Can. J. Math., Vol. XXXII, No. 5, 1980, pp. 1177–1205

COVER SET LATTICES

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Introduction. The proof of a main result in [**1**] concerning $(\mathbf{0},\mathbf{1})$-endomorphisms of finite lattices is based on properties of lattices $A(G)$ derived from the system of independent sets of an undirected loop-free graph G. For a number of questions naturally arising from [**1**] and [**2**], however, constructions employing only graph-induced complementation and properties of the lattices $A(G)$ associated with these are no longer adequate. The present paper introduces cover set lattices (a generalization of the lattices $A(G)$) to deal with some of these questions. A special case of the main result presented here states that for every $(\mathbf{0},\mathbf{1})$-lattice L and any monoid homomorphism $\varphi: M \rightarrow \mathrm{End}_{0,1}(L)$ there exists a lattice K containing L as a $(\mathbf{0},\mathbf{1})$-sublattice in such a way that the monoid $\mathrm{End}_{0,1}(K)$ of all $(\mathbf{0},\mathbf{1})$-endomorphisms of K is isomorphic to M, and the restriction to L of every $(\mathbf{0},\mathbf{1})$-endomorphism m of K is the $(\mathbf{0},\mathbf{1})$-endomorphism $\varphi(m)$ of L.

A significant feature of lattices $A(G)$, namely the ease of obtaining results on lattice complementations in proper lattice varieties, is inherited by cover set lattices. An example of this is the use of cover set lattices in a forthcoming paper which exhibitis $2^{\aleph_0}$ lattice varieties satisfying the famous theorem of Dilworth on uniquely complemented lattices.

The paper falls naturally into three sections: Section 1 introduces the notion of an $\mathscr{R}$-reduction and of an $\mathscr{R}$-reduced free product in an arbitrary lattice variety $\mathscr{V}$ as natural generalizations of a $\mathscr{C}$-reduction and of a $\mathscr{C}$-reduced free product ([**6**]), respectively. Associated with every $\mathscr{R}$-reduction is its cover set lattice. Lemma 1.1 establishes the crucial property of an $\mathscr{R}$-reduced $\mathscr{V}$-free product needed to prove the main result of this section, Theorem 1.6., that generalizes a result of [**4**] on $\mathscr{C}$-reduced free products to a result concerning the validity of an $\mathscr{R}$-reduction theorem for an $\mathscr{R}$-reduced $\mathscr{V}$-free product in any variety $\mathscr{V}$ containing the appropriate cover set lattice. Theorem 1.7 describes the complemented pairs of such $\mathscr{R}$-reduced $\mathscr{V}$-free products. Finally, Theorem 1.8 and Corollary 1.9 show that the reduction theorem often fails for reduced $\mathscr{V}$-free products whose cover set lattices do not belong to $\mathscr{V}$.

The second section returns to the special case of the graph-determined cover set lattices $A(G)$. The least variety $\mathscr{A}$ containing all lattices $A(G)$

Received December 18, 1978 and in revised form April 25, 1980. The support of the National Research Council of Canada is gratefully acknowledged.

is shown to be locally finite. The $(\mathbf{0}, \mathbf{1})$-lattices of $\mathscr{A}$ form a universal (binding) category, a result that considerably improves the conclusions of [**1**] and [**5**]. In addition, both subdirectly irreducible and simple upper (lower) cover set lattices of graphs are characterized here.

In the last section, a well-known question of E. Fried is answered by Theorem 3.1. Its proof uses the general concept of an $\mathscr{R}$-reduced free product and employs cover set lattices both as building blocks of its construction and as a means to establish the relevant structural properties of the resulting lattices. Theorem 3.9 shows that the proof of Theorem 1 will yield finite lattices provided all the initial data are finite; it also refines the main result of [**1**]. Theorems 3.7 and 3.8, in conjunction with [**3**], clarify the role of the set-theoretical axiom needed to strengthen Theorem 3.1; as stated, however, Theorem 3.1 requires no special set-theoretical considerations.

The paper also contains several unsolved problems; these are stated at the conclusion of the appropriate sections.

1. Cover set lattices. Before proceeding to the general definition, we consider a special case [1] to illustrate the general concept of a cover set lattice.

Let $G = (X, R)$ be an undirected graph without loops; that is, the set R of edges of G consists of two-element subsets of the vertex set X of G. A subset A of X is an *independent set* of G if it contains no element of R as a subset. The empty set $\emptyset$ is independent as is $\{x\}$ for every vertex $x \in X$. Independent sets of G are partially ordered by inclusion; let $I_*(G)$ be the poset of all finite independent sets of G extended by a largest element $\mathbf{1}$. It is easy to see that $I_*(G)$ is a bounded lattice in which $\emptyset$ serves as its smallest element, and that the join of two elements $A < \mathbf{1}$, $B < \mathbf{1}$ of $I_*(G)$ is $A \cup B$ if $A \cup B$ is an independent set of G, $A \vee B = \mathbf{1}$ if $A \cup B$ is dependent. The meet $A \wedge B$ is always the intersection of the independent sets A and B.

Another description of $I_*(G)$ is based on a concept of *lower weak direct product* $\Pi_*(C_x{:}x \in X)$ of three-element chains $C_x = \{\mathbf{0}, x, \mathbf{1}\}$: the lower weak direct product consists of all elements σ of $\Pi(C_x{:}x \in X)$ for which $\sigma(x) > \mathbf{0}$ for only finitely many indices $x \in X$. A set $R_* \subseteq \Pi_*(C_x{:}x \in X)$ (called the *upper reduction* associated with G) will consist of all elements $\sigma \in \Pi_*$ such that $\sigma(x) = \mathbf{1}$ for some $x \in X$ and of all those σ for which $\{x \in X{:}\sigma(x) > \mathbf{0}\}$ contains an edge of G. It is easy to see that adding a new unit $\mathbf{1}$ to the poset $\Pi_*(C_x{:}x \in X)\backslash R_*$ will produce a lattice isomorphic to $I_*(G)$. The zero $\emptyset$ of $I_*(G)$ is now represented by the zero $\mathbf{0}$ of $\Pi_*(C_x{:}x \in X)$ and, for every $x \in X$, the independent set $\{x\}$ corresponds to the sequence x_* of Π_* defined by $x_*(x) = x$, $x_*(y) = \mathbf{0}$ for $y \neq x$. The dual $I^*(G)$ of $I_*(G)$ can now be thought of as a subposet of the upper weak direct product $\Pi^*(C_x{:}x \in X)$ with a new zero added, from which a lower reduction R^* was removed; R^* consists of all $\tau \in \Pi^*$ with $\tau(x) = \mathbf{0}$ for some $x \in X$ together with all τ for which $\{x \in X : \tau(x) < \mathbf{1}\}$ is dependent. Observe that the unit of $I^*(G)$ is represented by the unit of Π^* and that the independent set $\{x\}$ of G corresponds to the sequence x^* of Π^* satisfying $x^*(x) = x$ and $x^*(y) = \mathbf{1}$ for $y \neq x$. The $(\mathbf{0}, \mathbf{1})$-sublattice $A(G)$ of $I_*(G) \times I^*(G)$ generated by all pairs (x_*, x^*) is the *cover set lattice* of the system $(C_x : x \in X)$ of lattices subject to the reduction $\mathscr{R} = (R_*, R^*)$. Lemma 7 of [**1**] indicates a reason for this terminology: if $\Lambda(G)$ is a bounded lattice generated by X in which $x \vee y = \mathbf{1}$ and $x \wedge y = \mathbf{0}$ whenever $\{x, y\} \in R$ and if the mapping $x \mapsto (x_*, x^*)$ extends to a $(\mathbf{0}, \mathbf{1})$-homomorphism $\varphi : \Lambda(G) \to A(G)$, then for every $\mathbf{0} < a < \mathbf{1}$ in $\Lambda(G)$ the nonzero values of the first component σ of $\varphi(a) = (\sigma, \tau)$ are exactly all $x \in X$ below a in $\Lambda(G)$; values of τ similarly form the set of all upper covers of a in X. An important corollary [**1**] of the existence of such a homomorphism φ states that the nontrivial complemented pairs of $\Lambda(G)$ are exactly the edges of G as described above.

The present section aims to extend the concept of such a cover set lattice and to apply the generalization to systems $\mathbf{L} = (L_i : i \in I)$ of lattices subject to a (possibly not symmetric) reduction $\mathscr{R} = (R_*, R^*)$. As a consequence, a generalization of the fundamental theorem [**4**] on $\mathscr{C}$-reduced free products will be proved in any variety of lattices containing the cover set lattice determined by $\mathbf{L}$ and $\mathscr{R}$.

For an arbitrary set $\mathbf{L} = (L_i : i \in I)$ of bounded lattices, let $\Pi_*\mathbf{L}$ denote their *lower weak direct product*, that is, the sublattice of $\Pi\mathbf{L} = \Pi(L_i : i \in I)$ consisting of all σ for which $\sigma(i) = \mathbf{0}$ for all but finitely many indices $i \in I$; similarly, let $\Pi^*\mathbf{L}$ (the *upper weak direct product*) be the lattice of all $\tau \in \Pi\mathbf{L}$ satisfying $\tau(i) = \mathbf{1}$ for all but finitely many $i \in I$. For every $i \in I$, set $Q_i = L_i \backslash \{\mathbf{0}, \mathbf{1}\}$, and let $Q = \cup (Q_i : i \in I)$ be the union of the posets Q_i in which no pair of elements of different components Q_i, Q_j is comparable. For $x \in Q_i$, let $x_* \in \Pi_*\mathbf{L}$ be the sequence defined by $x_*(i) = x$, $x_*(j) = \mathbf{0}$ for $j \neq i$; analogously, $x^*(i) = x$, $x^*(j) = \mathbf{1}$ for $j \neq i$ define an element of $\Pi^*\mathbf{L}$. An *upper reduction* R_* is a subset of $\Pi_*\mathbf{L}$ satisfying

(1) if $\sigma(i) = \mathbf{1}$ for some $i \in I$, then $\sigma \in R_*$,
(2) if $\sigma \in R_*$ and $\rho \geqq \sigma$, then $\rho \in R_*$,
(3) if $x \in Q$, then $x_* \notin R_*$.

A lower reduction $R^* \subseteq \Pi^*\mathbf{L}$ satisfies the dual conditions

(1′) if $\tau(i) = \mathbf{0}$ for some $i \in I$, then $\tau \in R^*$,
(2′) if $\tau \in R^*$ and $\rho \leqq \tau$, then $\rho \in R^*$,
(3′) if $x \in Q$, then $x^* \notin R^*$.

Extend the poset $\Pi_*\mathbf{L}\backslash R_*$ by adding a new unit $\mathbf{1}$. The resulting poset is a bounded lattice called the *lower cover set lattice* corresponding to $\mathbf{L}$ and R_*; we will denote it by $\Gamma_* = \Gamma_*(\mathbf{L}, R_*)$. The *upper cover set lattice* $\Gamma^* = \Gamma^*(\mathbf{L}, R^*)$ is obtained through an addition of a new zero $\mathbf{0}$ to the poset $\Pi^*\mathbf{L}\backslash R^*$. If $\mathscr{R} = (R_*, R^*)$, then the *cover set lattice* $\Gamma = \Gamma(\mathbf{L}, \mathscr{R})$ is defined as the $(\mathbf{0}, \mathbf{1})$-sublattice of $\Gamma_*(\mathbf{L}, R_*) \times \Gamma^*(\mathbf{L}, R^*)$ generated by all pairs (x_*, x^*) with $x \in Q$. It is easy to see that the mapping $x \mapsto x_*$ extends to a $(\mathbf{0}, \mathbf{1})$-embedding $(\varphi_i)_*:L_i \to \Gamma_*(\mathbf{L}, R_*)$ and that $x_* \leqq y_*$ if and only if $x \leqq y$ in a component Q_i of Q. Similar properties hold for $\Gamma^*(\mathbf{L}, R^*)$ and, consequently, the $(\mathbf{0}, \mathbf{1})$-sublattice of $\Gamma(\mathbf{L}, \mathscr{R})$ consisting of $(\mathbf{0}, \mathbf{0})$, $(\mathbf{1}, \mathbf{1})$, and of all (x_*, x^*) with $x \in Q_i$ is isomorphic to L_i. If $\varphi_i:L_i \to \Gamma(\mathbf{L}, \mathscr{R})$ is a $(\mathbf{0}, \mathbf{1})$-embedding thus defined, then $\varphi_i(x) \leqq \varphi_j(y)$ for $x, y \in Q$ if and only if $i = j$ and $x \leqq y$ in the lattice L_i.

Observe that for any $\sigma \in \Pi_*\mathbf{L}$ we have

$$\bigvee(\sigma(i)_*:\sigma(i) > \mathbf{0}) = \sigma;$$

Similarly, if $\tau \in \Pi^*\mathbf{L}$, then

$$\bigwedge(\tau(i)^*:\tau(i) < \mathbf{1}) = \tau.$$

Let $*\mathbf{L} = *(L_i:i \in I)$ denote the absolutely free product of lattices $L_i (i \in I)$ and let $(R_*, R^*) = \mathscr{R}$ satisfy (1)–(3′). Let

$$\varphi_*:*\mathbf{L} \to \Gamma_*(\mathbf{L}, R_*)$$

be the homomorphism uniquely extending the system $((\varphi_i)_*:i \in I)$ of $(\mathbf{0}, \mathbf{1})$-embeddings $(\varphi_1)_*:L_i \to \Gamma_*(\mathbf{L}, R_*)$ and let

$$\varphi^*:*\mathbf{L} \to \Gamma^*(\mathbf{L}, R^*)$$

be defined analogously. The mapping $\varphi = \varphi_* \times \varphi^*$ is a $(\mathbf{0}, \mathbf{1})$-homomorphism of $*\mathbf{L}$ onto $\Gamma(\mathbf{L}, \mathscr{R})$ such that $\varphi(x) = (x_*, x^*)$ for all $x \in Q$.

