$, $i = 1, \ldots, n$, $n \geqslant 1$ be as in Lemma 3.4.1. Then $[(A^0/A^1) \overset{M}{\otimes} \Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](k)$ is finite.*

Proof. Firstly, for every finite unramified extension L/K, the norm $N_{L/K}: (A^0/A^1)(L) \to (A^0/A^1)(K)$ is surjective (see [Ka2], p.1040; this is a consequence of Lang's theorem on connected algebraic groups over finite fields). From the definition of the tensor product of Mackey functors given in (3.2), we see that the map

$$N_{L/K}: [(A^0/A^1) \overset{M}{\otimes} \Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](L) \to [(A^0/A^1) \overset{M}{\otimes} \Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](K)$$

is also surjective. Therefore, we may assume that all $\Gamma_1, \ldots, \Gamma_n$ have trivial $\mathfrak{g}$-action.

If $n \geqslant 2$ then, as seen in the proof of Lemma 3.4.1, $[\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](K)$ is divisible, hence $[A^0/A^1](K) \otimes [\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](K) = 0$ for all finite extensions K/k. Thus $(A^0/A^1) \overset{M}{\otimes} \Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n = 0$. If $n = 1$ then we have for a symbol $(a, x + e_K\widetilde{\Gamma}_1)_{K/k}$, $x \in \Gamma_1$, $a \in \mathcal{A}_s^0(\kappa_K)$:

$$\begin{aligned}(a, x + e_K\widetilde{\Gamma}_1)_{K/k} &= (a, N_{K/K^t}(x + e_K\widetilde{\Gamma}_1))_{K^t/k} \\ &= (a, x + \widetilde{\Gamma}_1)_{K^t/k} = (N_{K^t/k}(a), x + \widetilde{\Gamma}_1)_{k/k},\end{aligned}$$

where K^t is the inertia field of K/k. Consequently, the map $[A^0/A^1](k) \otimes \Phi_1(k) \longrightarrow [(A^0/A^1) \overset{M}{\otimes} \Phi_1](k)$ is surjective. □

LEMMA 3.4.2. *Let A, B be Abelian varieties with semi-Abelian reduction. Then $(A^0/A^1) \overset{M}{\otimes} (B^0/B^1) = 0$.*

Proof. This follows at once from the main result of [Ka2]. Indeed $\mathcal{A}_s^0, \mathcal{B}_s^0$ are semi-Abelian varieties over a finite field, hence $\mathcal{A}_s^0 \overset{M}{\otimes} \mathcal{B}_s^0 = 0$. On the other hand, we have a surjection

$$\bigoplus_{L/K}(\mathcal{A}_s^0 \overset{\mathrm{M}}{\otimes} \mathcal{B}_s^0)(\kappa_L) \longrightarrow [(A^0/A^1) \overset{\mathrm{M}}{\otimes} (B^0/B^1)](K),$$

where L/K runs through all finite extensions. □

We also need the following lemma:

LEMMA 3.4.4. *Let S be a nonempty, multiplicatively closed subset of $\mathbb{N}$.*

(a) *For an Abelian group M, the following are equivalent:*

(i) $\varprojlim_{m\in S} M/m$ is finite.

(ii) $M \cong F \oplus D$ for an S-divisible group D and a finite group F.

We call a group with these properties finite-by-S-divisible. This is similar to the notion of torsion-by-divisible, which has been considered in a similar context by Colliot-Thélène ([CT3], *Lemma-Definition 3.1).*

(b) *If $M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of Abelian groups, and if M_1, M_3 are finite-by-S-divisible, then so is M_2. Also, if M_2 is finite-by-S-divisible, then so is M_3.*

Proof. (a) That (ii) implies (i) is clear. We show that (i) implies (ii). Let

$$\varphi : M \longrightarrow \varprojlim_{m\in S} M/m$$

be the natural map and let N be the order of $\varprojlim_{m\in S} M/m$. Note that $N \in S$. We claim that $\mathrm{Ker}(\varphi) = NM$. Indeed, if $\alpha \in NM$, then clearly $\alpha \in \mathrm{Ker}(\varphi)$. Conversely, if $\alpha \in \mathrm{Ker}(\varphi)$, then $\alpha \in \bigcap_{m\in S} mM \subseteq NM$. This proves the claim. Given $\alpha \in \mathrm{Ker}(\varphi)$ and $n \in S$, we can find $\beta \in M$ such that $\alpha = nN\beta$. Hence, NM is S-divisible.

Thus M is an extension of a finite group F that is killed by N by a group D that is S-divisible. Applying the functor $\mathrm{Ext}_{\mathbb{Z}}(F, -)$ to the exact sequence $0 \to {}_N D \to D \to D \to 0$ gives a long exact sequence:

$$\ldots \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(F, D) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(F, D) \longrightarrow \mathrm{Ext}^2_{\mathbb{Z}}(F, {}_N D) \longrightarrow \ldots$$

Now for any two $\mathbb{Z}$-modules A and B, we have $\mathrm{Ext}^i_{\mathbb{Z}}(A, B) = 0$ for $i \geqslant 2$ (see, e.g., [W] 3.3.1), and hence $\mathrm{Ext}^2_{\mathbb{Z}}(F, {}_N D) = 0$. From the exact sequence, we then see that the group $\mathrm{Ext}^1_{\mathbb{Z}}(F, D)$ is N-divisible. On the other hand, this group is killed by N (because F is) and, hence, it is trivial. Thus M is the direct sum of F and D, as claimed.

(b) is left to the reader. □

The main result of this section is

THEOREM 3.5. *Let $A_1, \ldots, A_n$ be Abelian varieties with semi-Abelian reduction. Then,*

$$K(k; A_1, \ldots, A_n) \cong F \oplus D,$$

where F is a finite group and D is m-divisible for all integers m which are prime to p. If for at least two different indices i, A_i has good reduction, then $F = 0$.

Proof. Let $S = \{m \in \mathbb{N} \mid (p, m) = 1\}$. We have to show that $K(k; A_1, \ldots, A_n)$ is finite-by-S-divisible. Because we have the surjection $[A_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} A_n](k) \to K(k; A_1, \ldots, A_n)$, it is enough to check this for $[A_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} A_n](k)$. Let us denote by $0 = F_i^0 \subseteq F_i^1 \subseteq F_i^2 \subseteq F_i^3 = A_i$ the filtrations $0 \subseteq A_i^1 \subseteq A_i^0 \subseteq A_i$ for $i = 1, \ldots, n$. Since $\overset{M}{\otimes}$ is right exact they induce an increasing filtration F^j of $[A_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} A_n](k)$ such that the successive quotients F^{j+1}/F^j are quotients of certain $[(F^{j_1+1}/F^{j_1}) \overset{M}{\otimes} \ldots \overset{M}{\otimes} (F^{j_n+1}/F^{j_n})](k)$. By Lemmas 3.4.1–3.4.4(b), we conclude that F^{j+1}/F^j is finite-by-S-divisible. Therefore, again by Lemma 3.4.4(b), $[A_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} A_n](k)$ is finite-by-S-divisible.

Now assume that for two different indices, say $i = 1, 2$, A_i has good reduction. Then $\Phi_1 = \Phi_2 = 0$. So

$$[(F^{j_1+1}/F^{j_1}) \overset{M}{\otimes} \ldots \overset{M}{\otimes} (F^{j_n+1}/F^{j_n})](k) = 0 \quad \text{if } (j_1, j_2) = (1, 1), (2, 1), (1, 2).$$

By Lemma (3.4.3) it is also $= 0$ for $j_1 = j_2 = 2$. In the remaining cases at least one factor is isomorphic to A_1^1 or A_2^1, hence is S-divisible. This implies that $[A_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} A_n](k)$ is S-divisible, too. □

From Theorem 3.5 and Corollary 2.4.1, we obtain:

COROLLARY 3.5.1. *Let $X_1, \ldots, X_d$ be smooth, projective, geometrically connected curves over k with Jacobians $J_1, \ldots, J_d$ such that $X_i(k) \neq \emptyset$ for each i. Let $A_0(X_1 \times \ldots \times X_d) = \mathrm{Ker}(CH_0(X_1 \times \ldots \times X_d) \xrightarrow{\deg} \mathbb{Z})$.*

(a) *If all $J_1, \ldots, J_d$ have semi-Abelian reduction, then*

$$A_0(X_1 \times \ldots \times X_d) \cong F \oplus D,$$

where F is finite and D is m-divisible for all m which are prime to p. In particular, $CH_0(X_1 \times \ldots \times X_d)/m$ is finite for such m.