Finally, let Λ be a bounded lattice for which there is an onto homomorphism g: $*\mathbf{L} \to \Lambda$ satisfying

(4) $g(\bigvee (\sigma(i):\sigma(i) > \mathbf{0})) = \mathbf{1}$ for all $\sigma \in R_*$,
(4′) $g(\bigwedge (\tau(i):\tau(i) < \mathbf{1})) = \mathbf{0}$ for all $\tau \in R^*$.

Thus, in particular, g is a $(\mathbf{0}, \mathbf{1})$-homomorphism.

The lemma that follows will be of central importance for most of our considerations.

LEMMA 1.1. *If $x \in Q$ and $X \in *\mathbf{L}$, then*

(5) $\quad \varphi_*(x) \leqq \varphi_*(X)$ *implies* $g(x) \leqq g(X)$,
(5′) $\quad \varphi^*(x) \geqq \varphi^*(X)$ *implies* $g(x) \geqq g(X)$.

Proof. We shall prove (5) by induction on the rank of X (see, for instance, [**6**]).

(a) Let $\operatorname{rank}(X) = 1$, that is, $X \in L_i$ for some $i \in I$:

If $X = \mathbf{0}$, then $\varphi_*(X) = \mathbf{0}$ and $\varphi_*(x) = x_* \leqq \mathbf{0}$ for no $x \in Q$. Hence (5) holds trivially. The same is true for $X = \mathbf{1}$, for $g(x) \leqq \mathbf{1} = g(\mathbf{1})$ is satisfied for all $x \in Q$. If $X \in Q_i$ and $x_* = \varphi_*(x) \leqq \varphi_*(X) = X_*$, then $x_*(j) \leqq X_*(j) = \mathbf{0}$ for all $j \neq i$. Since $x \in Q$, $x_*(i) > \mathbf{0}$ and, consequently, $x = x_*(i) \leqq X_*(i) = X$ in L_i. Hence $g(x) \leqq g(X)$, as required.

(b) Let $\operatorname{rank}(X) > 1$ and $X = Y \wedge Z$ for $Y, Z \in *\mathbf{L}$ of a smaller rank:

Hence $\varphi_*(x) \leqq \varphi_*(Y \wedge Z) = \varphi_*(Y) \wedge \varphi_*(Z)$ implies that $\varphi_*(x) \leqq \varphi_*(Y)$ and $\varphi_*(x) \leqq \varphi_*(Z)$. By the induction hypothesis, $g(x) \leqq g(Y)$ and $g(x) \leqq g(Z)$. Hence

$$g(x) \leqq g(Y) \wedge g(Z) = g(X).$$

(c) Let $\operatorname{rank}(X) > 1$ and $X = Y \vee Z$ for $Y, Z \in *\mathbf{L}$ of a smaller rank:

If $\varphi_*(Y) = \mathbf{1}$, then $\varphi_*(x) \leqq \varphi_*(Y)$ for all $x \in Q$ and the induction hypothesis yields $g(x) \leqq g(Y) \leqq g(X)$ for all $x \in Q$. We may, therefore, assume that $\varphi_*(Y) = \beta < \mathbf{1}$ and $\varphi_*(Z) = \gamma < \mathbf{1}$ holds true in $\Gamma_*(\mathbf{L}, R_*)$.

Assume first that $\beta \vee \gamma \in R_*$. If $\beta(i) > \mathbf{0}$, then

$$\varphi_*(\beta(i)) = \beta(i)_* \leqq \beta = \varphi_*(Y)$$

and, by the induction hypothesis, $g(\beta(i)) \leqq g(Y)$ for all $i \in I$ with $\beta(i) > \mathbf{0}$. Hence

$$\bigvee (g(\beta(i)) : \beta(i) > \mathbf{0}) \leqq g(Y)$$

and, similarly we find

$$\bigvee (g(\gamma(i)) : \gamma(i) > \mathbf{0}) \leqq g(Z).$$

Combining these two inequalities yields

$$\begin{aligned} g(X) = g(Y) \vee g(Z) \geqq \bigvee &(g(\beta(i)) : \beta(i) > \mathbf{0}) \\ &\vee \bigvee (g(\gamma(i)) : \gamma(i) > \mathbf{0}) \\ = g(\bigvee &((\beta \vee \gamma)(i) : (\beta \vee \gamma)(i) > \mathbf{0})). \end{aligned}$$

Since $\beta \vee \gamma \in R_*$, the latter inequality together with (4) imply $g(x) \leqq \mathbf{1} = g(X)$ for all $x \in Q$, thus proving (5) in this case.

Next, let $\beta \vee \gamma \notin R_*$. Then

$$\varphi_*(X) = \varphi_*(Y) \vee \varphi_*(Z) = \beta \vee \gamma.$$

If $x \in Q_i$, then $\varphi_*(x) \leqq \varphi_*(X)$ holds if and only if $x \leqq \beta(i) \vee \gamma(i) < \mathbf{1}$ in L_i. If $\beta(i) > \mathbf{0}$, the induction hypothesis and $\varphi_*(\beta(i)) = \beta(i)_* \leqq \beta = \varphi_*(Y)$ imply that $g(\beta(i)) \leqq g(Y)$; if $\beta(i) = \mathbf{0}$, then $g(\beta(i)) = g(\mathbf{0}) = \mathbf{0} \leqq g(Y)$ holds trivially. Altogether, $g(\beta(i)) \leqq g(Y)$ for each $i \in I$ and, similarly, $g(\gamma(i)) \leqq g(Z)$ for all indices $i \in I$. If $\varphi_*(x) \leqq \varphi_*(X)$, then, as stated above, $x \leqq \beta(i) \vee \gamma(i)$ and hence

$$g(x) \leqq g(\beta(i)) \vee g(\gamma(i)) \leqq g(Y) \vee g(Z) = g(X).$$

This finishes the proof, for (5′) is a claim dual to (5).

For a lattice homomorphism ψ, let Ker ψ denote the kernel congruence of ψ.

PROPOSITION 1.2. *Let $x \in Q$ and $X \in *\mathbf{L}$. If* Ker $g \subseteq$ Ker φ_*, *then*

(6) $\quad \varphi_*(x) \leqq \varphi_*(X)$ *if and only if* $g(x) \leqq g(X)$;

if Ker $g \subseteq$ Ker φ^*, *then*

(6′) $\quad \varphi^*(x) \geqq \varphi^*(X)$ *if and only if* $g(x) \geqq g(X)$.

Proof. Ker $g \subseteq$ Ker φ_* implies the existence of a $(\mathbf{0}, \mathbf{1})$-preserving homomorphism $\psi_*: \Lambda \rightarrow \Gamma_*(\mathbf{L}, R_*)$ such that $\varphi_* = \psi_* \circ g$. Hence $g(x) \leqq g(X)$ implies $\varphi_*(x) \leqq \varphi_*(X)$; the converse follows from Lemma 1.1. (6′) is a dual of (6).

It is easy to see that Ker $g \subseteq$ Ker φ_* implies that Λ is a bounded lattice generated by a copy of Q and containing each L_i as a $(\mathbf{0}, \mathbf{1})$-sublattice. If $\varphi_* = \psi_* \circ g$, then $\psi_*(a) = \mathbf{1}$ if and only if $a = \mathbf{1}$; dually, $\psi^*(a) = \mathbf{0}$ if and only if $a = \mathbf{0}$ in Λ, as long as $\varphi^* = \psi^* \circ g$.

PROPOSITION 1.3. *If $\varphi_* = \psi_* \circ g$ and if $a < \mathbf{1}$ is an element of Λ, then the sequence $\alpha = \psi_*(a) \in \Gamma_*(\mathbf{L}, R_*)$ is such that $\alpha(i)$ is the largest element of L_i below a in Λ. In other words, $\psi_*(a)$ is the sequence of lower covers of a in Λ. Dually, if $\varphi^* = \psi^* \circ g$, then $\psi^*(a)$ is the sequence of upper covers of an element $a > \mathbf{0}$ of Λ.*

Proof. If $a < \mathbf{1}$ is an element of Λ, the remark preceding Proposition 1.3 shows that $\alpha = \psi_*(a) < \mathbf{1}$ in $\Gamma_*(\mathbf{L}, R_*)$. Let $X \in *\mathbf{L}$ be such that $g(X) = a$. Then, for every $i \in I$,

$$\varphi_*(\alpha(i)) = \alpha(i)_* \leqq \alpha = \psi_*(a) = \varphi_*(X).$$

If $\alpha(i) > \mathbf{0}$, Lemma 1.1 yields

$$\alpha(i) = g(\alpha(i)) \leqq g(X) = a;$$

https://doi.org/10.4153/CJM-1980-089-0 Published online by Cambridge University Press

if $\alpha(i) = \mathbf{0}$, then $\mathbf{0} = g(\alpha(i)) \leqq a$ is satisfied trivially. In either case, $\alpha(i) \in L_i$ satisfies $\alpha(i) \leqq a$. To show that $\alpha(i)$ is the lower cover of a in L_i, choose $b \in L_i \subseteq \Lambda$ such that $b \leqq a = g(X)$. Since $b = g(b)$ if $b \in Q$, (6) gives

$$b_* = \varphi_*(b) \leqq \varphi_*(X) = \psi_*(a) = \alpha,$$

so that, in particular, $b = b_*(i) \leqq \alpha(i)$ in L_i. If $b = \mathbf{0}$, there is nothing to prove.

A similar argument proves the dual statement.

Remark. If $\operatorname{Ker} g \subseteq \operatorname{Ker} \varphi_* \cap \operatorname{Ker} \varphi^*$, then the factorizing homomorphism $\psi = \psi_* \times \psi^*$ satisfying $\psi \circ g = \varphi$ assigns to every $a \in \Lambda\backslash \{\mathbf{0}, \mathbf{1}\}$ a pair (σ, τ) of sequences that is an element of $\Gamma(\mathbf{L}, \mathscr{R})$: the sequence $\sigma = \psi_*(a)$ of lower covers of a in component lattices L_i of Λ and the sequence $\tau = \psi^*(a)$ of upper covers of a. This is the rationale for naming $\Gamma(\mathbf{L}, \mathscr{R})$ a cover set lattice.

Following is a useful reformulation of Proposition 1.2.

PROPOSITION 1.4. *Let* $\operatorname{Ker} g \subseteq \operatorname{Ker} \varphi_*$ *and let* $A \subseteq \Lambda\backslash\{\mathbf{1}\}$ *be a finite set. Then* $\bigvee A = \mathbf{1}$ *in* Λ *if and only if for every* $a \in A$ *there exists a sequence* $\sigma_a \in \Pi_*\mathbf{L}$ *such that* $\sigma_a(i) \leqq a$ *for all* $i \in I$ *and the sequence* $\sigma = \bigvee (\sigma_a : a \in A)$ *belongs to* R_*.

Dually, $\operatorname{Ker} g \subseteq \operatorname{Ker} \varphi^*$ *implies that for every finite* $A \subseteq \Lambda\backslash\{\mathbf{0}\}$ *there are* $\tau_a \in \Pi^*\mathbf{L}$ *such that* $\tau_a(i) \geqq a$ *for all* $i \in I$ *and* $a \in A$ *with* $\bigwedge (\tau_a : a \in A) \in R^*$ *if and only* if $\bigwedge A = \mathbf{0}$.

Proof. If $\operatorname{Ker} g \subseteq \operatorname{Ker} \varphi_*$, let ψ_* satisfy $\varphi_* = \psi_* \circ g$. If $\sigma_a(i) \leqq a$ for all $a \in A$ and all $i \in I$, then

$$\begin{aligned} \sigma_a &= \bigvee (\sigma_a(i)_* : \sigma_a(i) > \mathbf{0}) \\ &= \bigvee (\psi_*(\sigma_a(i)) : \sigma_a(i) > \mathbf{0}) \\ &= \psi_*(\bigvee (\sigma_a(i) : \sigma_a(i) > \mathbf{0})) \leqq \psi_*(a) \end{aligned}$$

for every $a \in A$. Since $\bigvee (\sigma_a : a \in A) \in R_*$ is assumed,

$$\mathbf{1} = \bigvee (\sigma_a : a \in A) \leqq \bigvee (\psi_*(a) : a \in A) = \psi_*(\bigvee A)$$

holds in $\Gamma_*(\mathbf{L}, R_*)$. Hence $\psi_*(\bigvee A) = \mathbf{1} = \psi_*(\mathbf{1})$ and $\bigvee A = \mathbf{1}$ follows from $\psi_*^{-1}\{\mathbf{1}\} = \{\mathbf{1}\}$.

Conversely, let $\bigvee A = \mathbf{1}$, $A \subseteq \Lambda\backslash\{\mathbf{1}\}$. Let $\sigma_a = \psi_*(a)$ for every $a \in A$. Proposition 1.3 implies that $\sigma_a(i) \leqq a$ for all $i \in I$. Furthermore,

$$\bigvee (\sigma_a : a \in A) = \bigvee (\psi_*(a) : a \in A) = \psi_*(\bigvee A) = \psi_*(\mathbf{1}) = \mathbf{1}$$

holds in $\Gamma_*(\mathbf{L}, R_*)$, and we conclude that $\bigvee (\sigma_a : a \in A) \in R_*$.