(b) *If $J_1, \ldots, J_d$ have good reduction, then the kernel of the Albanese map*

$$A_0(X_1 \times \ldots \times X_d) \to J_1(k) \times \ldots \times J_d(k)$$

is m-divisible for all integers m prime to p.

In the case $d = 2$ we can weaken our hypotheses.

COROLLARY 3.5.2. *Let X_1, X_2 be smooth, projective, geometrically connected curves over k. Then,*

$$A_0(X_1 \times X_2) \cong F \oplus D,$$

where F is finite and D is m-divisible for all m prime to p.

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Proof. Let K/k be a finite extension such that J_1, J_2 have semi-Abelian reduction, $X_i(K) \neq \emptyset$ for $i = 1, 2$ and $\mathfrak{g}_K$ operates trivially on Γ_1, Γ_2. Write $[K:k] = mp^r$ with $(m, p) = 1$ and let $Q = \mathrm{Coker}(N_{K/k}\colon A_0((X_1 \times X_2)_K) \to A_0(X_1 \times X_2))$. Then $[K:k]\,Q = 0$. On the other hand, we know that ${}_nCH_0(X_1 \times X_2)$ is finite for any nonzero integer n (see [CT1], Théorème 8.1). The exact sequence

$$A_0((X_1 \times X_2)_K) \longrightarrow A_0(X_1 \times X_2) \longrightarrow Q \longrightarrow 0$$

together with Corollary 3.5.1 show that ${}_mQ$ is finite. Now the assertion follows from Lemma 3.4.4(b). □

Remarks 3.5.3. (a) For a smooth, projective, geometrically integral variety X of dimension $\geqslant 3$ over a p-adic field, it is unknown at present whether the group ${}_nCH_0(X)$ is finite for any positive integer n. This is the basic reason why we are unable to deal with the case where the Jacobian of one of the curves has some additive reduction (this means that the linear part of the special fibre of the Néron model has a nontrivial unipotent part). Perhaps there is a way to handle additive reduction directly, rather than by trying to pass to a finite extension where the special fibre of the Néron model becomes semi-Abelian.
(b) Let Ker(alb) denote the kernel of the Albanese map $A_0(X) \to \mathrm{Alb}_X(k)$. We expect the following:

CONJECTURE 3.5.4*. *Let X be a smooth, projective, geometrically connected variety over a p-adic field k. Then*

$$\mathrm{Ker(alb)} \cong F \oplus V,$$

where F is a finite group and V is a uniquely divisible group. If X has good reduction with special fiber Y/κ then

$$F(\textit{non-p}) \cong \mathrm{Ker}(A_0(Y) \to \mathrm{Alb}_Y(\kappa))(\textit{non-p}).$$

In particular, $CH_0(X)/m$ should be finite for all nonzero m. If X is a surface and m is prime to p, this finiteness statement was proved by Saito and Sujatha ([SaSu], Theorem 2.5). For surfaces X with the property that the Albanese mapping is an isomorphism for $\overline{X}$, this finiteness was proved by Colliot-Thélène ([CT1], Théorème 8.5; see also Remark 4.5.8). Corollary 3.5.1 provides us with other examples.

In general, the group V should be very large. For a surface, one should expect (according to Mumford's Theorem and Bloch's conjecture; see, e.g., ([Ja], sect. 1)):

$$V = 0 \Longleftrightarrow H^2(X, \mathcal{O}_X) = 0.$$

(c) In (b), if we take instead of a p-adic field the Henselization of an algebraic number field at a prime ideal, then the Albanese kernel should have the same struc-

*This is formulated as a question in ([CT4], 1.4.(d),(e),(f))

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ture of the direct sum of a finite group F and a uniquely divisible group V. However, in this case the Beilinson–Bloch conjecture [Be], [Bl1] predicts that the Albanese kernel should be a torsion group, and hence we expect that the group V above should be zero.

4. The Group $CH_0(X_1 \times \ldots \times X_d)$ Modulo p^m

(4.1) We keep the notation and hypotheses of the last section. Now we are going to study the groups $K(k; A_1, \ldots, A_d)/p^n$ for Abelian varieties $A_1, \ldots, A_d$ over k. We can prove finiteness of $K(k; A_1, \ldots, A_d)/p^n$ in the case where $A_1, \ldots, A_d$ have split semi-ordinary reduction (see (4.4)). The result is based on the fact that in this case the groups $[A_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} A_d](k)/p^n$ are related to the Milnor K-groups $K_*^M(k)/p^n$.

To illustrate our method, consider the case of two Tate curves E_1, E_2 with periods q_1, q_2. The uniformizations $\mathbb{G}_m/q_i^{\mathbb{Z}} \longrightarrow E_i$ induce a surjection $[\mathbb{G}_m \overset{M}{\otimes} \mathbb{G}_m](k)/p^n \to [E_1 \overset{M}{\otimes} E_2](k)/p^n \to K(k; E_1, E_2)/p^n$, and it is easy to see that $[\mathbb{G}_m \overset{M}{\otimes} \mathbb{G}_m](k)/p^n \cong K_2^M(k)/p^n$ (see Remark 4.2.5 below). The group $K_2^M(k)/p^n$, being a quotient of $(k^* \otimes k^*)/p^n$, is easily seen to be finite and, hence, $K(k; E_1, E_2)/p^n$ is finite.

(4.2) To begin with, we investigate tensor products of the Mackey functors $U\colon K \mapsto U(K) = U_K$ (U_K are the units in $\mathcal{O}_K$), $\mathbb{G}_m\colon K \mapsto K^*$ and $\mathcal{Z}\colon K \mapsto \mathbb{Z}$, where for finite extensions $L/K/k$, the norm is multiplication by $e_{L/K}$, and $\mathrm{res}_{L/K}$ is given by multiplication by $f_{L/K}$.

LEMMA 4.2.1. *Let i, j, l be nonnegative integers such that $i + j + l \geqslant 2$. Then,*

$$[U^{\overset{M}{\otimes} i} \overset{M}{\otimes} \mathbb{G}_m^{\overset{M}{\otimes} j} \overset{M}{\otimes} \mathcal{Z}^{\overset{M}{\otimes} l}]/p \cong \begin{cases} H^2(\ldots, \mu_p^{\otimes 2}), \textit{if } i + j = 2, l = 0, \\ \widetilde{\mathcal{Z}}/p, \textit{if } i = 0, j = 1 = l, \\ 0, \textit{otherwise}. \end{cases}$$

Here $H^2(\ldots, \mu_p^{\otimes 2})$ denotes the Mackey functor $K \mapsto H^2(K, \mu_p^{\otimes 2})$, and $\widetilde{\mathcal{Z}}$ is $K \mapsto \mathbb{Z}$, but with all norms equal to the identity.

Proof. We will show only that $(U/p) \overset{M}{\otimes} (U/p) \cong H^2(\ldots, \mu_p^{\otimes 2})$. The proofs of

$$(U/p) \overset{M}{\otimes} (\mathbb{G}_m/p) \cong H^2(\ldots, \mu_p^{\otimes 2}) \quad \text{and} \quad \mathbb{G}_m/p \overset{M}{\otimes} \mathbb{G}_m/p \cong H^2(\ldots, \mu_p^{\otimes 2})$$

are analogous. All other cases can either be proved directly by simple calculations or can be easily deduced from the first case.

The maps

$$(U_K/p) \otimes (U_K/p) \to H^1(K, \mu_p) \otimes H^1(K, \mu_p) \xrightarrow{\cup} H^2(K, \mu_p^{\otimes 2})$$

for finite extensions K/k induce by the universal property of the tensor product for

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Mackey functors a map

$$\widetilde{h} : (U/p)\overset{\mathrm{M}}{\otimes}(U/p)\longrightarrow H^2(\dots, \mu_p^{\otimes 2}).$$

We will show that $\widetilde{h}$ is an isomorphism by modifying Tate's proof of the bijectivity of the Galois symbol for a local field ([Ta], Corollary to Prop. 4.5). By a transfer argument, we may assume that k contains the pth roots of unity. Then $H^2(K, \mu_p^{\otimes 2})$ is a cyclic group of order p, and the norm maps $H^2(L, \mu_p^{\otimes 2}) \to H^2(K, \mu_p^{\otimes 2})$ are isomorphisms for all finite extensions $L/K/k$. For a finite extension K/k and $a, b \in U_K$, we set

$$(a, b)'_K = \widetilde{h}((\overline{a}, \overline{b})_{K/k}) \in H^2(K, \mu_p^{\otimes 2}) \quad (\overline{a} \overset{\mathrm{def}}{=} a \bmod U_K^p)$$

We have with $L = K(\sqrt[p]{a})$,

$$(a, b)'_K = 0 \Longleftrightarrow b \in N_{L/K}(U_L) \Longleftrightarrow (\overline{a}, \overline{b})_{K/k} = 0.$$

In particular, if L/K is totally ramified of degree p and $b \notin N_{L/K}(U_L)$, then $(a, b)'_K$ is a generator of $H^2(K, \mu_p^{\otimes 2})$. Hence, $\widetilde{h}$ is surjective. As in the proof of ([Ta], Prop.4.5), one can show that the symbol $(a, b)'_K$ satisfies the properties (i), (ii) of ([Ta], Corollary to Theorem 4.4), and consequently, the map $\widetilde{h}_K$ restricted to the subgroup S_K, generated by the symbols $(\overline{a}, \overline{b})_{K/K}$, $a, b \in U_K$, is bijective. It remains to prove that $S_K = [U/p\overset{\mathrm{M}}{\otimes}U/p](K)$. Let L/K be a finite extension. We will show by induction on $\mathrm{ord}_p(e_{L/K})$ — the exponent of p in $e_{L/K}$ — that any symbol $(\overline{a}, \overline{b})_{L/K}$, $a, b \in U_L$, lies in S_K. In the case $p \nmid e_{L/K}$, $N_{L/K}\colon U_L/p \to U_K/p$ is surjective. So there are $c \in U_K/p$ and $d \in U_L/p$ such that $(c, N_{L/K}(d))'_K$ is a generator of $H^2(K, \mu_p^{\otimes 2})$. Thus $(\mathrm{res}_{L/K}(c), d)'_L$ generates $H^2(L, \mu_p^{\otimes 2})$ and the bijectivity of $\widetilde{h}_L|_{S_L}$ yields $(\overline{a}, \overline{b})_{L/L} = (\mathrm{res}_{L/K}(\overline{c}), \overline{d^i})_{L/L}$ for some i. Therefore $(\overline{a}, \overline{b})_{L/K} = (\overline{c}, N_{L/K}(\overline{d^i}))_{K/K} \in S_K$.

Now assume $\mathrm{ord}_p(e_{L/K}) \geqslant 1$. There exists a finite extension M/L of degree prime to p and an intermediate field M_1 of M/K such that M/M_1 is a cyclic and totally ramified extension of degree p (this can be seen by considering the extension LK_p/K_p where K_p is the fixed field of a p-Sylow group of G_K). The surjectivity of $N_{M/L} : U_M/p \to U_L/p$ allows us to write $(\overline{a}, \overline{b})_{L/K} = (\mathrm{res}_{M/L}(\overline{a}), \overline{b}_1)_{M/K}$ for some $b_1 \in U_M$. Let $c \in U_{M_1}$ be such that $\Sigma = M_1(\sqrt[p]{c})$ is a totally ramified nontrivial extension of M_1 and $\Sigma \neq M$ (such a c obviously exists if we choose M large enough). Then $N_{\Sigma/M_1}(U_\Sigma) + N_{M/M_1}(U_M) = U_{M_1}$, so there is a $d \in U_M$ such that $(c, N_{M/M_1}(d))'_{M_1}$ generates $H^2(M_1, \mu_p^{\otimes 2})$. As before, we have $(\overline{a}, \overline{b}_1)_{M/M} = (\mathrm{res}_{M/M_1}(\overline{c}), \overline{d^i})_{M/M}$ for some i. Thus

$$(\overline{a}, \overline{b})_{L/K} = (\mathrm{res}_{M/L}(\overline{a}), \overline{b}_1)_{M/K} = (\mathrm{res}_{M/M_1}(\overline{c}), \overline{d^i})_{M/K} = (\overline{c}, N_{M/M_1}(\overline{d^i}))_{M_1/K},$$

and since $\mathrm{ord}_p(e_{M_1/K}) < \mathrm{ord}_p(e_{L/K})$, the latter symbol lies in S_K by the induction hypothesis. □

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Let $\widetilde{k}/k$ be the maximal unramified extension of k and let $n_0 \geqslant 0$ be the largest integer such that $\mu_{p^{n_0}}(\overline{k}) \subseteq \widetilde{k}$.

LEMMA 4.2.2. *Let i, j, l, n be nonnegative integers such that $i + j + l \geqslant 2$. Then,*

$$[U^{\overset{M}{\otimes} i} \overset{M}{\otimes} \mathbb{G}_m^{\overset{M}{\otimes} j} \overset{M}{\otimes} \mathcal{Z}^{\overset{M}{\otimes} l}](k)/p^n \cong \begin{cases} \mathbb{Z}/p^{n_1} \quad \textit{for some } n_1 \leqslant n_0, \textit{ if } i+j=2, l=0, \\ \mathbb{Z}/p^n, \textit{ if } i=0, j=1=l, \\ 0, \textit{otherwise}. \end{cases}$$

Proof. Again, we will give the proof only for $[U \overset{M}{\otimes} U](k)/p^n$. The products $[U \overset{M}{\otimes} \mathbb{G}_m](k)/p^n$ and $[\mathbb{G}_m \overset{M}{\otimes} \mathbb{G}_m](k)/p^n$ can be handled similarly. All other assertions follow from (4.2.1) or by simple computations.

We may assume that k contains the pth roots of unity, since otherwise we have $[U \overset{M}{\otimes} U](k)/p = 0$ by Lemma 4.2.1, hence $[U \overset{M}{\otimes} U](k)/p^n = 0$. Since for a finite unramified extension K/k, the norm $[U \overset{M}{\otimes} U](K) \to [U \overset{M}{\otimes} U](k)$ is surjective, we can replace k by K and thus may assume from the beginning that k (not only $\widetilde{k}$) contains the p^{n_0}-roots of unity. Under this hypothesis we show by induction that

$$\widetilde{h}_k \colon [U \overset{M}{\otimes} U](k)/p^n \longrightarrow H^2(k, \mu_{p^n}^{\otimes 2}) \tag{4.2.3}$$

is an isomorphism. By Tate local duality ([Mi2], Chapter 1, §2, Corollary 2.3), the latter group is isomorphic to the dual of $\mathrm{Hom}(\mu_{p^n}, \mathbb{Z}/p^n)^{G_k}$, i.e. to $\mu_{p^n}(k)$, hence is cyclic of order at most p^{n_0}. This yields the assertion. For $n = 1$, $\widetilde{h}_k$ is an isomorphism by Lemma 4.2.1. For arbitrary n, we have a commutative diagram with exact rows except possibly at $[U \overset{M}{\otimes} U](k)/p^n$:

$$\begin{array}{ccccccccc} U_k \otimes \mu_p & \longrightarrow & [U \overset{M}{\otimes} U](k)/p^n & \longrightarrow & [U \overset{M}{\otimes} U](k)/p^{n+1} & \longrightarrow & [U \overset{M}{\otimes} U](k)/p & \longrightarrow & 0 \\ \downarrow & & \downarrow 1 & & \downarrow 2 & & \downarrow 3 & & \\ H^1(k, \mu_p^{\otimes 2}) & \longrightarrow & H^2(k, \mu_{p^n}^{\otimes 2}) & \longrightarrow & H^2(k, \mu_{p^{n+1}}^{\otimes 2}) & \longrightarrow & H^2(k, \mu_p^{\otimes 2}) & \longrightarrow & 0 \end{array}$$