The dual statement is proved analogously.

Proposition 1.4 claims that the join of a finite set $A \subseteq \Lambda\backslash\{\mathbf{1}\}$ is $\mathbf{1}$ if and only if it is forced to be by the prescribed upper reduction R_*. We say that "the reduction theorem for R_* holds in a lattice Λ" if the equivalence stated by the first part of Proposition 1.4 is valid in Λ; similarly for R_* and $\mathscr{R} = (R_*, R^*)$. To formulate a generalization of the result of [**4**], we define the concept of an $\mathscr{R}$-reduced $\mathscr{V}$-free product of lattices as follows.

Definition. Let $\mathscr{V}$ be a variety of lattices, let $\mathbf{L} = (L_i : i \in I)$ be a system of bounded lattices from $\mathscr{V}$. Let $\mathscr{R} = (R_*, R^*)$ be a reduction. A bounded lattice $L \in \mathscr{V}$ is an *$\mathscr{R}$-reduced $\mathscr{V}$-free product* of $\mathbf{L}$, denoted $L = \mathscr{V}(\mathbf{L}, \mathscr{R})$, if there is a system $m = (m_i : i \in I)$ of $(\mathbf{0}, \mathbf{1})$-preserving homomorphisms, $m_i : L_i \rightarrow L$, such that

(7) $\quad \bigvee (m_i(\sigma(i)) : \sigma(i) > \mathbf{0}) = \mathbf{1}$ whenever $\sigma \in R_*$,

(7′) $\quad \bigvee (m_i(\tau(i)) : \tau(i) < \mathbf{1}) = \mathbf{0}$ whenever $\tau \in R^*$,

(8′) $\quad$ if $m' = (m_i' : i \in I)$ is a system of $(\mathbf{0}, \mathbf{1})$-preserving homomorphisms $m_i' : L_i \rightarrow L' \in \mathscr{V}$ satisfying (7) and (7′), then there is a unique homomorphism $e : L \rightarrow L'$ such that $e \circ m_i = m_i'$ for every $i \in I$.

Remark. It is easy to see that $L = \mathscr{V}(\mathbf{L}, \mathscr{R})$ is determined uniquely up to isomorphism and that it is a bounded lattice generated by $\bigcup (m_i(L_i) : i \in I)$. Furthermore, let $\mathscr{V}(\mathbf{L})$ be the $\mathscr{V}$-free product of $\mathbf{L} = (L_i : i \in I)$ extended to a bounded lattice through an addition of a new $\mathbf{0}$ and a new $\mathbf{1}$; let Θ be the smallest congruence on $\mathscr{V}(\mathbf{L})$ such that

$$\bigvee (\sigma(i) : \sigma(i) > \mathbf{0})\ \Theta\ \mathbf{1} \text{ for all } \sigma \in R_*$$

and

$$\bigwedge (\tau(i) : \tau(i) < \mathbf{1})\ \Theta\ \mathbf{0} \text{ for all } \tau \in R^*.$$

A straightforward argument now shows that $\mathscr{V}(\mathbf{L}, \mathscr{R}) \cong \mathscr{V}(\mathbf{L})/\Theta$.

Let $\mathbf{K} = (K_j : j \in J)$ be another system of lattices from $\mathscr{V}$ and let $\mathscr{S} = (S_*, S^*)$ be a reduction of $\mathbf{K}$. Let $n_j : K_j \rightarrow K = \mathscr{V}(\mathbf{K}, \mathscr{S})$ be the $(\mathbf{0}, \mathbf{1})$-preserving homomorphisms such that $n = (n_j : j \in J)$ satisfies the requirements of the definition of $\mathscr{V}(\mathbf{K}, \mathscr{S})$.

The following simple lemma on homomorphisms $\mathscr{V}(\mathbf{L}, \mathscr{R}) \rightarrow \mathscr{V}(\mathbf{K}, \mathscr{S})$ will be used in the last section of the paper.

PROPOSITION 1.5. *Let $\alpha : I \rightarrow J$ be an arbitrary mapping and let $f = (f_i : L_i \rightarrow K_{\alpha(i)} : i \in I)$ be a system of $(\mathbf{0}, \mathbf{1})$-preserving lattice homomorphisms.*

For every $\sigma \in \Pi_*\mathbf{L}$ define $f_*(\sigma) \in \Pi_*\mathbf{K}$ by

$$[f_*(\sigma)](\alpha(i)) = f_i(\sigma(i)) \text{ and}$$

$$[f_*(\sigma)](j) = \mathbf{0} \text{ for } j \in J\backslash\alpha(I);$$

similarly, if $\tau \in \Pi^*\mathbf{L}$, *define*

$$[f^*(\tau)](\alpha(i)) = f_i(\tau(i)) \text{ and}$$
$$[f^*(\tau)](j) = \mathbf{1} \text{ for } j \in J\backslash\alpha(I).$$

If

(9) $f_*(\sigma) \in S_*$ *for every* $\sigma \in R_*$, *and*
(9′) $f^*(\tau) \in S^*$ *for every* $\tau \in R^*$,

then there is a unique $(\mathbf{0}, \mathbf{1})$*-preserving homomorphism* $F:L \rightarrow K$ *such that* $F \circ m_i = n_{\alpha(i)} \circ f_i$ *holds for all* $i \in I$.

Proof. The mapping $m_i' = n_{\alpha(i)} \circ f_i:L_i \rightarrow K$ is a $(\mathbf{0}, \mathbf{1})$-preserving homomorphism and

$$m_i'(\sigma(i)) = n_{\alpha(i)} \circ f_i(\sigma(i)) = n_{\alpha(i)}([f_*(\sigma)](\alpha(i)))$$

for each $i \in I$. Since $f_i(\mathbf{0}) = \mathbf{0}$ for every $i \in I$, we have

$$\bigvee (m_i'(\sigma(i)):\sigma(i) > \mathbf{0}) \geqq \bigvee (m_i'(\sigma(i)):[f_*(\sigma)](\alpha(i)) > \mathbf{0})$$

and the latter join equals

$$\bigvee (n_{\alpha(i)}(f_*(\sigma)(\alpha(i))):[f_*(\sigma)](\alpha(i)) > \mathbf{0}).$$

If $\sigma \in R_*$, then $f_*(\sigma) \in S_*$ by (9) and hence the last join must equal $\mathbf{1}$. Consequently,

$$\bigvee (m_i'(\sigma(i)):\sigma(i) > \mathbf{0}) = \mathbf{1} \text{ for every } \sigma \in R_*$$

and this proves (7) for the system $m' = (m_i':i \in I)$. An analogous argument shows that m' also satisfies (7′). From (8) we now conclude that there is a unique $F:L \rightarrow K$ such that

$$F \circ m_i = m_i' = n_{\alpha(i)} \circ f_i \text{ for all } i \in I.$$

THEOREM 1.6. *If* $\Gamma(\mathbf{L}, \mathscr{R}) \in \mathscr{V}$, *then all homomorphisms* $m_i:L_i \rightarrow \mathscr{V}(\mathbf{L}, \mathscr{R})$ *from the definition of* $\mathscr{V}(\mathbf{L}, \mathscr{R})$ *are one-to-one and* $\mathscr{V}(\mathbf{L}, \mathscr{R})$ *is generated by the poset* $\bigcup (m_i(L_i\backslash\{\mathbf{0}, \mathbf{1}\}):i \in I)$ *canonically isomorphic to* Q. *Furthermore, the reduction theorem for* $\mathscr{R}$ *holds in* $\mathscr{V}(\mathbf{L}, \mathscr{R})$.

Proof. Let $r:\mathscr{V}(\mathbf{L}) \rightarrow (\mathbf{L}, \mathscr{R})$ be the homomorphism whose kernel is the congruence Θ defined in the remark preceding Proposition 1.5. If $v:*\mathbf{L} \rightarrow \mathscr{V}(\mathbf{L})$ is the canonical homomorphism (that is, v is the common extension of all identity homomorphisms $L_i \rightarrow L_i$), then clearly

$$\operatorname{Ker}(r \circ v) \subseteq \operatorname{Ker}(\varphi).$$

This, in turn, implies that the copy of Q described in the statement of Theorem 1.6 generates $\mathscr{V}(\mathbf{L}, \mathscr{R})$. The lattice $\mathscr{V}(\mathbf{L}, \mathscr{R})$ satisfies the reduction theorem for $\mathscr{R}$ because Proposition 1.4 applies to $\Lambda = \mathscr{V}(\mathbf{L}, \mathscr{R})$ and $g = r \circ v$.

Remark. If $\Gamma(\mathbf{L}, \mathscr{R})$ and $\Gamma(\mathbf{K}, \mathscr{S})$ belong to $\mathscr{V}$, then the conclusion of Proposition 1.5 asserts, in view of Theorem 1.6, that F is the unique

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extension of all homomorphisms $f_i : L_i \to K_{\alpha(i)}$ of component sublattices of respective reduced $\mathscr{V}$-free products. To illustrate a simple consequence of Proposition 1.5, consider three-element chains for all lattices L_i and K_j and isomorphisms $f_i : L_i \to K_{\alpha(i)}$ for every $i \in I$ in the variety $\mathscr{L}$ of all lattices. If the reductions $\mathscr{R}, \mathscr{S}$ are those determined by graphs G, H on I, J respectively, then (9) and (9′) simply say that α is a compatible mapping of the graph G into the graph H. The extension F determined by Proposition 1.5 is then the lattice homomorphism $M(\alpha)$ of $\mathscr{L}(\mathbf{L}, \mathscr{R}) = M(G)$ into $M(H) = \mathscr{L}(\mathbf{K}, \mathscr{S})$ considered in [5]. The lattice $\Gamma(\mathbf{L}, \mathscr{R})$ then becomes $A(G)$, and $\Gamma(\mathbf{K}, \mathscr{S}) \cong A(H)$. Theorem 1.6 implies that $M(G)$ is a bounded lattice generated by an antichain isomorphic to I; the validity of the relevant reduction theorem in $M(G)$ means that the complemented pairs of $M(G)$ are $\{\mathbf{0}, \mathbf{1}\}$ and all pairs $\{i, i'\}$ of vertices of G that are edges of G.

A nontrivial use of Proposition 1.5 appears in the last section of the paper.

THEOREM 1.7. *Let $L = \mathscr{V}(\mathbf{L}, \mathscr{R})$ be an $\mathscr{R}$-reduced $\mathscr{V}$-free product in a variety $\mathscr{V}$ of lattices containing $\Gamma(\mathbf{L}, \mathscr{R})$. Then $\{a, b\} \neq \{\mathbf{0}, \mathbf{1}\}$ is a complemented pair in L if and only if there are $\{x, z\} \subseteq Q_i$, $\{y, t\} \subseteq Q_j$ satisfying $x \leqq a \leqq z$ and $y \leqq b \leqq t$ such that either $i = j$ and $\{v, y\}$, $\{z, t\}$ are complemented pairs in L_i, or $i \neq j$ and $x_* \vee y_* \in R_*$, $z^* \wedge t^* \in R^*$.*

Proof. By Theorem 1.6, the reduction theorem for $\mathscr{R}$ holds in $\mathscr{V}(\mathbf{L}, \mathscr{R})$. Hence there exist σ_a, σ_b in $\Pi_*\mathbf{L}$ and τ_a, τ_b in $\Pi^*\mathbf{L}$ such that $\sigma_a \vee \sigma_b \in R_*$ and $\tau_a \wedge \tau_b \in R^*$, and $\sigma_a(i) \leqq a \leqq \tau_a(j)$, $\sigma_b(i) \leqq b \leqq \tau_b(j)$ hold for all $i, j \in I$. Since $\{a, b\} \cap \{\mathbf{0}, \mathbf{1}\} = \emptyset$, neither σ_a or σ_b belong to R_*; also, $\tau_a, \tau_b \notin R^*$ and, consequently, $\sigma_a > \mathbf{0}$, $\sigma_b > \mathbf{0}$, $\tau_a < \mathbf{1}$, and $\tau_b < \mathbf{1}$.

Observe that $\sigma_a(i) > \mathbf{0}$ and $\tau_a(j) < \mathbf{1}$ can hold simultaneously only if $i = j$, for the elements of Q_i and Q_j are pairwise incomparable in $\mathscr{V}(\mathbf{L}, \mathscr{R})$ for $i \neq j$. Since $\sigma_a > \mathbf{0}$, there is an $i \in I$ with $\sigma_a(i) > \mathbf{0}$ and our observation yields $\tau_a = \tau_a(i)^*$; this, in turn, implies that $\sigma_a = \sigma_a(i)_*$. A similar argument shows that $\sigma_b = \sigma_b(j)_*$ and $\tau_b = \tau_b(j)^*$. Let $\sigma_a(i) = x$, $\tau_a(i) = z$, $\sigma_b(j) = y$, and $\tau_b(j) = t$.

First of all, assume that $i = j$. We may have $\sigma_a \vee \sigma_b \in R_*$ only if $x \vee y = \mathbf{1}$ in L_i, for $(\sigma_a \vee \sigma_b)(k) = \mathbf{0}$ for $k \neq i$. Analogously, $z \wedge t = \mathbf{0}$ holds in L_i; since $x \leqq a \leqq z$ and $y \leqq b \leqq t$, $\{x, y\}$, $\{z, t\}$ are complemented pairs of L_i.

Secondly, let $i \neq j$. Then $x_* \vee y_* = \sigma_a \vee \sigma_b \in R_*$ and $z^* \wedge t^* = \tau_a \wedge \tau_b \in R^*$.

This finishes the proof, for the converse implication is trivial.