The maps 1 and 3 are isomorphisms (the latter by the induction hypothesis). A diagram chase shows that the map 2 is an isomorphism if

$$U_k \otimes \mu_p \longrightarrow H^1(k, \mu_p^{\otimes 2}) \longrightarrow \mathrm{Ker}(H^2(k, \mu_{p^n}^{\otimes 2}) \longrightarrow H^2(k, \mu_{p^{n+1}}^{\otimes 2})) \tag{4.2.4}$$

is surjective. If $n + 1 \leqslant n_0$, i.e. $\mu_{p^{n+1}}(\overline{k}) \subseteq k$, then the computation of $H^2(k, \mu_{p^n}^{\otimes 2})$ done above using Tate duality shows that $\mathrm{Ker}(H^2(k, \mu_{p^n}^{\otimes 2}) \longrightarrow H^2(k, \mu_{p^{n+1}}^{\otimes 2})) = 0$, so (4.2.4) is surjective. If $n \geqslant n_0$, then $\mathrm{Ker}(H^2(k, \mu_{p^n}^{\otimes 2}) \longrightarrow H^2(k, \mu_{p^{n+1}}^{\otimes 2}))$ is a cyclic group of order p, so we only need to verify that the map (4.2.4) is not trivial. If we identify $H^2(k, \mu_{p^n}^{\otimes 2})$ with $\mu_{p^{n_0}}$, then the map

$$U_k \otimes \mu_p \longrightarrow H^1(k, \mu_p^{\otimes 2}) \longrightarrow H^2(k, \mu_{p^n}^{\otimes 2}) \cong \mu_{p^{n_0}}$$

is just given by the Hilbert symbol $u \otimes \zeta \mapsto (u, \zeta/\mathfrak{p}) \in \mu_{p^{n_0}}$. Let ζ be a primitive p-th root of unity. Then $k(\sqrt[p^{n_0}]{\zeta})/k$ is a ramified extension of degree p (since $\sqrt[p^{n_0}]{\zeta} \notin \tilde{k}$). It follows from the definition of the Hilbert symbol that there exists $u \in U_k$ such that $(u, \zeta/\mathfrak{p})$ is non-trivial. Therefore (4.2.4) is surjective and so (4.2.3) is an isomorphism for all n. □

Remarks 4.2.5. (a) The above proof for $[U \overset{M}{\otimes} \mathbb{G}_m]/p^n$ and $[\mathbb{G}_m \overset{M}{\otimes} \mathbb{G}_m]/p^n$ instead of $[U \overset{M}{\otimes} U]/p^n$ also yields

$$[U \overset{M}{\otimes} \mathbb{G}_m]/p^n \cong H^2(\ldots, \mu_{p^n}^{\otimes 2}) \cong [\mathbb{G}_m \overset{M}{\otimes} \mathbb{G}_m]/p^n.$$

(b) In an unpublished work, Kahn has shown that for any field k and any positive integer n prime to the characteristic of k, the natural map:

$$[\mathbb{G}_m \overset{M}{\otimes} \mathbb{G}_m](k)/n \longrightarrow K_2^M(k)/n$$

is an isomorphism. The proof requires a lot of calculation with symbols. Note, however, that using an argument similar to ([Ta], Proposition 3.1), one can easily show the following:

LEMMA 4.2.6. *Let k be a field and let F be any Mackey functor over k. Let $f: \mathbb{G}_m \overset{M}{\otimes} \mathbb{G}_m \longrightarrow F$ be a homomorphism of Mackey functors. Then for any positive integer n prime to the characteristic of k, f induces a homomorphism: $K_2^M(k)/n \longrightarrow F(k)/n$.*

The basic idea of the proof is that the image of the subgroup of $k^* \otimes k^*$ generated by tensors of the form $a \otimes (1-a)$ is n-divisible in $F(k)$.

(4.3) Let G be a commutative finite flat group scheme over Spec $\mathcal{O}$. Such a G represents a sheaf for the fppf topology on Spec $(\mathcal{O})$, which we also denote by G.

LEMMA 4.3.1. *Let G be a commutative finite flat group scheme over Spec $\mathcal{O}$.*

(a) *There exists an exact sequence (for the fppf topology) of flat group schemes*

$$0 \longrightarrow G \longrightarrow \mathcal{A} \xrightarrow{f} \mathcal{B} \longrightarrow 0$$

where $\mathcal{A}, \mathcal{B}$ are Abelian schemes over $\mathcal{O}$. Let A, B be their generic fibers. For every finite extension K/k, we have

$$H^1_{\mathrm{fl}}(\mathcal{O}_K, G) \cong \mathrm{Coker}(f: A(K) \longrightarrow B(K))$$

(b) *The assignment $K \mapsto H^1_{\mathrm{fl}}(\mathcal{O}_K, G)$ defines in a natural way a Mackey functor, which we denote by $H^1_{\mathrm{fl}}(\mathcal{O}_{\_}, G)$. It is isomorphic to* Coker$(f: A \to B)$ *(viewed as a Mackey functor). Consequently, for an unramified extension L/K, the norm map $N_{L/K}: H^1_{\mathrm{fl}}(\mathcal{O}_L, G) \to H^1_{\mathrm{fl}}(\mathcal{O}_K, G)$ is surjective.*

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Proof. We note first that for any smooth commutative group scheme $\mathcal{G}$ over $\mathcal{O}$, we have a canonical isomorphism

$$H^1_{\mathrm{fl}}(\mathcal{O}_K, \mathcal{G}) \cong H^1_{\mathrm{\acute{e}t}}(\kappa_K, \mathcal{G}_s) \tag{4.3.2}$$

where $\mathcal{G}_s$ denotes the special fiber of $\mathcal{G}$. Indeed it is known (see, e.g., [Mi1], Chapter III, Theorem 3.9) that for a smooth group scheme over $\mathcal{O}_K$, flat and étale cohomology coincide, i.e. $H^1_{\mathrm{fl}}(\mathcal{O}_K, \mathcal{G}) \cong H^1_{\mathrm{\acute{e}t}}(\mathcal{O}_K, \mathcal{G})$. On the other hand, as a consequence of the proper base change theorem, the latter group is isomorphic to $H^1_{\mathrm{\acute{e}t}}(\kappa_K, \mathcal{G}_s)$ (see e.g. [Mi1], Chapter VI, Corollary 2.7).

(a) For the existence of the sequence $0 \to G \to \mathcal{A} \to \mathcal{B} \to 0$ see, e.g., ([Mi2], Appendix, Remark A7). The associated long exact sequence of *fppf* cohomology

$$0 \longrightarrow G(\mathcal{O}_K) \longrightarrow \mathcal{A}(\mathcal{O}_K) \longrightarrow \mathcal{B}(\mathcal{O}_K) \longrightarrow H^1_{\mathrm{fl}}(\mathcal{O}_K, G) \longrightarrow H^1_{\mathrm{fl}}(\mathcal{O}_K, \mathcal{A})$$

yields the assertion since $\mathcal{A}(\mathcal{O}_K) = A(K)$, $\mathcal{B}(\mathcal{O}_K) = B(K)$ and since we have by (4.3.2) and by Lang's theorem (see, e.g., [Se], VI.6)

$$H^1_{\mathrm{fl}}(\mathcal{O}_K, \mathcal{A}) \cong H^1(\kappa_K, \mathcal{A}_s) = 0.$$

(b) By ([Mi2], Chapter III, Lemma 1.1 (a)) the natural morphism $H^1_{\mathrm{fl}}(\mathcal{O}_K, G) \to H^1_{\mathrm{\acute{e}t}}(K, G_K)$ is injective. Therefore, to see that $K \mapsto H^1_{\mathrm{fl}}(\mathcal{O}_K, G)$ defines a Mackey functor, it is enough to show that for finite extensions $L/K/k$, the corestriction $H^1_{\mathrm{\acute{e}t}}(L, G_L) \to H^1_{\mathrm{\acute{e}t}}(K, G_K)$ maps $H^1_{\mathrm{fl}}(\mathcal{O}_L, G)$ into $H^1_{\mathrm{fl}}(\mathcal{O}_K, G)$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & G(\mathcal{O}_K) & \longrightarrow & \mathcal{A}(\mathcal{O}_K) & \longrightarrow & \mathcal{B}(\mathcal{O}_K) & \longrightarrow & H^1_{\mathrm{fl}}(\mathcal{O}_K, G) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \\
0 & \longrightarrow & G(K) & \longrightarrow & A(K) & \longrightarrow & B(K) & \longrightarrow & H^1_{\mathrm{\acute{e}t}}(K, G_K) & \longrightarrow & \dots
\end{array}$$

It shows that

$$\begin{aligned}
&\mathrm{Im}(H^1_{\mathrm{fl}}(\mathcal{O}_K, G) \to H^1_{\mathrm{\acute{e}t}}(K, G_K)) = \\
&\quad \mathrm{Im}(B(K) \to H^1_{\mathrm{\acute{e}t}}(K, G_K)) \cong \mathrm{Coker}(f\colon A(K) \to B(K)).
\end{aligned}$$

Since the lower row is natural with respect to corestrictions, the assertion follows immediately. For the last statement it is enough to remark that $N_{L/K}\colon B(L) \to B(K)$ is surjective (see [Maz], Corollary 4.4). □

LEMMA 4.3.3. (a) *Let G_i $(i = 1, 2, 3)$ be commutative finite flat group schemes over $\mathcal{O}$, and assume that $0 \to G_1 \to G_2 \to G_3 \to 0$ is a short exact sequence of sheaves for the fppf topology. Then the sequence*

$$H^1_{\mathrm{fl}}(\mathcal{O}_-, G_1) \longrightarrow H^1_{\mathrm{fl}}(\mathcal{O}_-, G_2) \longrightarrow H^1_{\mathrm{fl}}(\mathcal{O}_-, G_3) \longrightarrow 0$$

is exact.

(b) $H^1_{\mathrm{fl}}(\mathcal{O}_-, \mathbb{Z}/p^n) \cong \mathcal{Z}/p^n$ *and* $H^1_{\mathrm{fl}}(\mathcal{O}_-, \mu_{p^n}) \cong U/p^n$.

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(c) *Let* $\mathcal{A}$ *be an Abelian scheme over* Spec $\mathcal{O}$ *with generic fiber* A. *Then,* $H^1_{\mathrm{fl}}(\mathcal{O}_-, \mathcal{A}[p^n]) \cong A/p^n$.

Proof. We have $H^2(\mathcal{O}_K, G) = 0$ for any commutative finite flat group-scheme G over $\mathcal{O}_K$ (see, e.g., ([Mi2], Ch.III, Lemma 1.1(a))). This and the long exact sequence of fppf cohomology prove (a). The first assertion of (b) is a consequence of

$$H^1_{\mathrm{fl}}(\mathcal{O}_K, \mathbb{Z}/p^n) \cong H^1(\kappa_K, \mathbb{Z}/p^n) \cong \mathbb{Z}/p^n.$$

The first isomorphism is (4.3.2). The second isomorphism is given by evaluating a cocycle in $H^1(\kappa_K, \mathbb{Z}/p^m)$ at the Frobenius automorphism in $\mathfrak{g}_K$. The second assertion of (b) follows from $H^1_{\mathrm{fl}}(\mathcal{O}_K, \mathbb{G}_m) \cong \mathrm{Pic}(\mathcal{O}_K) = 0$ by using the Kummer sequence. Finally (c) is a special case of Lemma 4.3.1(b) above. □

LEMMA 4.3.4. *Let G be a commutative finite étale group scheme over* Spec $\mathcal{O}$. *Then the natural map*

$$k^* \otimes H^1(\kappa, G_s) \cong k^* \otimes H^1_{\mathrm{fl}}(\mathcal{O}, G) \longrightarrow [\mathbb{G}_m \overset{M}{\otimes} H^1_{\mathrm{fl}}(\mathcal{O}_-, G)](k)$$

is surjective.

Proof. By (4.3.2), we can identify $H^1_{\mathrm{fl}}(\mathcal{O}_K, G)$ with $H^1_{\mathrm{\acute{e}t}}(\kappa_K, G_s)$. It is an easy exercise to verify that under this identification the norm and the restrictions maps for finite extensions are given as $e_{L/K}\,\mathrm{cor}_{\kappa_L/\kappa_K}\colon H^1_{\mathrm{\acute{e}t}}(\kappa_L, G_s) \to H^1_{\mathrm{\acute{e}t}}(\kappa_K, G_s)$ and $\mathrm{res}_{\kappa_L/\kappa_K}\colon H^1_{\mathrm{\acute{e}t}}(\kappa_K, G_s) \to H^1_{\mathrm{\acute{e}t}}(\kappa_L, G_s)$. In particular, we note that if L/K is totally ramified, then $\mathrm{res}_{L/K}$ is an isomorphism, and for every $x \in H^1_{\mathrm{fl}}(\mathcal{O}_K, G)$, there exists a finite unramified extension M/K such that $\mathrm{res}_{M/K}(x) = 0$.

Let K/k be a finite extension with inertia field $K' \subseteq K$ and let $(a, b)_{K/k} \in [\mathbb{G}_m \overset{M}{\otimes} H^1_{\mathrm{fl}}(\mathcal{O}_-, G)](k)$ be a symbol. Then there is a $b' \in H^1_{\mathrm{fl}}(\mathcal{O}_{K'}, G)$ with $\mathrm{res}_{K/K'}(b') = b$, hence $(a, b)_{K/k} = (N_{K/K'}(a), b')_{K'/k}$. Let $\pi \in k^*$ be a prime element and write $N_{K/K'}(a) = \pi^m u$ for $u \in U_{K'}$. Then

$$(a, b)_{K/k} = m(\pi, b')_{K'/k} + (u, b')_{K'/k} = m(\pi, N_{K'/k}(b'))_{k/k} + (u, b')_{K'/k}.$$

To finish the proof it is enough to show that $(u, b')_{K'/k} = 0$. Choose a finite unramified extension M/K' with $\mathrm{res}_{M/K'}(b') = 0$. Let $u' \in U_M$ with $N_{M/K'}(u') = u$. We obtain $(u, b')_{K'/k} = (N_{M/K'}(u'), b')_{K'/k} = (u', \mathrm{res}_{M/K'}(b'))_{M/k} = 0$. □

(4.4) Recall that an Abelian variety of dimension d over a field F of characteristic $p > 0$ is said to be *ordinary* if it has p^d points of order p over an algebraic closure $\overline{F}$ of F. Equivalently, this can be also characterized by saying that the connected component of its group scheme of p-division points over $\overline{F}$ is isomorphic to μ_p^d. We will say that A/k has *semi-ordinary reduction* if A has semi-Abelian reduction and $\mathcal{A}_s^0[p]^0 = \mu_p^{\dim A}$ over $\overline{F}$ (here the second 0 denotes the connected component of the neutral element of a finite group scheme). The second condition is equivalent to requiring that the maximal Abelian quotient of $\mathcal{A}_s^0$ is ordinary. If, furthermore,

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the maximal torus of $\mathcal{A}_s^0$ splits over the residue field F, we say that A has *split semi-ordinary reduction*. Finally, A is said to have *potentially semi-ordinary reduction* if it has semi-ordinary reduction after base change by a finite extension K/k.