Remark. Theorem 1.7 extends the Chen–Grätzer theorem on $\mathscr{C}$-reduced free products [4] in two ways. First, the reduction $\mathscr{R}$ is not

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necessarily determined by a $\mathscr{C}$-relation (i.e., a relation imposing complementation on elements from distinct factors L_i, L_j). Secondly, Theorem 1.7 is valid in any variety $\mathscr{V}$ of lattices containing the relevant cover set lattice $\Gamma(\mathbf{L}, \mathscr{R})$. Both of these features play an essential role in the proof of the main result of the last section.

It should be pointed out that the proof of Theorem 1.7 only interprets the reduction theorem in $\mathscr{V}(\mathbf{L}, \mathscr{R})$ and does not use the stronger assumption $\Gamma(\mathbf{L}, \mathscr{R}) \in \mathscr{V}$ directly. The existence of covers in $\mathscr{V}(\mathbf{L}, \mathscr{R})$ also is a consequence of $\Gamma(\mathbf{L}, \mathscr{R}) \in \mathscr{V}$ (see Proposition 1.3) and there appears to be no reason why a reduction theorem should fail in the absence of covers. (Examples exist of varieties of lattices for which not all elements of a $(\mathbf{0}, \mathbf{1})$-free product have covers in every lattice component.) In a special case of separating reductions described below, however, the presence of $\Gamma(\mathbf{L}, \mathscr{R})$ in $\mathscr{V}$ becomes also necessary for the validity of a respective reduction theorem in $\mathscr{V}(\mathbf{L}, \mathscr{R})$.

Definition. An upper reduction $R_* \subseteq \Pi_*\mathbf{L}$ has the *separating property* if for every ideal I of $\Pi_*\mathbf{L}$ and every $q_* \notin I$ such that $((q_*] \vee I) \cap R_* = \emptyset$ there is an ideal $J \supset I$ satisfying both $J \cap R_* = \emptyset$ and $((q_*] \vee J) \cap R_* \neq \emptyset$.

The separating property of a lower reduction R^* is defined dually. A reduction $\mathscr{R} = (R_*, R^*)$ is *separating* if both R_* and R^* are.

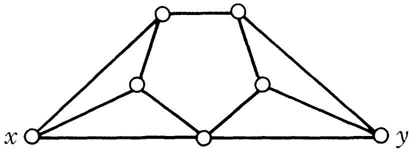

FIG. 1

The separating property has a particularly transparent interpretation in the case of a reduction R_* associated with a graph G: here it says that for every independent set $I \cup \{v\}$ such that $v \notin I$ there is another independent set $J \supset I$ such that $J \cup \{v\}$ is dependent. Thus every disjoint union of complete graphs has a separating reduction. Also, every graph G is a full subgraph of another one, H, such that the reduction associated with H is separating. For every pair of vertices $x, y \in G$ extend the graph by a copy of the graph shown in Fig. 1. It is a simple matter to check that the graph H thus obtained induces a separating reduction.

THEOREM 1.8. *Let $\mathscr{R} = (R_*, R^*)$ be a separating reduction for $\mathbf{L} = (L_i : i \in I)$. If Λ is a bounded lattice generated by Q in which the reduction theorem for $\mathscr{R}$ is valid, then there is an onto homomorphism $\psi: \Lambda \to \Gamma(\mathbf{L}, \mathscr{R})$ such that $\psi(q) = (q_*, q^*)$ for all $q \in Q$.*

Proof. Assuming the separating property of R_*, we will show that the

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(upper) reduction theorem implies the existence of an onto homomorphism $\lambda: \Lambda \to \Gamma_*(\mathbf{L}. R_*)$ such that $\lambda(q) = q_*$ for all $q \in Q$. To this end, define a mapping h of Λ into the ideal lattice Δ_* of $\Gamma_*(\mathbf{L}, R_*)$ as follows:

$$h(a) = \{\sigma \in \Gamma_*:\sigma(i) \leqq a \text{ for all } i \in I\} \text{ for every } a < \mathbf{1} \text{ in } \Lambda,$$
$$h(\mathbf{1}) = \Gamma.$$

It is easy to see that every $h(a)$ is an ideal of Γ_*, that $h(q) = (q_*]$ for all $q \in Q$, and that h is a meet-preserving mapping. Were h also join-preserving, it would be a homomorphism of Λ onto a sublattice of Δ_* isomorphic to Γ_*. Let us assume that, on the contrary, there are $a, b \in \Lambda$ such that $h(a \vee b) \supset h(a) \vee h(b)$. Clearly $a, b < \mathbf{1}$.

Assume first that $h(a \vee b) = \Gamma_*$. Then $q \leqq a \vee b$ for all $q \in Q$. Since a separating R_* is nonempty, there is a $\sigma \in R_*$ and

$$\bigvee (\sigma(i):\sigma(i) > \mathbf{0}) \leqq a \vee b.$$

Because the reduction theorem for R_* holds in Λ, $a \vee b$ must be the unit of Λ. However, $h(a) \vee h(b)$ is a proper ideal of Γ_* and, therefore, $\sigma_a \vee \sigma_b \notin R_*$ for all σ_a, σ_b satisfying $\sigma_a(i) \leqq a$ and $\sigma_b(i) \leqq b$ for all $i \in I$. The finite set $\{a, b\} \subseteq \Lambda\backslash\{\mathbf{1}\}$ violates the reduction theorem.

$h(a \vee b)$ must, therefore, be a proper ideal of Γ_*, so that $a \vee b < \mathbf{1}$ in Λ. Let γ be an arbitrary sequence in $h(a \vee b)\backslash(h(a) \vee h(b))$. Clearly $\gamma(i)_* \notin h(a) \vee h(b)$ for at least one index $i \in I$, for otherwise $\gamma \in h(a) \vee h(b)$. Let $q = \gamma(i)$ and apply the separating property to $h(a) \vee h(b) = I$ and q_*. There is an ideal $J \supset h(a) \vee h(b)$ such that $J \cap R_* = \emptyset$ and $((q_*] \vee J) \cap \mathrm{R}_* \neq \emptyset$; the latter property implies the existence of a $\sigma \in J$ such that $\sigma \vee q_* \in R_*$. Let

$$S = \{\sigma(i):\sigma(i) > \mathbf{0}\},$$

and set

$$A = S \cup \{a, b\}.$$

Clearly, $A \subseteq \wedge\backslash\{\mathbf{1}\}$ is a finite set and

$$\bigvee A = (\bigvee S) \vee a \vee b \geqq \bigvee S \vee q = \mathbf{1}$$

because $q \leqq a \vee b$ and $\sigma \vee q_* \in R_*$. Simultaneously, however, if $\sigma_z(i) \leqq z$ for $i \in I$ and each $z \in A$, then

$$\bigvee (\sigma_z:z \in A) = \sigma_a \vee \sigma_b \vee \bigvee (\sigma_s:s \in S) \leqq \sigma_a \vee \sigma_b \vee \sigma \in J;$$

hence

$$\bigvee (\sigma_z:z \in A) \notin R_*.$$

The set A thus exhibits a failure of the reduction theorem in Λ.

As a result, h is a lattice homomorphism.

COROLLARY 1.9. *If $\mathscr{R}$ is a separating reduction for a system* $\mathbf{L}$ *of lattices from a variety* $\mathscr{V}$ *and if* $\mathscr{V}(\mathbf{L}, \mathscr{R})$ *is generated by a copy of* Q*, then the reduction theorem for* $\mathscr{R}$ *holds in* $\mathscr{V}(\mathbf{L}, \mathscr{R})$ *if and only if* $\mathscr{V}$ *contains* $\Gamma(\mathbf{L}, \mathscr{R})$.

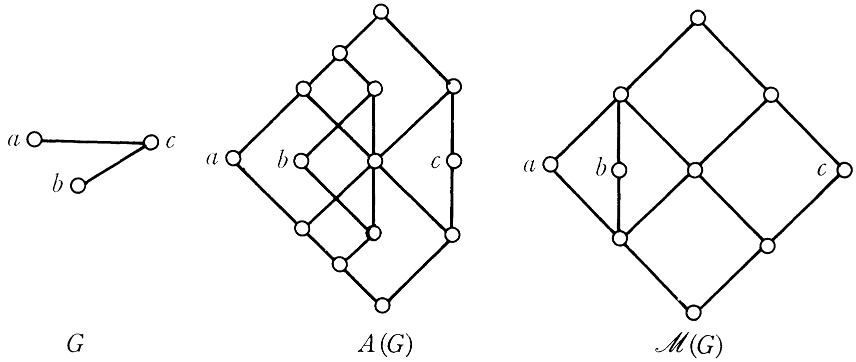

FIG. 2

A reduction theorem may be valid in $\mathscr{V}(\mathbf{L}, \mathscr{R})$ even though $\Gamma(\mathbf{L}, \mathscr{R})$ is not a member of the variety $\mathscr{V}$. In other words, some non-separating reductions are "natural" for a given variety $\mathscr{V}$. Fig. 2 shows, in order, a non-separating graph G, its (non-modular) cover set lattice $A(G)$, and a modular lattice that satisfies the reduction theorem associated with G. The lattice $\mathscr{M}(G)$ is the $\mathscr{R}$-reduced $\mathscr{M}$-free product of three 3-element chains and $\mathscr{R}$ is determined by G; $\mathscr{M}$ denotes the variety of all modular lattices.

We conclude by listing some of the problems arising from the above results.

Problem 1.1. Are there properties of reductions, other than the separating property, that force $\Gamma(\mathbf{L}, \mathscr{R})$ to belong to a variety $\mathscr{V}$ under the assumption of the validity of the reduction theorem in $\mathscr{V}(\mathbf{L}, \mathscr{R})$?

Problem 1.2. Are there lattices, other than $\Gamma(\mathbf{L}, \mathscr{R})$, testing the validity of a reduction theorem in $\mathscr{V}(\mathbf{L}, \mathscr{R})$ in the sense of Corollary 1.9?

Problem 1.3. Can the reductions "natural" for a given variety $\mathscr{V}$ be characterized in terms of the identities of $\mathscr{V}$?

2. Cover set lattices associated with graphs. In this section we investigate the properties of lattices $A(G)$ and the variety $\mathscr{A}$ they generate in more detail.

Recall that, for a given undirected graph $G = (X, R)$, the lattice $A(G)$ is isomorphic to the sublattice of $I_*(G) \times I^*(G)$ generated by all pairs $(\{x\}, \{x\})$ with $x \in X$. The lattice $I_*(G)$ is obtained from the poset of all finite independent sets of G by adding a largest element $\mathbf{1}$, and $I^*(G)$ is the dual of $I_*(G)$.

Let $\mathscr{A}_*$ denote the variety of lattices generated by all $I_*(G)$, let $\mathscr{A}^*$ be generated by the class of all $I^*(G)$. Then $\mathscr{A}$ denotes the join of these two varieties, $\mathscr{A} = \mathscr{A}_* \vee \mathscr{A}^*$.

LEMMA 2.1. *$\mathscr{A}$ is a locally finite variety.*

Proof. Let $L \subseteq I_*(G)$ be generated by a set A of at most n elements. If M is the set of all possible meets of elements of A, then every element $l < \mathbf{1}$ of L is a join of members of M. Hence L has at most 2^{2^n} elements and there are only finitely many pairwise nonisomorphic sublattices L of $I_*(G)$ or of $I^*(G)$ with no more than n generators. Since the n-generated $\mathscr{A}$-free lattice $\mathscr{A}(n)$ is a subdirect product of n-generated sublattices of $I_*(G)$ or $I^*(G)$ and there are only finitely many of these, $\mathscr{A}(n)$ belongs to a variety generated by finitely many lattices; such a variety is locally finite and thus $\mathscr{A}(n)$ is a finite lattice.

The local finiteness of $\mathscr{A}$ enables us to stengthen the main result of [**1**]. Recall that a category $\mathscr{C}$ is *binding* or *universal* if every category of algebras is isomorphic to a full subcategory of $\mathscr{C}$. Equivalently, $\mathscr{C}$ is binding if there is a full and one-to-one functor $F: \mathscr{G}_3 \to \mathscr{C}$, where $\mathscr{G}_3$ is the category of all undirected 3-completely connected graphs [**10**]; for a complete definition of $\mathscr{G}_3$, see Section 3 of this paper.

THEOREM 2.2. *The category $\mathscr{C}$ of all bounded lattices in the variety $\mathscr{A}$ and all their* $(\mathbf{0}, \mathbf{1})$*-preserving homomorphisms is universal: there is a full embedding $M: \mathscr{G}_3 \to \mathscr{C}$ such that $M(G)$ is finite for any finite graph $G \in \mathscr{G}_3$. Consequently, every finite category is isomorphic to a full category of $\mathscr{C}_{\text{fin}}$, where $\mathscr{C}_{\text{fin}}$ is the full subcategory of $\mathscr{C}$ determined by all finite bounded lattices of $\mathscr{A}$.*

Proof. Given a graph $G = (X, R) \in \mathscr{G}_3$, let $M(G)$ be a homomorphic image of the $\mathscr{A}$-free bounded lattice $\mathscr{A}(X)$ over X under a homomorphism whose kernel $\equiv$ is the smallest congruence for which $x \vee y \equiv \mathbf{1}$ and $x \wedge y \equiv \mathbf{0}$ whenever $\{x, y\} \in R$. The reader is referred to [**1**], where it is shown that the nontrivial complemented pairs of $M(G)$ are exactly the pairs $\{x, y\}$ of (incomparable) generators that belong to R; this conclusion can also be arrived at by using Theorem 1.7. It is easy to verify that M extends naturally to a one-to-one functor; the complete description of complemented pairs of $M(G)$ then implies the fullness of M by an argument contained essentially in [**5**]. Since finite members of $\mathscr{G}_3$ determine a full subcategory of $\mathscr{G}_3$ containing all finite categories as full subcategories (see also Section 3 of this paper), the local finiteness of $\mathscr{A}$ yields the second claim of the theorem.