Now let A be an Abelian variety over a p-adic field k. Then we have the following generalization of Tate's p-adic uniformization theorem for elliptic curves with multiplicative reduction (due to Raynaud and Faltings/Chai):

THEOREM 4.4.1 (See [SGA VII], Part I, Exposé IX, §§2, 5, 7 or [FaCh], Ch.II and III for (i)-(iii); (iv) is proved in [FaCh], III.8.1). *Assume that A has semi-Abelian reduction. Let K/k be a finite unramified extension such that the maximal torus of $\mathcal{A}_s^0 \otimes_\kappa \kappa_K$ splits. Then there exists an exact sequence of commutative smooth group schemes over Spec $\mathcal{O}$*

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{A}^\sharp \longrightarrow \mathcal{B} \longrightarrow 0 \tag{4.4.2}$$

and a subgroup $\Gamma \subseteq \mathcal{A}^\sharp(K)$ such that

(i) *$\mathcal{T}_{\mathcal{O}_K} \cong \mathbb{G}_m^r$ is a split torus.*
(ii) *$\mathcal{B}$ is an Abelian scheme.*
(iii) *The special fibers of $\mathcal{A}^\sharp \otimes_\mathcal{O} \mathcal{O}_K$ and $\mathcal{A}^0 \otimes_\mathcal{O} \mathcal{O}_K$ coincide.*
(iv) *Γ is a free-Abelian group of rank r. For every finite extension L/K there is an isomorphism $\mathcal{A}^\sharp(L)/\Gamma \cong A(L)$*

The semi-Abelian scheme $\mathcal{A}^\sharp$ is called the *Raynaud extension* associated to A/k.

THEOREM 4.5. *Let $n \geqslant 2$ and let $A_1, \ldots, A_n/k$ be Abelian varieties with split semi-ordinary reduction. Then $K(k; A_1, \ldots, A_n) \cong F \oplus D$, where F is a finite group and D is divisible.*

Proof. Let $\mathcal{A}_i^\sharp, \mathcal{T}_i \cong \mathbb{G}_m^{r_i}, \mathcal{B}_i$ be as in (4.4.1) for A_i and $K = k$. Let $A_i^\sharp$ and B_i be the generic fibers of $\mathcal{A}_i^\sharp$ and $\mathcal{B}_i$ respectively. According to Theorem 3.5, Theorem 4.4.1(iv) and Lemma 3.4.4, it is enough to prove that $[A_1^\sharp \overset{M}{\otimes} \ldots \overset{M}{\otimes} A_n^\sharp](k)/p^n$ is finite and of order bounded independently of n. By Theorem 4.4.1 we have exact sequences of Mackey functors

$$0 \longrightarrow \mathbb{G}_m^{r_i} \longrightarrow A_i^\sharp \longrightarrow B_i \longrightarrow 0.$$

They yield a filtration on $A_1^\sharp \overset{M}{\otimes} \ldots \overset{M}{\otimes} A_n^\sharp$ whose successive quotients are quotients of $\bigotimes^M_{i \in I} B_i \overset{M}{\otimes} \bigotimes^M_{i \in J} \mathbb{G}_m^{r_i}$ for $I \dot{\cup} J = \{1, \ldots, n\}$. Therefore it is enough to show that $[\bigotimes^M_{i \in I} B_i \overset{M}{\otimes} \mathbb{G}_m^{\overset{M}{\otimes} j}](k)/p^n$ is finite and of order bounded independently of n for every subset $I \subseteq \{1, \ldots, n\}$ and $j = n - \#(I)$.

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Denote by $\mathcal{B}_i^t$ the dual Abelian scheme of $\mathcal{B}_i$. Let

$$0 \longrightarrow \mathcal{B}_i[p^n]^0 \longrightarrow \mathcal{B}_i[p^n] \longrightarrow \mathcal{B}_i[p^n]^{\text{ét}} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{B}_i^t[p^n]^0 \longrightarrow \mathcal{B}_i^t[p^n] \longrightarrow \mathcal{B}_i^t[p^n]^{\text{ét}} \longrightarrow 0$$

be the 'decomposition' of $\mathcal{B}_i[p^m]$, $\mathcal{B}_i^t[p^n]$ into a connected and an étale $\mathcal{O}$-scheme (see e.g., [Sh], §3, Proposition on p. 43). By Lemma 4.3.3(c), we obtain an exact sequence of Mackey functors

$$H^1_{\text{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^0) \longrightarrow B_i/p^n \longrightarrow H^1_{\text{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^{\text{ét}}) \longrightarrow 0.$$

Again we obtain filtrations on $\bigotimes^{M}_{i \in I} B_i \overset{M}{\otimes} \mathbb{G}_m^{\overset{M}{\otimes} j}/p^n$ which show that it is enough to prove that the following holds for every pair of disjoint subsets $I_1, I_2 \subseteq \{1, \ldots, n\}$ and $j = n - \#(I_1 \cup I_2)$:

(4.5.1) The group

$$\left[\bigotimes^{M}_{i \in I_1} H^1_{\text{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^{\text{ét}}) \overset{M}{\otimes} \bigotimes^{M}_{i \in I_2} H^1_{\text{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^0) \overset{M}{\otimes} \mathbb{G}_m^{\overset{M}{\otimes} j}/p^n\right](k)$$

is finite and of order bounded independently of n.

If $I_1 = \emptyset = I_2$, then $j = n \geqslant 2$ and (4.5.1) follows from Lemma 4.2.2 in this case. Now assume $I_1 \cup I_2 \neq \emptyset$. If K/k is an unramified extension the norm $N_{K/k}$: $H^1_{\text{fl}}(\mathcal{O}_L, \mathcal{B}_i[p^n]^*) \to H^1_{\text{fl}}(\mathcal{O}_K, \mathcal{B}_i[p^n]^*)$ (with $* = 0$ or *ét*) is surjective for every i according to Lemma 4.3.1. Therefore

$$N_{K/k}\colon \left[\bigotimes^{M}_{i \in I_1} H^1_{\text{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^{\text{ét}}) \overset{M}{\otimes} \bigotimes^{M}_{i \in I_2} H^1_{\text{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^0) \overset{M}{\otimes} \mathbb{G}_m^{\overset{M}{\otimes} j}/p^n\right](K) \longrightarrow$$
$$\left[\bigotimes^{M}_{i \in I_1} H^1_{\text{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^{\text{ét}}) \overset{M}{\otimes} \bigotimes^{M}_{i \in I_1} H^1_{\text{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^0) \overset{M}{\otimes} \mathbb{G}_m^{\overset{M}{\otimes} j}/p^n\right](k)$$

is also surjective. Hence we can replace k by a finite unramified extension if necessary. The assumption that each B_i has ordinary reduction is equivalent to the following fact about the Cartier duals (see, e.g., [Mu], §15, p. 147)

$$(\mathcal{B}_i[p^n]^{\text{ét}})^D \cong \mathcal{B}_i^t[p^n]^0, \quad (\mathcal{B}_i^t[p^n]^{\text{ét}})^D \cong \mathcal{B}_i[p^n]^0. \tag{4.5.2}$$

We choose a finite unramified extension K/k such that all $\mathcal{B}_i[p^n]^{\text{ét}} \otimes_{\mathcal{O}} \mathcal{O}_K$, $\mathcal{B}_i^t[p^n]^{\text{ét}} \otimes_{\mathcal{O}} \mathcal{O}_K$ are constant group schemes, hence $\cong (\mathbb{Z}/p^n)^{\dim(B_i)}$. The isomorphisms in (4.5.2) show then that $\mathcal{B}_i[p^n]^0 \otimes_{\mathcal{O}} \mathcal{O}_K \cong (\mu_{p^n})^{\dim(B_i)}$ for every i. From Lemma

(4.3.3), we get

$$\left[\overset{\mathrm{M}}{\bigotimes_{i\in I_1}} H^1_{\mathrm{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^{\text{ét}}) \overset{\mathrm{M}}{\otimes} \overset{\mathrm{M}}{\bigotimes_{i\in I_2}} H^1_{\mathrm{fl}}(\mathcal{O}_{-}, \mathcal{B}_i[p^n]^0) \overset{\mathrm{M}}{\otimes} \mathbb{G}_m^{\overset{\mathrm{M}}{\otimes} j}/p^n\right](K)$$
$$\cong \left[\overset{\mathrm{M}}{\bigotimes_{i\in I_1}} \mathcal{Z} \overset{\mathrm{M}}{\otimes} \overset{\mathrm{M}}{\bigotimes_{i\in I_2}} \mathcal{U} \overset{\mathrm{M}}{\otimes} \mathbb{G}_m^{\overset{\mathrm{M}}{\otimes} j}\right](K)/p^n.$$

According to Lemma 4.2.2, the latter group is finite and of bounded order (independently of n) as long as we are not in the case $\#(I_1) = 1, \#(I_2) = 0$ and $j = 1$.