The remainder of the section establishes results needed for the proof of Theorem 3.1.

A graph $H = (Y, S)$ is a *full subgraph* of $G = (X, R)$ if $Y \subseteq X$ and S consists of the two-element subsets of Y that belong to R. Equivalently, a finite $F \subseteq Y$ is an independent set of H if and only if it is an independent set of G. The lemma below easily follows.

LEMMA 2.3. *If H is a full subgraph of G, then $I_*(H)$ is a $(\mathbf{0}, \mathbf{1})$-sublattice of $I_*(G)$ and $I^*(H)$ is a $\mathbf{0}, \mathbf{1})$-sublattice of $I^*(G)$.*

We say that $i \in X$ is an *isolated vertex of* $G = (X, R)$ if $i \notin r$ for all $r \in R$.

PROPOSITION 2.4. *Let $G = (X, R)$ be a nonempty graph and let I denote the set of isolated vertices of G. Then $I_*(G)$ is subdirectly irreducible if and only if*

(s) $\operatorname{card}(X) > 2 > \operatorname{card}(I)$.

Proof. If $\operatorname{card}(X) \leqq 2$, then $I_*(G)$ is a distributive lattice with more than two elements. For every isolated vertex i of G, the congruence $\Theta_i = \Theta(\{i\}, \emptyset)$ has sets $\{\mathbf{1}\}$ and all $\{a, a \cup \{i\}\}$ with $a < \mathbf{1}$ and $i \notin a$ for its congruence classes. If $i, j \in I$ are distinct, then $\Theta_i \cap \Theta_j = \omega$; the lattice $I_*(G)$ is a subdirect power of $I_*(G_i)$, where G_i is the full subgraph of G on a vertex set $(X \backslash I) \cup \{i\}$. This establishes the necessity of (s).

Let G satisfy (s) and let $\omega < \Theta < \iota$ be a congruence of $I_*(G)$. Assume that $a\Theta b$ for $a, b \in I_*(G)$ with $a < b$. There is an $x_0 \in X$ such that $\{x_0\} \leqq b$ and $a \cap \{x_0\} = \emptyset$; consequently, $\{x_0\}\Theta\emptyset$ and hence Θ contains a principal congruence $\Theta(\{x_0\}, \emptyset)$. If $\{x_0, y\} \in R$ for some $y \in X$, then $\{y\}\Theta\mathbf{1}$ and $\{x\}\Theta\emptyset$ for every $x \neq y$. We see that every nonextremal congruence Θ contains Θ_i whenever $I = \{i\}$ and G has more than two vertices. If $I = \emptyset$, then $\Theta < \iota$ implies that $\{x, x'\} \notin R$ if x, x' are distinct from y and hence $\{y, x\} \in R$ for every $x \neq y$. The congruence Θ has just two classes: $\{\mathbf{1}\} \cup \{a : y \in a\}$ and $\{a : y \notin a\}$. If $\varphi > \omega$ is a congruence not containing Θ, then $\varphi \supset \Theta(\{y\}, \emptyset)$; because $\{x, y\} \in R$ for all $x \neq y$, $\{x\}\varphi\mathbf{1}$ for all these x. Since G has more than two vertices, the latter claim yields $\emptyset\varphi\mathbf{1}$. Hence Θ is the only nonextremal congruence of $I_*(G)$ if G has no isolated points.

A graph $G = (X, R)$ is a *star* if there is a vertex y_0 of G for which $R = \{\{x, y_0\} : x \neq y_0\}$. The above proof also yields the following.

PROPOSITION 2.5. *$I_*(G)$ is a simple lattice if and only if G has no isolated vertices and is not a star.*

We conclude this section by a list of problems suggested by the investigations presented here.

Problem 2.1. Theorem 2.2 states that the variety $\mathscr{A}$ is universal as a category. Are there proper universal subvarieties of $\mathscr{A}$?

Problem 2.2. Find the defining identities of $\mathscr{A}$. Is $\mathscr{A}$ finitely based?

3. An application: homomorphisms of bounded lattices and of their (0, 1)-sublattices. Throughout this section, we will restrict ourselves to $(\mathbf{0}, \mathbf{1})$-preserving homomorphisms of bounded lattices, thus eliminating constant homomorphisms from our considerations. $\mathscr{L}$ will denote the category of all bounded lattices with more than one element and all $(\mathbf{0}, \mathbf{1})$-preserving homomorphisms of these lattices. If L is a lattice in $\mathscr{L}$, $\mathrm{End}_{\mathbf{0},\mathbf{1}}(L)$ will denote the monoid of all $(\mathbf{0}, \mathbf{1})$-preserving endomorphisms of L.

A consequence of the main result of [**5**] states that the endomorphism monoids of a bounded lattice and its $(\mathbf{0}, \mathbf{1})$-sublattice are independent: for every pair M_1, M_2 of monoids there are bounded lattices L_1, L_2 such that L_1 is a $(\mathbf{0}, \mathbf{1})$-sublattice of L_2 and $\mathrm{End}_{\mathbf{0},\mathbf{1}}(L_1) \cong M_i$ for $i = 1, 2$. A closer examination of this claim shows that for any general construction of such lattices every endomorphism of L_2 that maps L_1 into itself must leave L_1 pointwise fixed. To see this, it is sufficient to take different prime-order cyclic groups for the monoids M_1, M_2 to be represented. This independence result is strengthened in [**2**], where it is shown that every nontrivial bounded lattice L occurs as a $(\mathbf{0}, \mathbf{1})$-sublattice of some lattice L' with a prescribed endomorphism monoid. It is clear that the endomorphism monoids of the pair $L \subseteq L'$ are subject to the same general requirement.

If, in general, $L_1 \subseteq L_2$ are bounded lattices such that every endomorphism f of L_2 preserves the sublattice L_1, then the restriction $f \restriction L_1$ is an endomorphism of L_1 and the mapping $f \mapsto f \restriction L_1$ is a monoid homomorphism of $\mathrm{End}_{\mathbf{0},\mathbf{1}}(L_2)$ into $\mathrm{End}_{\mathbf{0},\mathbf{1}}(L_1)$; we may ask what are the monoid homomorphisms $\varphi: M_1 \rightarrow M_2$ representable in this manner. [**2**] shows that any constant φ (i.e., φ defined by $\varphi(m_1) = 1_{M_2}$ for all $m_1 \in M_1$) is representable. A special case of the main theorem of this section generalizes both these results as follows: given a nontrivial bounded lattice L and a monoid homomorphism $\varphi: M' \rightarrow \mathrm{End}_{\mathbf{0},\mathbf{1}}(L)$, there is a lattice L' containing L as a $(\mathbf{0}, \mathbf{1})$-sublattice such that $M' \cong \mathrm{End}_{\mathbf{0},\mathbf{1}}(L')$ and the restriction to L of the endomorphism f_m of L' representing $m \in M'$ is the endomorphism $\varphi(m)$ of L.

To formulate the main theorem, we recall some additional categorical concepts.

A category $\mathbf{K}$ is *small* if its morphism class is a set. A *full embedding* is a (covariant) functor $\Phi: \mathbf{A} \rightarrow \mathbf{B}$ which is one-to-one and maps $\mathbf{A}$ onto a full subcategory of $\mathbf{B}$. If $U: \mathbf{A} \rightarrow \mathbf{Set}$ and $V: \mathbf{B} \rightarrow \mathbf{Set}$ are fixed faithful functors (i.e., functors that are one-to-one on all hom-sets of respective domain categories), we say that a full embedding $\Phi: \mathbf{A} \rightarrow \mathbf{B}$ is an *extension* of $\mathbf{A}$ into $\mathbf{B}$ if there is a natural transformation $\mu: U \to V \circ \Phi$ consisting of one-to-one mappings $\mu_A: U(A) \to V(\Phi(A))$; such a μ will be called a *monotransformation*. Given functors $F, G: \mathbf{K} \to \mathscr{L}$, we say that F is a *subfunctor* of G if there is a system of one-to-one $(\mathbf{0}, \mathbf{1})$-preserving homomorphisms $h_A: F(A) \to G(A)$ forming a natural transformation $h: F \to G$, again called a monotransformation.

THEOREM 3.1. *If* $\mathbf{K}$ *is a small category and* $F: \mathbf{K} \to \mathscr{L}$ *is a functor, then there is a full embedding* $\Phi: \mathbf{K} \to \mathscr{L}$ *containing* F *as a subfunctor.*

The restriction of smallness of $\mathbf{K}$ cannot be removed, for the inclusion functor $F: \mathscr{L}' \to \mathscr{L}$ of any proper subcategory $\mathscr{L}'$ of $\mathscr{L}$ whose objects are all objects of $\mathscr{L}$ is not a subfunctor of any full embedding [**3**]. The main result of [**2**] says that, on the other hand, any constant functor $F: \mathbf{K} \to \mathscr{L}$ whose domain is fully embeddable into a category of algebras does occur as a subfunctor of a suitable full embedding. Under the set-theoretical assumption

(M) there is a cardinal δ such that every δ-complete ultrafilter is principal,

every concrete category $\mathbf{K}$ (i.e., category with a given faithful functor $H: \mathbf{K} \to \mathbf{Set}$) can be extended into any universal category of algebras or graphs [**13**]. Hence any constant functor F from a concrete category $\mathbf{K}$ into $\mathscr{L}$ is a subfunctor of a full embedding Φ if (M) is assumed. The concretizability of $\mathbf{K}$ is, of course, also necessary.

The above results naturally suggest the question of whether Theorem 3.1 holds for functors F whose domain $\mathbf{K}$ is a category of algebras (and thus fully embeddable into $\mathscr{L}$; see [**5**]) and such that the range of F has only a set of objects. Results of [**3**] show that, surprisingly, (M) is a necessary requirement for the validity of Theorem 3.1 even for these functors F. On the other hand, Theorem 3.8 below states that (M) is also sufficient. Let us emphasize that no set-theoretical restrictions are needed in the proof of Theorem 3.1 as stated above.

Theorem 3.1 is better illustrated by considering some of its special cases.

If $\mathbf{K}$ is a one-object category, i.e., if the morphism set $\mathbf{K}^m$ of $\mathbf{K}$ is a monoid, then F really is a monoid homomorphism of $\mathbf{K}^m$ into $\mathrm{End}_{\mathbf{0},\mathbf{1}}(L)$, where L is the image of the single object of $\mathbf{K}$. Theorem 3.1 then gives the extension of particular results of [**5**] and [**2**] discussed earlier.

If F is a constant functor, Theorem 3.1 describes a special case of the principal result of [**2**].

For a small (not necessarily full) subcategory $\mathbf{L}$ of $\mathscr{L}$ and its inclusion functor $F: \mathbf{L} \to \mathscr{L}$, Theorem 3.1 claims the existence of lattices $L' = \Phi(L)$ containing objects L of $\mathbf{L}$ as $(\mathbf{0}, \mathbf{1})$-sublattices in such a way that each homomorphism $f: L_1 \to L_2$ in $\mathbf{L}$ is uniquely extended to a homomorphism

$$\Phi(f) = f': L_1' \to L_2',$$

and the latter are all morphisms from L_1' to L_2' in $\mathscr{L}$. Thus if, for instance, $\mathscr{L}^m$ consists of identity homomorphisms only (i.e., if $\mathbf{L}$ is a discrete category), then lattices $L' \supseteq L$ determine a discrete full subcategory of $\mathscr{L}$. Loosely stated, this consequence expresses the possibility of elimination of all nontrivial lattice homomorphisms by enlarging each of the given lattices. If, in particular, $\mathbf{L}$ consists of bounded free lattices, then mutually rigid extensions of these free lattices can be found in $\mathscr{L}$ (an object R of a category $\mathscr{C}$ is *rigid* if $\mathrm{Hom}_{\mathscr{C}}(R, R) = \{1_R\}$).

The remaining part of this section is concerned with a proof of Theorem 3.1 and of its variations stated at the end of this section.

Let $n \geqq 3$ be an integer. An undirected graph $G = (X, R)$ is *n-completely connected* if for every pair a, b of vertices of G there are full subgraphs (X_i, R_i) of G isomorphic to the complete graph K_n on n vertices satisfying

$$a \in X_1, b \in X_m, \text{ and } X_i \cap X_{i+1} \neq \emptyset$$

for $i = 1, \ldots, m - 1$. Let $\mathscr{G}_n$ be the category of all n-completely connected graphs and their compatible mappings; let $V: \mathscr{G} \to \mathbf{Set}$ denote the standard faithful vertex-set functor, that is, the functor V determined by $V(X, R) = X$, for any category $\mathscr{G}$ of directed or undirected graphs.

In [**10**], an extension $\chi_n: \mathbf{R}(2) \to \mathscr{G}_n$ of the binding category $\mathbf{R}(2)$ of all directed graphs is constructed for every $n \geqq 3$. The monotransformation $\eta: V \to V \circ \chi_n$ of the extension χ_n is such that for every $(X, R) \in \mathbf{R}(2)$, the set

$$\eta_{(X,R)}(X) \subseteq V(\chi_n(X, R))$$

is an independent set of $\chi_n(X, R)$. Furthermore, if $a, b \in \eta_{(X,R)}(X)$, then $\{a, z\}, \{b, z\}$ are edges of $\chi_n(X, \mathbf{R}) = (Y, S)$ only if $a = b$. In other words, the neighbourhoods

$$N(a) = \{y \in Y: \{a, y\} \in S\}$$

and $N(b)$ of vertices $a, b \in \eta_{(X,R)}(X)$ are disjoint whenever $a \neq b$.

Let $U: \mathscr{L} \to \mathbf{Set}$ denote the standard underlying-set functor. U is, clearly, faithful.

If $A \neq \emptyset$ is a set, define a functor $K_A: \mathbf{Set} \to \mathbf{Set}$ by

$$K_A(X) = X \times A$$

for every set X, and by

$$K_A(f)(x, a) = (f(x), a)$$

for all $(x, a) \in X \times A$. Let $V_A(X)$ be the disjoint union of X and A, define $V_A(f)(x) = f(x)$ for all $x \in X$ and $V_A(f)(a) = a$ for all $a \in A$. It is easy to see that K_A, V_A are faithful functors.

LEMMA 3.2. *Let $F:\mathbf{K} \to \mathscr{L}$ be a functor. If $\mathbf{K}$ is small or if $\mathbf{K}$ is a concrete category and* (M) *is assumed, then there is a full embedding $\Psi:\mathbf{K} \to \mathscr{G}_3$ and a monotransformation*

$$\mu: V_1 \circ K_4 \circ U \circ F \to V \circ \Psi$$

such that for every object K of $\mathbf{K}$ and any distinct elements a, b of $V_1 \circ K_4 \circ U \circ F(K)$

(10) $\{\mu_K(a), \mu_K(b)\}$ *is not an edge of* $\Psi(K)$,

(11) $N(\mu_K(a)) \cap N(\mu_K(b)) = \emptyset$.

Furthermore, if $\mathbf{K}$ is a finite category and $F(K)$ is a finite lattice for every object K of $\mathbf{K}$, then all graphs $\Psi(K)$ can be chosen to be finite.

Proof. Let $C:\mathbf{K} \to \mathbf{Set}$ be a faithful functor which is the left Cayley representation of $\mathbf{K}$ if $\mathbf{K}$ is a small category. Thus if, in particular, $\mathbf{K}$ is a finite category, then $C(K)$ is a finite set for every object K of $\mathbf{K}$. Let a functor $W:\mathbf{K} \to \mathbf{Set}$ be defined as the disjoint union of C and $V_1 \circ K_4 \circ U \circ F$. The resulting W is faithful and there is an obvious monotransformation $V_1 \circ K_4 \circ U \circ F \to W$. Furthermore, if every lattice $F(K)$ is finite and $\mathbf{K}$ is a finite category, then every $W(K)$ is a finite set. There is an extension $\Sigma:\mathbf{K} \to \mathbf{R}(2)$ accompanied by a monotransformation $W \to V \circ \Sigma$. If $\mathbf{K}$ is not small and (M) is assumed, the existence of such a Σ is the result by L. Kučera and Z. Hedrlín announced in [**9**] and presented in [**13**]. If $\mathbf{K}$ is a small category, the existence of such a Σ is proved in [**7**] and, in this case, $\Sigma(K)$ is a finite graph in $\mathbf{R}(2)$ whenever $W(K)$ is a finite set. To obtain the desired extension, set $\Psi = \chi_3 \circ \Sigma$; since χ_3 and Σ are extensions, so is Ψ. Because $\chi_3(G)$ is finite for any finite directed graph G [**10**], we conclude that, under our finiteness assumptions, $\Psi(K)$ is a finite graph for all objects K of $\mathbf{K}$. Using the properties of χ_n stated before Lemma 3.2, we see that the composite monotransformation

$$\mu: V_1 \circ K_4 \circ U \circ F \to W \to V \circ \Sigma \to V \circ \Psi$$

also satisfies (10) and (11).

Expressed in more intuitive terms, the properties of the extension Ψ described by Lemma 3.2 are as follows.

The vertex set X_K of each graph $\Psi(K) = (X_K, R_K)$ contains four mutually disjoint copies of the underlying set $U(L)$ of the lattice $L = F(K)$ and a vertex a_K of $\Psi(K)$ not belonging to any of these four copies. To simplify our notation, let $F(X) \times 4$ denote the union of these four

https://doi.org/10.4153/CJM-1980-089-0 Published online by Cambridge University Press

copies of $UF(K)$. Furthermore, (10) claims that $\{a_K\} \cup (F(K) \times 4)$ is an independent set of $\Psi(K)$, while (11) implies that $\{x, (l, i)\}$ and $\{x, (l', j)\}$ are in R_K only if $l = l'$ and $i = j$. If $\kappa : K \to K'$ is a morphism of $\mathbf{K}$, then

$$\Psi(\kappa)(a_K) = a_{K'} \text{ and}$$

$$\Psi(\kappa)(l, i) = (F(\kappa)(l), i) \text{ for all } l \in F(K) \text{ and } i \in 4.$$

From now on, we will assume that the range of the functor $F : \mathbf{K} \to \mathscr{L}$ is a set, and denote by β the supremum of the set of all $\operatorname{card}(UF(K))$ for K ranging over the class $\mathbf{K}^0$ of objects of $\mathbf{K}$.

Let $H_j = (Z_j, T_j)$ be mutually rigid graphs from $\mathscr{G}_4$ such that $\beta < \operatorname{card}(Z_j)$ for $j = 1, 2$. A combination of the results of [**14**], [**8**], and [**10**] guarantees the existence of such graphs; H_1 and H_2 can be chosen finite if β is finite. It is easy to see that the disjoint union $H = (Z, T)$ of H_1 and H_2 is a rigid graph, and that H_1 and H_2 are the 4-components (i.e., maximal 4-completely connected subgraphs) of H. Let $A(H)$ be the cover set lattice determined by H; that is, the sublattice of $I_*(H) \times I^*(H)$ generated by all pairs $(\{z\}, \{z\})$ with $z \in Z$. To simplify the notation, we will often write z instead of $(\{z\}, \{z\})$ and say that $A(H)$ is generated by a copy of Z.

Choose once and for all arbitrary vertices $b_i \in Z_i$ for $i = 1, 2$.

LEMMA 3.3. *$I_*(H_1)$ and $I_*(H_2)$ are simple lattices. Furthermore, $I_*(H_j) \times \{\emptyset\}$ and $\{\emptyset\} \times I^*(H_j)$ are sublattices of $A(H)$ such that*

$$(\mathbf{1}, \emptyset) \in (I_*(H_1) \times \{\emptyset\}) \cap (I_*(H_2) \times \{\emptyset\})$$

is the unit of $A(H)$, and

$$(\emptyset, \mathbf{1}) \in (\{\emptyset\} \times I^*(H_i)) \cap (\{\emptyset\} \times I^*(H_2))$$

is the zero of $A(H)$.

Proof. A nonempty graph in $\mathscr{G}_4$ contains a copy of the complete graph K_4 on four vertices and is connected. By Proposition 2.5, both $I_*(H_1)$ and $I_*(H_2)$ are simple lattices.

Since no nontrivial complete graph is rigid, there are independent sets $\{a, b\} \subseteq Z_1$ and $\{c, d\} \subseteq Z_2$. The lattice $A(H)$ generated by all pairs $(\{z\}, \{z\})$ with $z \in Z_1 \cup Z_2$ thus contains the element

$$[(\{a\}, \{a\}) \vee (\{b\}, \{b\})] \wedge [(\{c\}, \{c\}) \vee (\{d\}, \{d\})]$$
$$= (\{a, b\}, \emptyset) \wedge (\{c, d\}, \emptyset) = (\emptyset, \emptyset).$$

For every $z \in Z$, the element $(\{z\}, \emptyset) = (\{z\}, \{z\}) \vee (\emptyset, \emptyset)$ belongs to $A(H)$. Hence $A(H)$ contains the copy $I_*(H_1) \times \{\emptyset\}$ of $I_*(H_1)$ as a sublattice. The rest of the proof is trivial, for $(\emptyset, \mathbf{1}) < (\{z\}, \{z\}) < (\mathbf{1}, \emptyset)$ for all $z \in Z$.

A lattice $\Phi(K)$ will be constructed as an $\mathscr{R}$-reduced free product of lattices $F(K)$, $A(H)$, and a system $(C_x : x \in X_K)$ of three-element chains, where $\Psi(K) = (X_K, R_K)$ is the graph in $\mathscr{G}_3$ from Lemma 3.2. Following is the reduction $\mathscr{R} = \mathscr{R}(K) = (R_*(K), R^*(K))$ used to define $\Phi(K)$.

To simplify the notation, an element $\sigma \in R_* = R_*(K)$ will be written as the list of all components $\sigma(i) > \mathbf{0}$. The entries in these lists are elements of the disjoint union Q_K of posets $F(K)\backslash\{\mathbf{0}, \mathbf{1}\}$, X_K, and $A(H)\backslash\{\mathbf{0}, \mathbf{1}\}$. A similar convention will apply to the description of $R^* = R^*(K)$.

R_* is the reduction generated by the set

$$\begin{aligned}&\{\{l, (l, i), (l, j)\} : l \in F(K)\backslash\{\mathbf{0}, \mathbf{1}\}, i, j \in 4, i \neq j\}\\ &\cup \{\{(\mathbf{0}, i), (\mathbf{0}, j)\} : i, j \in 4, i \neq j\}\\ &\cup \{\{y_i, y_j, y_k\} : \operatorname{card}\{i, j, k\} = 3, y_r \in N(\mathbf{1}, r)\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } r \in \{i, j, k\} \subseteq 4\}\\ &\cup R_K \cup \{\{a_K, b_1\}, \{a_K, b_2\}\}\end{aligned}$$

in the following sense: a sequence $\{w_1, \ldots, w_n\}$ of elements $w_i \in Q_K$ from different factors belongs to R_* if and only if there is an element $\{a_1, \ldots, a_m\}$ of this generating set such that for every a_i there is a w_j with $a_i \leqq w_j$ holding true in Q_K; R_* contains also all sequences containing the unit of one of the factors.

The reduction R^* is dually generated by the set

$$\begin{aligned}&\{\{l, (l, i), (l, j)\} : l \in F(K)\backslash\{\mathbf{0}, \mathbf{1}\}, i, j \in 4, i \neq j\}\\ &\cup \{\{(\mathbf{1}, i), (\mathbf{1}, j)\} : i, j \in 4, i \neq j\}\\ &\cup \{\{x_i, x_j, x_k\} : \operatorname{card}\{i, j, k\} = 3, x_r \in N(\mathbf{0}, r) \text{ for } r \in \{\mathrm{i}, j, k\} \subseteq 4\}\\ &\cup R_K \cup \{\{a_K, b_1\}, \{a_K, b_2\}\}.\end{aligned}$$

Define $\Phi(K)$ as an $(R_*(K), R^*(K))$-reduced free product of $F(K)$, $A(H)$, and all three-element chains C_x with $x \in X_K$. According to the remarks preceding Proposition 1.5, $\Phi(K)$ is a quotient lattice of the $(\mathbf{0}, \mathbf{1})$-free product of these lattices modulo the smallest congruence relation $\equiv$ satisfying

$$\begin{aligned}&l \vee (l, i) \vee (l, j) \equiv \mathbf{1} \text{ and } l \wedge (l, i) \wedge (l, j) \equiv \mathbf{0}\\ &\text{if } l \in F(K)\backslash\{\mathbf{0}, \mathbf{1}\} \text{ and } i, j \in 4 \text{ are distinct,}\\ &(\mathbf{0}, i) \vee (\mathbf{0}, j) \equiv \mathbf{1} \text{ and } (\mathbf{1}, i) \wedge (\mathbf{1}, j) \equiv \mathbf{0} \text{ for distinct } i, j \in 4,\\ &y_i \vee y_j \vee y_k \equiv \mathbf{1} \text{ if } i, j, k \text{ are distinct elements of } 4 \quad\text{and}\\ &y_i \in N(\mathbf{1}, i), y_j \in N(\mathbf{1}, j), y_k \in N(\mathbf{1}, k),\\ &x_i \wedge x_j \wedge x_k \equiv \mathbf{0} \text{ if } i, j, k \text{ are distinct elements of } 4 \quad\text{and}\\ &x_i \in N(\mathbf{0}, i), x_j \in N(\mathbf{0}, j), x_k \in N(\mathbf{0}, k),\\ &x \vee x' \equiv \mathbf{1} \text{ and } x \wedge x' \equiv \mathbf{0} \text{ if } \{x, x'\} \in R_K \cup \{\{a_K, b_1\}, \{a_K, b_2\}\}.\end{aligned}$$

Let Γ_K denote the cover set lattice determined by the reduction $\mathscr{R}(K)$, let $(\Gamma_K)_*$ and $(\Gamma_K)^*$ stand for the corresponding upper and lower cover set lattice, respectively. Let

$$(\gamma_K)_*:\Phi(K) \to (\Gamma_K)_*$$

be the canonical homomorphism extending the identity mapping of Q_K onto its copy in $(\Gamma_K)_*$; the homomorphism $(\gamma_K)^*:\Phi(K) \to (\Gamma_K)^*$ is defined analogously. These definitions are justified by Theorem 1.6; $\Phi(K)$ is generated by a copy of Q_K and contains all component lattices as $(\mathbf{0}, \mathbf{1})$-sublattices.