This case of 4.5.1 has to be treated separately. We may assume $I_1 = \{1\}$ and set $\mathcal{B} = \mathcal{B}_1$ for simplicity. According to Lemma 4.3.4, the natural map

$$k^* \otimes H^1(\kappa, \mathcal{B}[p^n]^{\text{ét}}_s) \cong k^* \otimes H^1_{\mathrm{fl}}(\mathcal{O}, \mathcal{B}[p^n]^{\text{ét}}) \longrightarrow [\mathbb{G}_m \overset{\mathrm{M}}{\otimes} H^1_{\mathrm{fl}}(\mathcal{O}_{-}, \mathcal{B}[p^n]^{\text{ét}})](k) \qquad (4.5.3)$$

is surjective. Since $\#(H^1(\kappa, \mathcal{B}[p^n]^{\text{ét}}_s)) = \#(\mathcal{B}_s[p^n]^{\text{ét}}(\kappa)) \leqslant \#(\mathcal{B}_s(\kappa))$ we obtain a bound for the left hand side of (4.5.3) which is independent of n. This completes the proof of (4.5.1) and of the theorem. □

Remark 4.4.5. Under the assumptions of Theorem 4.5, the proofs of Theorems 3.5 and 4.5 show that $K(k; A_1, \ldots, A_d)$ is divisible for $d \geqslant 3$.

Theorem 4.5, Corollary 3.5.1 and Corollary 2.4.1 imply:

COROLLARY 4.5.6. *Let $X_1, \ldots, X_d$ be smooth, projective, geometrically connected curves over k with Jacobians $J_1, \ldots, J_d$ such that $X_i(k) \neq \emptyset$ for each i.*

(a) *Assume that $J_1, \ldots, J_d$ have split semi-ordinary reduction. Then the kernel of the Albanese map $A_0(X_1 \times \ldots \times X_d) \to J_1(k) \times \ldots \times J_d(k)$ is of the form $F \oplus D$ for a finite group F and a divisible group D. In particular, $CH_0(X_1 \times \ldots \times X_d)/m$ is finite for every positive integer m.*

(b) *If $J_1, \ldots, J_d$ have good ordinary reduction then F is a p-group.*

Again in the case $d = 2$, we can weaken the hypotheses:

COROLLARY 4.5.7. *Let $X_1, X_2/k$ be smooth, projective, geometrically connected curves. Assume that J_1, J_2 have potentially semi-ordinary reduction. Then,*

$$\mathrm{Ker}(alb : A_0(X_1 \times X_2) \longrightarrow J_1(k) \times J_2(k)) \cong F \oplus D,$$

where F is finite and D is divisible.

This is proved in the same way as Corollary 3.5.2 by using the finiteness of the p-torsion of $CH_0(X_1 \times X_2)$.

Remarks 4.5.8. (a) For m a postive integer and $i \geqslant 2$, there are as yet very few finiteness results for the group $CH^i(X)/m$ for varieties X over number fields or local fields (but see [CT2], Sect. 4.3 and [SaSu]). In the case of a smooth projective surface

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X over a p-adic field k, the only result concerning the mod p^n-part of the Chow group of zero cycles is due to Colliot-Thélène ([CT1], Théorème 8.5), who has shown that $CH_0(X)/p^n$ is finite if the Albanese mapping is injective for X over $\overline{k}$.⋆ The same result was obtained by Saito and Sujatha ([SaSu], Theorem 2.5) under the assumptions $H^2(X, \mathcal{O}_X) = 0$ and X is not of general type (in fact these assumptions imply that the Albanese map is injective). Since for a product of curves $X = X_1 \times X_2$ of genus $\geqslant 1$, we have $H^2(X, \mathcal{O}_X) \neq 0$, these results do not cover Corollary (4.5.7).

(b) Let k be a field and let $A_1, \dots, A_n$ be Abelian varieties over k. Somekawa has constructed a natural map

$$c\colon K(k; A_1, \dots, A_d)/m \longrightarrow H^n(k, A_1[m] \otimes \dots \otimes A_d[m]) \tag{4.5.9}$$

for every nonzero integer m prime to the characteristic of k ([So], Prop. 1.5) and has conjectured that it is always injective. This would imply the finiteness of the groups $K(k; A_1, \dots, A_n)/m$ (hence also of $CH_0(X_1 \times \dots \times X_d)/m$ for a product $X_1 \times \dots \times X_d$ of curves) if k is a local or global field. Indeed, in the latter case one can show that c factors through the finite group $H^n(G_S, A_1[m] \otimes \dots \otimes A_d[m])$, where S is a finite set of primes of k including all infinite primes, all primes dividing m and all primes where at least one of the A_i's has bad reduction and where G_S denotes the Galois group of the maximal extension of k which is unramified outside of S . In the situation considered in Theorem 4.5, one can show that (4.5.9) is injective if all m-division points of $A_1, \dots, A_n$ are k-rational.

(c) Let $k/\mathbb{Q}_p$ be a finite extension. We will give now an example which shows that it is really necessary to restrict the second assertion of Conjecture 3.5.4 to the prime to p part. Let $E_1, E_2/k$ be elliptic curves with good ordinary reduction, and assume that their p-division points are k-rational. Let $X = E_1 \times E_2$ and let $Y = \widetilde{E}_1 \times \widetilde{E}_2$ be its special fiber. Here $\widetilde{E}_i/\kappa$ denotes the special fiber of E_i, $i = 1, 2$. By (4.5.7) we have

$$K(k; E_1, E_2) = \mathrm{Ker}(\mathrm{alb} : A_0(X) \to E_1(k) \times E_2(k)) \cong F \oplus D$$

for a finite group F and a divisible group D (which should be uniquely divisible according to Conjecture 3.5.4). By analyzing the proof of Theorem 4.5, one can show that the composition

$$[E_1 \overset{\mathrm{M}}{\otimes} E_2](k)/p \longrightarrow\!\!\!\!\rightarrow K(k; E_1, E_2)/p \xrightarrow{c} H^2(k, E_1[p] \otimes E_2[p])$$

is injective and its image is cyclic of order p. This implies $F \neq 0$. On the other hand,

⋆Note that in [CT1], Théorème 8.5 it is also assumed that $H^2(X, O_X) = 0$. This is implied by the assumption that the Albanese mapping is injective over $\overline{k}$, as follows from the main result of [Ro].

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we have

$$\mathrm{Ker}(\mathrm{alb}: A_0(Y) \to \widetilde{E}_1(\kappa) \times \widetilde{E}_2(\kappa)) = K(\kappa; \widetilde{E}_1, \widetilde{E}_2) = 0$$

by (see [KS] or [Ka2]).

(d) The reason why we cannot say anything at the moment about the case where the Jacobian of one of the curves does not have semi-ordinary reduction is that the formal group of the Jacobian will then not be of multiplicative type, hence we cannot relate the corresponding K-group to a Milnor K-group of a field.

Appendix

We now finish the proof of Lemma 3.4.1. For a torsion group A and a prime number ℓ, denote by $A\{\ell\}$ the ℓ-primary component A. We have to show:

(A 1) $[\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](k)\{\ell\}$ *is divisible for almost all* ℓ.

(A 2) $[\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](k)\{\ell\}$ *is finite-by-($\mathbb{N}$-)divisible for all primes* $\ell \neq p$.

To see (A 1), let K/k be a finite unramified extension such that $\mathfrak{g}_K$ operates trivially on each Γ_i. We have already shown that $[\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](K)$ is divisible. For a prime ℓ not dividing $[K:k]$, the norm $N_{K/k}$: $[\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](K)\{\ell\} \to [\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](k)\{\ell\}$ is surjective. Thus (A 1) holds.

Now fix a prime $\ell \neq p$ and let $K = k_\ell$ be the fixed field of an ℓ-Sylow group of G_k. Then

$$\mathrm{res}_{K/k}\colon \varprojlim_{m \in \mathbb{N}} [\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](k)/\ell^m \to \varprojlim_{m \in \mathbb{N}} [\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](K)/\ell^m$$

is injective, so by Lemma 3.4.4, it is enough to show (A 2) for K instead of k. Since we are only interested in the ℓ-primary part of $[\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](K)$, we may replace Φ_i by $\Phi_i \otimes \mathbb{Z}_\ell$ for $i = 1, \ldots, n$ or assume from the beginning that we are dealing with $\mathfrak{g}_K$-$\mathbb{Z}_\ell$-lattices $\Gamma_i, \widetilde{\Gamma}_i$. For $m \geqslant 0$ let K_m/K be the unramified extension of degree ℓ^m. We choose m minimal with the property that $\mathfrak{g}_K$ operates trivially on $\Gamma_1, \ldots, \Gamma_n$, and we will show by induction on m and n that $[\Phi_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} \Phi_n](K)$ is finite-by-divisible.

The case $m = 0$, n arbitrary was done in the part of Lemma 3.4.1 we have already proved. Therefore let $m > 0, n \geqslant 2$. Let ϕ be a generator of $\mathfrak{g}_K \cong \mathbb{Z}_\ell$. We put

$$\Gamma_i^{(1)} = \Gamma_i^{\phi - 1}, \qquad \widetilde{\Gamma}_i^{(1)} = \widetilde{\Gamma}_i \cap \Gamma_i^{(1)}, \qquad \Gamma_i^{(2)} = \Gamma_i/\Gamma_i^{(1)}, \qquad \widetilde{\Gamma}_i^{(2)} = \widetilde{\Gamma}_i + \Gamma_i^{(1)}/\Gamma_i^{(1)}.$$

Note that $\Gamma_i^{(2)}, \widetilde{\Gamma}_i^{(2)}$ are again $\mathfrak{g}_K$-$\mathbb{Z}_\ell$-lattices and that we have exact sequences

$$0 \longrightarrow \Phi_i^{(1)} \longrightarrow \Phi_i \longrightarrow \Phi_i^{(2)}$$

for $i = 1, \dots, n$, where we have set $\Phi_i^{(1)} = \Phi_{(\Gamma_i^{(1)}, \widetilde{\Gamma}_i^{(1)})}$, $\Phi_i^{(2)} = \Phi_{(\Gamma_i^{(2)}, \widetilde{\Gamma}_i^{(2)})}$. Let Ψ_i be the image of Φ_i in $\Phi_i^{(2)}$. We need the following fact:

(A 3) For each $i = 1, \dots n$ there is a constant $c = c_i > 0$ such that for every finite totally ramified extension L/K, we have $\#(\Psi_i(L)) \leqslant c$.

Proof. Since $\Psi_i \subseteq \Phi_i^{(2)}$, it is enough to verify this for $\Phi_i^{(2)}$. We have for a finite totally ramified extension L/K,

$$\#(\Phi_i^{(2)}(L)) = \#(\Gamma_i^{(2)}/e_{L/K} \cdot \widetilde{\Gamma}_i^{(2)})^{\mathfrak{g}_L} = \#(\Gamma_i^{(2)}/[L:K] \cdot \widetilde{\Gamma}_i^{(2)})^{\phi - 1}$$
$$= \#(\Gamma_i^{(2)}/[L:K] \cdot \widetilde{\Gamma}_i^{(2)})_{(\phi-1)} \leqslant \#(\Gamma_i^{(2)})_{(\phi-1)} \stackrel{\text{def}}{=} c$$

Note that $(\Gamma_i^{(2)})_{(\phi-1)}$ is finite because — by definition of $\Gamma_i^{(1)}$ — 1 is not an eigenvalue of $\phi : \Gamma_i^{(2)} \otimes \mathbb{Q}_\ell \to \Gamma_i^{(2)} \otimes \mathbb{Q}_\ell$. □

The short exact sequences $0 \to \Phi_i^{(1)} \to \Phi_i \to \Psi_i \to 0$ induce a filtration on $[\Phi_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \Phi_n](K)$ whose successive quotients are quotients of $[\bigotimes_{i\in I}^{M} \Phi_i^{(1)} \overset{M}{\otimes} \bigotimes_{i\in J}^{M} \Psi_i](K)$ for partitions $I \dot{\cup} J = \{1, \dots, n\}$, and it remains to prove that the latter groups are finite-by divisible. For $I = \{1, \dots, n\}$, $\bigotimes_{i\in J}^{M} \Phi_i^{(1)}(K)$ is divisible, since $\mathfrak{g}_K$-operates trivially on each $\Gamma_i^{(1)}$.
Now assume $J \neq \emptyset$, say $n \in J$. By using the surjection

$$\left[\bigotimes_{i\in I}^{M} \Phi_i^{(1)} \overset{M}{\otimes} \bigotimes_{i\in J}^{M} \Phi_i\right](K_1) \to \left[\bigotimes_{i\in I}^{M} \Phi_i^{(1)} \overset{M}{\otimes} \bigotimes_{i\in J}^{M} \Psi_i\right](K_1)$$

and the induction hypothesis, we see that $[\bigotimes_{i\in I}^{M} \Phi_i^{(1)} \overset{M}{\otimes} \bigotimes_{i\in I}^{M} \Psi_i](K_1)$ is finite-by-divisible. If we put $\widetilde{\Phi} = \bigotimes_{i\in I}^{M} \Phi_i^{(1)} \overset{M}{\otimes} \bigotimes_{i\in J}^{M} \Psi_i$ and $\Psi = \Psi_n$, it remains to show that $C = \mathrm{Coker}(N_{K_1/K} : [\widetilde{\Phi} \overset{M}{\otimes} \Psi](K_1) \to [\widetilde{\Phi} \overset{M}{\otimes} \Psi](K))$ is finite. It is generated by the images of the symbols $(x, y)_{L/K}$, $x \in \widetilde{\Phi}(L)$, $y \in \Psi(L)$ for L/K finite and totally ramified. By (A 3) we know that the groups $\Psi(L)$ for such L are bounded. On the other hand, for every tower of finite extensions $M/L/K$ with M/L totally ramified, the restriction $\Phi_n^{(2)}(L) \to \Phi_n^{(2)}(M)$ is injective, hence $\Psi(L) \to \Psi(M)$ is injective. Note also that $\Psi(L)$ depends only on $e_{L/K}$ and $f_{L/K}$. Putting these facts together, we get

(A 4) *There is an integer N (a power of ℓ) such that for all totally ramified finite extensions $M/L/K$ with $[L:K] \geqslant N$, the restriction $\Psi(L) \to \Psi(M)$ is an isomorphism.*

Since G_K is solvable, any totally ramified extension M/K of degree $> N$ contains a subextension L/K of degree N. Therefore any symbol $(x, y)_{M/K}$, $x \in \widetilde{\Phi}(M)$, $y \in \Psi(M)$ can be written in the form $(x', y')_{L/K}$ with $[L:K] \leqslant N$, i.e. the natural map $\bigoplus_{L/K} \widetilde{\Phi}(L) \otimes \Psi(L) \longrightarrow C$ is surjective, where L/K runs through the totally ramified finite extensions of K of degree $\leqslant N$. By the induction hypotheses, $\widetilde{\Phi}(L)$ is finite-by-divisible ($\widetilde{\Phi}$ is a quotient of $\bigotimes_{i\in I}^{M} \Phi_i^{(1)} \overset{M}{\otimes} \bigotimes_{i\in J-\{n\}}^{M} \Phi_i$, which has only

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$n-1$ factors), hence $\widetilde{\Phi}(L)\otimes\Psi(L)$ is finite. Since there are only finitely many such extensions L/K (here we use the hypothesis $\ell\neq p$), we conclude that C is finite. This completes the proof of (A 2) and of Lemma (3.4.1).

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