For every morphism $\kappa:K \to K'$ of $\mathbf{K}$ we now define a mapping

$$\kappa_1:Q_K \cup \{\mathbf{0}, \mathbf{1}\} \to Q_{K'} \cup \{\mathbf{0}, \mathbf{1}\}$$

by

$$\kappa_1 = F(\kappa) \quad \text{on } F(K) \subseteq \Phi(K),$$

$$\kappa_1 = \Psi(\kappa) \quad \text{on } X_K \subseteq \Phi(K),$$

$$\kappa_1 = \mathrm{id}_{A(H)} \text{ on } A(H) \subseteq \Phi(K).$$

Thus, in particular,

$$\kappa(l, i) = \Psi(\kappa)(l, i) = (F(\kappa)(l), i)$$

for $(l, i) \in F(K) \times 4 \subseteq X_K$, so that also $\kappa_1(\mathbf{0}, i) = (\mathbf{0}, i)$ and $\kappa_1(\mathbf{1}, i) = (\mathbf{1}, i)$ for every $i \in 4$. Furthermore, $\kappa_1(b_1) = b_1$ and $\kappa_1(b_2) = b_2$, $\kappa_1(a_K) = \Psi(\kappa)(a_K) = a_{K'}$.

LEMMA 3.4. *For every morphism $\kappa:K \to K'$ of $\mathbf{K}$ there is a unique $(\mathbf{0}, \mathbf{1})$-preserving homomorphism $\Phi(\kappa):\Phi(K) \to \Phi(K')$ extending κ_1. Moreover, Φ is a one-to-one functor containing F as a subfunctor.*

Proof. By Proposition 1.5, it is enough to show that

$$\{\kappa_1(x), \ldots, \kappa_1(t)\} \in R_*(K')$$

whenever $\{x, \ldots, t\} \in R_*(K)$ and a similar statement concerning the lower reductions in order to conclude the existence of the homomorphism $\Phi(\kappa)$ extending κ_1. It is easily seen that, to prove the first implication, we only need to show that $\{\kappa_1(x), \ldots, \kappa_1(t)\} \in R_*(K')$ for every generating sequence $\{x, \ldots, t\}$ of $R_*(K)$.

Let $l \in F(K)\backslash\{\mathbf{0}, \mathbf{1}\}$, and consider $\{l, (l, i), (l, j)\} \in R_*(K)$. By the definition of κ_1,

$$\{\kappa_1(l), \kappa_1(l, i), \kappa_1(l, j)\} = \{F(\kappa)(l), (F(\kappa)(l), i), (F(\kappa)(l), j)\}.$$

If $F(\kappa)(l) = \mathbf{0}$, then

$$\{\kappa_1(l), \kappa_1(l, i), \kappa_1(l, j)\} = \{\mathbf{0}, (\mathbf{0}, i), (\mathbf{0}, j)\} \in R_*(K').$$

If $F(\kappa)(l) = \mathbf{1}$, there is nothing to prove. If $F(\kappa)(l) \neq \mathbf{0}, \mathbf{1}$, then

$$\{\kappa_1(l), \kappa_1(l, i), \kappa_1(l, j)\} \in R_*(K')$$

is obviously true.

According to the observations preceding Lemma 3.4,

$$\{\kappa_1(\mathbf{0}, i), \kappa_1(\mathbf{0}, j)\} = \{(\mathbf{0}, i), (\mathbf{0}, j)\} \in R_*(K') \text{ for } i \neq j.$$

Since $\kappa_1(\mathbf{1}, i) = (\mathbf{1}, i)$ for all $i \in 4$ and because κ_1 is a graph homomorphism of $\Psi(K)$ into $\Psi(K')$,

$$\kappa_1(N(\mathbf{1}, r)) \subseteq N(\mathbf{1}, r) \subseteq X_{K'}$$

for all $r \in 4$. Hence if $\{y_i, y_j, y_k\} \in R_*(K)$ and $y_r \in N(\mathbf{1}, r)$ for $r \in \{i, j, k\}$, then

$$\{\kappa_1(y_i), \kappa_1(y_j), \kappa_1(y_k)\} \in R_*(K').$$

If $\{x, x'\} \in R_K$, then $\{\kappa_1(x), \kappa_1(x')\} \in R_{K'} \subseteq R_*(K')$ is true because $\kappa_1 = \Psi(\kappa)$ on X_K.

Finally, observe that $\{\kappa_1(a_K), \kappa_1(b_j)\} = \{a_{K'}, b_j\}$ for $j = 1, 2$. We conclude that κ_1 maps $R_*(K)$ into $R_*(K')$ as required; a similar argument for the other pair of reductions would show that the hypothesis of Proposition 1.5 is satisfied. Thus there is a unique $(\mathbf{0}, \mathbf{1})$-preserving homomorphism $\Phi(\kappa)$ extending κ_1. It is easily seen that Φ is a functor; since the restriction of $\Phi(\kappa)$ to the sublattice $F(K)$ of $\Phi(K)$ is the $(\mathbf{0}, \mathbf{1})$-preserving homomorphism $F(\kappa): F(K) \to F(K') \subseteq \Phi(K')$, the functor Φ contains F as a subfunctor. The restriction of $\Phi(\kappa)$ to the antichain $X_K \subseteq \Phi(K)$ is the graph homomorphism $\Psi(\kappa)$. Since Ψ is a faithful functor, so is Φ. Any faithful functor $\Phi: \mathbf{K} \to \mathscr{L}$ is naturally equivalent to a one-to-one functor. We may thus replace Φ by a one-to-one functor sharing all other properties of Φ. To avoid notational inconvenience, we denote the one-to-one functor thus obtained by Φ again.

The proof of Theorem 3.1 will be completed once it is shown that every $(\mathbf{0}, \mathbf{1})$-preserving homomorphism $h: \Phi(K) \to \Phi(K')$ has the form $h = \Phi(\kappa)$ for some morphism $\kappa: K \to K'$ of $\mathbf{K}$; that is, once the fullness of Φ is established. It will be enough to show that h coincides with some κ_1 on Q_K, for Q_K is the generating set of $\Phi(K)$.

A first step in the proof of fullness of Φ is the following description of complemented pairs of $\Phi(K)$.

LEMMA 3.5. *If $\{c_1, c_2\} \neq \{\mathbf{0}, \mathbf{1}\}$ is a complemented pair in $\Phi(K)$, then either*

$$\{c_1, c_2\} \in R_K \cup T \cup \{\{a_K, b_1\}, \{a_K, b_2\}\}$$

or there are complemented pairs $\{x_1, x_2\}, \{y_1, y_2\}$ of the lattice $F(K)$ such that $x_1 \leqq c_1 \leqq y_1$ and $x_2 \leqq c_2 \leqq y_2$.

Proof. The lattice $\Phi(K)$ is an $(R_*(K), R^*(K))$-reduced free product of lattices $F(K)$, $A(H)$, and the system $(C_x : x \in X_K)$ of three-element chains. We apply Theorem 1.7 to $\Phi(K)$ and find that there are $\{x_1, y_1\} \subseteq Q_i$, $\{x_2, y_2\} \subseteq Q_j$ such that $x_1 \leqq c_1 \leqq y_1$ and $x_2 \leqq c_2 \leqq y_2$. Theorem 1.7 classifies the complemented pairs of $\Phi(L)$ into two groups as follows.

In the first group, $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are contained in the same factor of $\Phi(K)$ and $\{x_1, x_2\}$, $\{y_1, y_2\}$ are complemented pairs of this factor. If the factor in question is different from $F(K)$, then it must be $A(H)$, for lattices C_x have no nontrivial complements. The only nontrivial complemented pairs of $A(H)$ are pairs $\{z, z'\} \in T$ of generators of $A(H)$; since the generating set $Z \subseteq A(H)$ is an antichain, $x_1 = c_1 = y_1$ and $x_2 = c_2 = y_2$ and, consequently, $\{c_1, c_2\} \in T$.

In the second case, $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are subsets of distinct factors and $(x_1)_* \vee (x_2)_* \in R_*$, $(y_1)^* \wedge (y_2)^* \in R^*$. If $\{x_1, y_1\} \subseteq C_x$ for some $x \in X_K$, then $x_1 = x = y_1$ and hence $c_1 = x$ as well. If, in addition, $\{x_2, y_2\} \subseteq A(H)$, then $x_* \vee (x_2)_* \in R_*(K)$ implies that $x = a_K$ and $x_2 \geqq b_j$ for $j = 1$ or $j = 2$; similarly, $x^* \wedge (y_2)^* \in R^*(K)$ only if $x = a_K$ and $y_2 \leqq b_1$ or $y_2 \leqq b_2$. Since b_1, b_2 are incomparable, $x_2 = b_j = y_2$ follows for a unique $j \in \{1, 2\}$. Hence

$$c_2 = b_j \text{ and } \{c_1, c_2\} \in \{\{a_K, b_1\}, \{a_K, b_2\}\}.$$

If $\{x_1, y_1\} \subseteq C_x$ and $\{x_2, y_2\} \subseteq C_v$ for some $v \in X_K$, then again $x_1 = y_1 = c_1 = x$ and $x_2 = y_2 = c_2 = v$. A pair $\{x, v\} \subseteq X_K$ belongs to $R_* \cap R^*$ if and only if $\{x, v\} \in R_K$. There are no elements of R_* that are pairs $\{x, x_2\}$ with $x \in X_K$ and $x_2 \in F(K)\backslash\{\mathbf{0}, \mathbf{1}\}$. The proof is finished by observing that $\{a_K, b_j\}$ are the only sequences in $R_* \cap R^*$ involving an element of $A(H)$ together with an element of another factor.

Notation and Remark. Let $C(L)$ denote the set of all complemented pairs of a bounded lattice L; the undirected graph $G(L) = (L, C(L))$ will be called the *complementation graph* of L. Every $(\mathbf{0}, \mathbf{1})$-preserving homomorphism $f : L \rightarrow L_1$ is a compatible mapping of $G(L)$ into $G(L_1)$, a property to be frequently used in what follows.

It is clear that Lemma 3.5 describes the complemented pairs of $\Phi(K)$ completely. As a consequence ,the subgraphs H_1, H_2 of $G(\Phi(K))$ are the only 4-components of $G(\Phi(K))$ not contained in $G(F(K)')$; where $F(K)'$ is the convex closure of $F(K)\backslash\{\mathbf{0}, \mathbf{1}\}$ in $\Phi(K)$ extended by $\mathbf{0}, \mathbf{1}$. Furthermore, every 3-component of $G(\Phi(K))$ other than (X_K, R_K), H_1, H_2 must be contained in $G(F(K)')$.

The homomorphic image of an n-component of a graph G must clearly be n-completely connected.

Let $h : \Phi(K) \rightarrow \Phi(K')$ be a $(\mathbf{0}, \mathbf{1})$-preserving lattice homomorphism. Since H_1 is a full subgraph of H, the cover set lattice $A(H_1)$ is a $(\mathbf{0}, \mathbf{1})$-sublattice of $A(H)$ by Lemma 2.3. Let h_1 denote the restriction of h to

$A(H_1) \subseteq \Phi(K)$. The generating set Z_1 of $A(H_1)$ determines the 4-component H_1 of $G(\Phi(K))$ and, therefore, $h_1(Z_1)$ must be contained in a 4-component of $G(\Phi(K'))$; that is, $h_1(Z_1) \subseteq F(K')'$ or $h_1(Z_1) \subseteq Z_j$ for $j = 1$ or 2.

Assume $h_1(Z_1) \subseteq F(K')'$ first. Since Z_1 generates $A(H_1)$,

$$h_1(A(H_1)) \subseteq F(K')'.$$

The image of $F(K')'$ under the canonical homomorphism

$$\gamma_* = (\gamma_{K'})_*: \Phi(K') \to (\Gamma_{K'})_*$$

is a sublattice of $(\Gamma_{K'})_*$ isomorphic to $F(K')$. It follows that the composite homomorphism $\gamma_* \circ h_1$ maps $A(H_1)$ into $F(K')$. By Lemma 3.3, $A(H_1)$ contains the copy $I_*(H_1) \times \{\emptyset\}$ of $I_*(H_1)$ as a sublattice. $I_*(H_1) \times \{\emptyset\}$ is generated by the set $\{(\{z_1\}, \emptyset): z_1 \in Z_1\}$ bijective to Z_1 and

$$\operatorname{card}(Z_1) > \beta \geqq \operatorname{card}(F(K')).$$

The homomorphism $\gamma_* \circ h_1$ must, therefore, collapse a pair of elements of this generating set. By Lemma 3.3 again, $I_*(H_1)$ is a simple lattice, so that $\gamma_* \circ h_1$ is a constant mapping and, in particular,

$$\mathbf{1} = \gamma_* h_1(\mathbf{1}, \emptyset) = \gamma_* h_1(\emptyset, \emptyset).$$

Since the unit $\mathbf{1} \in \Phi(K')$ is the only element of $\Phi(K')$ mapped to $\mathbf{1}$ by γ_*, we conclude that $h_1(\emptyset, \emptyset) = \mathbf{1}$. A similar argument involving $\gamma^* = (\gamma_{K'})^*$ and the sublattice $\{\emptyset\} \times I^*(H_1)$ of $A(H_1)$ leads to $h_1(\emptyset, \emptyset) = \mathbf{0}$. This contradiction shows that $h(Z_1) \subseteq Z_1$ or $h(Z_1) \subseteq Z_2$.

If $h(Z_1) \subseteq Z_2$, then h defines a compatible mapping of the 4-component H_1 of $G(A(H_1))$ into H_2. However, H_1 and H_2 are mutually rigid, so that $h(z_1) = z_1$ and, analogously, $h(z_2) = z_2$ for all $z_j \in Z_j$ $(j = 1, 2)$. Because these copies of Z_1, Z_2 generate respective sublattices $A(H_1)$, $A(H_2)$, we see that h maps $A(H) \subseteq \Phi(K)$ identically onto the copy of $A(H)$ in $\Phi(K')$. In particular, $h(b_j) = b_j$ for $j = 1, 2$.

Observe that a_K is the only element of $\Phi(K)$ that has b_1, b_2 as its complements; it follows that $h(a_K) = a_{K'}$ because $a_{K'}$ plays the same role in $\Phi(K')$ and $h(b_1) = b_1$, $h(b_2) = b_2$. Recall that X_K determines the 3-component of $G(\Phi(K))$ containing a_K and that $X_{K'}$ is the vertex set of the 3-component of $G(\Phi(K'))$ that contains $a_{K'}$. Thus $h(X_K) \subseteq X_K{}'$ now follows from $h(a_K) = a_K{}'$. If $\{x, x'\} \in R_K$, then $x \vee x' = \mathbf{1}$ and $x \wedge x' = \mathbf{0}$ in $\Phi(K)$ and, consequently, $\{h(x), h(x')\} \subseteq X_{K'}$ must be a complemented pair of $\Phi(K')$. Lemma 3.5 now implies that $\{h(x), h(x')\} \in R_{K'}$. In other words, the restriction of h to X_K is a graph homomorphism $\Psi(K) \to \Psi(K')$. Since Ψ is a full embedding, $h = \Psi(\kappa)$ for some $\kappa: K \to K'$ in $\mathbf{K}$ uniquely determined by h. Altogether, h coincides with κ_1 on $X_K \cup A(H)$.

To complete the proof of fullness of Φ, we proceed to show that h maps $F(K) \subseteq \Phi(K)$ into $F(K') \subseteq \Phi(K')$ and that h coincides with $\kappa_1 = F(\kappa)$ on $F(K)$. Since $h = \kappa_1$ on X_K, we obtain the validity of the following formulae:

$$h(l, i) = (F(\kappa)(l), i) \text{ for all } (l, i) \in F(K) \times 4;$$

in particular,

$$h(\mathbf{0}, i) = (\mathbf{0}, i), h(\mathbf{1}, i) = (\mathbf{1}, i) \text{ for } i \in 4.$$

Furthermore,

$$h(x, i) \in N(\mathbf{0}, i) \subseteq X_{K'} \text{ whenever } x_i \in N(\mathbf{0}, i) \subseteq X_K$$

and, similarly,

$$y_i \in N(\mathbf{1}, i) \subseteq X_K \text{ implies } h(y_i) \in N(\mathbf{1}, i) \subseteq X_{K'} \text{ for all } i \in 4.$$

Let $l \in F(K)\backslash\{\mathbf{0}, \mathbf{1}\}$ be such that $h(l) = \mathbf{1}$. Since

$$\mathbf{0} = l \wedge (l, i) \wedge (l, j)$$

for distinct $i, j \in 4$, it follows that

$$\mathbf{0} = h(l) \wedge h(l, i) \wedge h(l, j) = (F(\kappa)(l), i) \wedge (F(\kappa)(l), j).$$

By (10) and Proposition 1.4, a pair $\{(l', i), (l', j)\} \subseteq X_{K'}$ meets to zero only if it belongs to the reduction $R^*(K')$; that is, only if $l' = \mathbf{1}$. Hence $F(\kappa)(l) = \mathbf{1}$ if $l \neq \mathbf{0}, \mathbf{1}$ and $h(l) = \mathbf{1}$.

Conversely, let $l \in F(K)\backslash\{\mathbf{0}, \mathbf{1}\}$ and $F(\kappa)(l) = \mathbf{1}$. Then

$$h(l) \vee (\mathbf{1}, i) \vee (\mathbf{1}, j) = \mathbf{1}$$

holds in $\Phi(K')$ for all distinct $i, j \in 4$. Assume $h(l) < \mathbf{1}$ and let B be the set of all nonzero lower covers of $h(l)$ in $\Phi(K')$. Then

$$B_{ij} = B \cup \{(\mathbf{1}, i), (\mathbf{1}, j)\} \in R_*(K')$$

by Proposition 1.4; $h(l) < \mathbf{1}$ implies that $B_{ij} \subseteq Q_{K'}$. B is nonempty, for otherwise $\{(\mathbf{1}, i), (\mathbf{1}, j)\} \in R_*(K')$ would contradict (10). By the definition of $R_*(K')$, there is a generating sequence S_{ij} in $R_*(K')$ such that for every $s \in S_{ij}$ there is a $b \geqq s$ in B_{ij}. Since B is the set of lower covers of $h(l) < \mathbf{1}$, S_{ij} is not a subset of B. Because $Q_{K'} \cap C_{(\mathbf{1},k)} = \{(\mathbf{1}, k)\}$ for every $k \in 4$, it follows that $(\mathbf{1}, i) \in S_{ij}$ or $(\mathbf{1}, j) \in S_{ij}$. Recall that every generating sequence S from $R_*(K')$ containing $(\mathbf{1}, k)$ has the form

$$S = \{(\mathbf{1}, k), y_k\} \in R_{K'},$$

that is, $y_k \in N(\mathbf{1}, k)$. Hence if $(\mathbf{1}, i) \in S_{ij}$, then there is a $y_i \in N(\mathbf{1}, i)$ such that $y_i \leqq a$ for some $a \in B \cup \{(\mathbf{1}, j)\}$. The possibility of $a = (\mathbf{1}, j)$ is excluded by (10); thus $y_i \leqq b_i \in B$. We see that for every two-element subset D of 4 there is an $r \in D$ such that $y_r \leqq b_r \in B$ and $y_r \in N(\mathbf{1}, r)$.

In view of (11), the sets $N(\mathbf{1}, r)$ are pairwise disjoint for $r \in \mathbf{4}$; hence there is a three-element subset T of 4 such that, for every $r \in T$, both $y_r \leqq b_r \in B$ and $y_r \in N(\mathbf{1}, r)$. However, the set $\{y_r : r \in D\}$ is a generating sequence in $R_*(K')$ and thus

$$\mathbf{1} = \bigvee \{y_r : r \in D\} \leqq h(l).$$

This shows that, for $l \in F(K)\backslash\{\mathbf{0}, \mathbf{1}\}$, $F(\kappa)(l) = \mathbf{1}$ implies $h(l) = \mathbf{1}$.

These arguments show that $h(l) = \mathbf{1}$ is equivalent to $F(\kappa)(l) = \mathbf{1}$; a proof dual to the above yields the equivalence of $h(l) = \mathbf{0}$ and $F(\kappa)(l) = \mathbf{0}$. Both equivalences are, clearly, valid for every $l \in F(K)$.

If $h(l) \in \Phi(K')\backslash\{\mathbf{0}, \mathbf{1}\}$, then, by the above equivalences,

$$F(\kappa)(l) \in F(K')\backslash\{\mathbf{0}, \mathbf{1}\}.$$

Since $l \in F(K)\backslash\{\mathbf{0}, \mathbf{1}\}$, the equations

$$l \vee (l, i) \vee (l, j) = \mathbf{1},$$

$$l \wedge (l, i) \wedge (l, j) = \mathbf{0}$$

hold in $\Phi(K)$. Therefore

$$h(l) \vee (F(\kappa)(l), i) \vee (F(\kappa)(l), j) = \mathbf{1}$$

and

$$h(l) \wedge (F(\kappa)(l), i) \wedge (F(\kappa)(l), j) = \mathbf{0}$$

in $\Phi(K')$ for all distinct $i, j \in 4$. It follows, from (10) and Proposition 1.4, that $F(\kappa)(l)$ is both an upper and a lower cover of $h(l)$ in $F(K')$. This proves that $h(l) = F(\kappa)(l)$ whenever $h(l) \neq \mathbf{0}, \mathbf{1}$.

Altogether, the restriction of h to $F(K) \subseteq \Phi(K)$ is the homomorphism

$$F(\kappa) : F(K) \to F(K') \subseteq \Phi(K').$$

This finishes the proof of $h = \Phi(\kappa)$.

LEMMA 3.6. *The functor Φ is full.*

THEOREM 3.7. *The set-theoretical requirement* (M) *is equivalent to*

(XC) *if* $\mathbf{K}$ *is a concrete category and* $F : \mathbf{K} \to \mathscr{L}$ *is a functor whose range is small, then there is a full embedding* $\Phi : \mathbf{K} \to \mathscr{L}$ *containing* F *as a subfunctor.*

Proof. Under the assumption (M), the proof of Theorem 3.1 presented here can be used to derive (XC); see Lemma 3.2.

Conversely, if (XC) is assumed, choose the dual $\mathbf{Set}^{\text{opp}}$ of the category $\mathbf{Set}$ for $\mathbf{K}$ and any constant functor $F : \mathbf{Set}^{\text{opp}} \to \mathscr{L}$. The full embedding Φ guaranteed by (XC) exists only if (M) is satisfied [**12**].

Remark. Surprisingly enough, the set-theoretical assumption (M) is necessary even if (XC) is weakened to a condition (XA) requiring that

https://doi.org/10.4153/CJM-1980-089-0 Published online by Cambridge University Press

K be a full subcategory of a category of algebras instead of a general concrete category. Using this result, proved in [**3**], together with the implication (M) $\rightarrow$ (XC) discussed earlier, we obtain immediately the following theorem.

THEOREM 3.8. (M) *is equivalent to* (XA).

If **K** is a finite category and every lattice $F(K)$ is finite, Lemma 3.2 states that all graphs $\Psi(K) = (X_K, R_K)$ may be chosen to be finite and the graph H may be finite as well. The cover set lattice Γ_K of finitely many finite lattices $F(K)$, $A(H)$, and C_x for $x \in X_K$ is then obviously finite. The variety $\mathscr{V}$ generated by finitely many finite lattices Γ_K is locally finite and, consequently, each $\mathscr{R}(K)$-reduced $\mathscr{V}$-free product $\Phi'(K)$ of $F(K)$, $A(H)$, and C_x for $x \in X_K$ is a finite lattice. Since $\Gamma_K \in \mathscr{V}$ for each cover set lattice of $\Phi'(K)$, the proof of Theorem 3.1 gives rise to the following generalization of a result from [**1**].

THEOREM 3.9. *If* **K** *is a finite category and if* $F:\mathbf{K} \rightarrow \mathscr{L}$ *is a functor such that* $F(K)$ *is a finite lattice for every object* K *of* **K**, *then there is a full embedding* $\Phi':\mathbf{K} \rightarrow \mathscr{L}$ *containing* F *as a subfunctor and such that every* $\Phi'(K)$ *is a finite lattice.*

Following is a list of problems concerning full embeddings into categories of lattices.

Problem 3.1. A full embedding Φ of a category **A** of algebras into another one, **B**, is called *strong*, if there is a functor S:**Set** $\rightarrow$ **Set** such that $U \circ \Phi = S \circ U$, where U is the standard underlying-set functor. Is there a strong embedding of the category $\mathbf{A}(2)$ of all groupoids into $\mathscr{L}$?

Problem 3.2 A functor $F:\mathbf{K} \rightarrow \mathscr{L}$ is a *quotient* of $\Phi:\mathbf{K} \rightarrow \mathscr{L}$ if there is a natural transformation $\epsilon:\Phi \rightarrow F$ consisting of onto lattice homomorphisms. The problem of characterization of quotients of full embeddings into $\mathscr{L}$ appears to be more complex than that of subfunctors. It is easily seen that no constant functor F whose image is a lattice possessing a prime ideal can be a quotient of a full embedding Φ if the domain category **K** has a rigid object. A simpler question of this kind may thus ask for a characterization of constant quotients of full embeddings into $\mathscr{L}$.

Remark. Arguments in a forthcoming paper of the authors show that there are $2^{\aleph_0}$ varieties $\mathscr{V}$ of lattices such that Theorem 3.1 remains valid if $\mathscr{L}$ is replaced by $\mathscr{L} \cap \mathscr{V}$.

REFERENCES

1. M. E. Adams and J. Sichler, *Bounded endomorphisms of lattices of finite height*, Can. J. Math. *29* (1977), 1254–1263.

2. ——— *Homomorphisms of bounded lattices with a given sublattice*, Arch. Math. (Basel) *30* (1978), 122–128.
3. ——— *Subfunctors of full embeddings of algebraic categories*, to appear in Alg. Universalis.
4. C. C. Chen and G. Grätzer, *On the construction of complemented lattices*, J. Alg. *11* (1969), 56–63.
5. G. Grätzer and J. Sichler, *On the endomorphism semigroup (and category) of bounded lattices*, Pacific J. Math. *35* (1970), 639–647.
6. G. Grätzer, *General lattice theory* (Birkhäuser Verlag, 1978).
7. Z. Hedrlín and A. Pultr, *On full embeddings of categories of algebras*, Ill. J. Math. *10* (1966), 392–405.
8. Z. Hedrlín and J. Sichler, *Any boundable binding category contains a proper class of mutually disjoint copies of itself*, Alg. Universalis *1* (1971), 97–103.
9. Z. Hedrlín, *Extensions of structures and full embeddings of categories*, in Proc. Intern. Congr. of Mathematicians, Nice (Gauthier-Villars, Paris, 1971).
10. P. Hell, *Full embeddings into some categories of graphs*, Alg. Universalis *2* (1972), 129–141.
11. B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. *21* (1967), 110–121.
12. L. Kučera and A. Pultr, *Non-algebraic concrete categories*, J. Pure Appl. Alg. *3* (1973), 95–102.
13. A. Pultr and V. Trnková, *Combinatorial, algebraic, and topological representations of categories* (North Holland, 1980).
14. P. Vopěnka, A. Pultr, and Z. Hendrlín, *A rigid relation exists on any set*, Comment. Math. Univ. Carolinae *6* (1965), 149–155.

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