A LYAPUNOV INEQUALITY AND FORCED OSCILLATIONS IN GENERAL NONLINEAR nTH ORDER DIFFERENTIAL-DIFFERENCE EQUATIONS†

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1. Introduction. The purpose of this paper is to consider the general nonlinear nth order differential-difference equation

$$[r(t)h(y'(t))]^{(n-1)}+a(t)y(t)f(y(t-\tau(t)))=b(t) \tag{1}$$

and derive an inequality of Lyapunov type. Later we use this inequality to find conditions to ensure that the oscillatory solutions of equation (1) tend to zero as $t\to\infty$. The conditions that ensure that the oscillatory solutions of equation (1) tend to zero, also cause all solutions of equation

$$[r(t)h(y'(t))]^{(n-1)}+a(t)y(t)f(y(t-\tau(t)))=0 \tag{2}$$

to be non-oscillatory.

The classical Lyapunov inequality states that if $y(t)$ is a non-trivial solution of the second order linear equation

$$y''(t)+a(t)y(t)=0,$$

where $a(t)$ is real and continuous, and if $y(t)$ vanishes at least twice on the interval $[t_1, t_2]$, then

$$(t_2-t_1)\int_{t_1}^{t_2} a^+(t)\,dt>4, \quad \text{where} \quad a^+(t)=\max(a(t),0).$$

This inequality is well known to be the sharpest possible, so that 4 cannot be replaced by larger constant, cf. **[1]**. In general, this inequality is not true for delay equations. As an example, the equation

$$y''(t)-y(t-\pi)=0$$

has as a nontrivial solution $y(t)=\sin t$ on $(0,\infty)$ subject to $y(t)=\sin t$, $t\in[-\pi,0]$, but taking $t_1=0$, $t_2=\pi$, $a(t)=-1$, we find that the conclusion of the inequality is not true.

Eliason [2] considered the equation

$$[r(t)y'(t)]'+a(t)y(t)f(y(t))=0$$

and proved a more general version of Lyapunov inequality. Recently, Dahiya-Singh **[4]** considered the equation

$$[r(t)h(y'(t))]'+a(t)y(t)f(y(t-\tau(t)))=0,$$

and, more recently, Singh **[5]** also considered the equation

$$[r(t)y'(t)]^{(n-1)}+a(t)y(t)=f(t)$$

and proved an extension of this inequality which is a particular case of our result.

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We shall consider only those solutions of the equation (1) which exist on some half-line $[t_\xi, \infty)$, where t_ξ may depend on the particular solution, and are nontrivial in any neighbourhood of infinity. Such a solution is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *non-oscillatory*. In addition the following assumptions will be made for the rest of this paper.

ASSUMPTIONS. (i) $a(t)$ and $b(t)$ are continuous real-valued functions on $[0, \infty)$.

(ii) $r(t)$ is a continuous and positive real-valued function on $[0, \infty)$.

(iii) $\tau(t)$ is continuous positive and bounded so that there exists some positive constant m such that $0 < \tau(t) \leqq m$.

(iv) $h(x)$ is continuously differentiable on $(-\infty, \infty)$ and is an odd function such that $\operatorname{sgn} h(x) = \operatorname{sgn} x$; there exists $\beta > 0$ such that $0 < x/h(x) \leqq \beta$, and $\lim_{x \to 0} (x/h(x))$ exists finitely so that $x/h(x)$ is continuously differentiable on $[0, \infty)$.

(v) $f(x)$ is a continuous, even, real positive function on $(-\infty, \infty)$ and increasing on $[0, \infty)$ with $f(0) = 0$.

To obtain our results we make use of the following lemma adapted from Singh [5].

LEMMA. *Assume that* $\alpha_1 > \alpha_2 > \alpha_3 > \ldots > \alpha_{n-2}$ *are, respectively, zeros of*

$$[r(t)h(y'(t))]', \quad [r(t)h(y'(t))]'', \quad \ldots, \quad [r(t)h(y'(t))]^{(n-3)}, \quad [r(t)h(y'(t))]^{(n-2)},$$

where $y(t)$ *is a solution of equation* (1). *Furthermore, suppose that* $t_1 < \alpha_{n-2}$ *and* $t_2 > \alpha_1$ *are zeros of* $y(t)$. *Let*

$$L = \sup\{y(t) : t \in (t_1 - m, t_2), t_1, t_2 > m\} \quad \text{and} \quad M = \sup\{|y(t)| : t \in [t_1, t_2]\}.$$

Then

$$4 \leqq \beta \int_{t_1}^{t_2} \frac{dt}{r(t)} \left\{ f(L) \int_{t_1}^{t_2} \frac{(t-t_1)^{n-2}}{(n-2)!} |a(t)|\, dt + \frac{1}{M} \int_{t_1}^{t_2} \frac{(t-t_1)^{n-2}}{(n-2)!} |b(t)|\, dt \right\}. \tag{3}$$

Proof. Integration of (1) $n-2$ times gives

$$(-1)^n [r(t)h(y'(t))]' + \int_t^{\alpha_1} \int_{s_2}^{\alpha_2} \cdots \int_{s_{n-2}}^{\alpha_{n-2}} a(s)y(s)f(y(s-\tau(s)))\, ds\, ds_{n-2} \ldots ds_2$$

$$= \int_t^{\alpha_1} \int_{s_2}^{\alpha_2} \cdots \int_{s_{n-2}}^{\alpha_{n-2}} b(s)\, ds\, ds_{n-2} \ldots ds_2. \tag{4}$$

Since $\alpha_1 > \alpha_2 > \alpha_3 > \ldots > \alpha_{n-2}$, we obtain from (4),

$$|[r(t)h(y'(t))]'| \leqq \int_t^{\alpha_1} \int_{s_2}^{\alpha_1} \cdots \int_{s_{n-2}}^{\alpha_1} |a(s)| \, |y(s)| \, |f(y(s-\tau(s)))|\, ds\, ds_{n-2} \ldots ds_2$$

$$+ \int_t^{\alpha_1} \int_{s_2}^{\alpha_1} \cdots \int_{s_{n-2}}^{\alpha_1} |b(s)|\, ds\, ds_{n-2} \ldots ds_2,$$

which implies

$$|[r(t)h(y'(t))]'| \leqq \int_t^{\alpha_1} \frac{(s-t)^{n-3}}{(n-3)!} |a(s)| \, |y(s)| \, |f(y(s-\tau(s)))|\, ds + \int_t^{\alpha_1} \frac{(s-t)^{n-3}}{(n-3)!} |b(s)|\, ds. \tag{5}$$

Let

$$M = |y(t_0)|, \quad t_0 \in [t_1, t_2]. \tag{6}$$

Now

$$\pm M = y(t_0) = \int_{t_1}^{t_0} y'(t)\, dt,$$

which implies

$$M \leqq \int_{t_1}^{t_0} |y'(t)|\, dt. \tag{7}$$

Similarly

$$M \leqq \int_{t_0}^{t_2} |y'(t)|\, dt. \tag{8}$$

From (7) and (8),

$$2M \leqq \int_{t_1}^{t_2} |y'(t)|\, dt.$$

By Schwarz's inequality, we get

$$4M^2 \leqq \int_{t_1}^{t_2} \frac{y'(t)}{h(y'(t))} \frac{dt}{r(t)} \int_{t_1}^{t_2} [r(t)h(y'(t))] y'(t)\, dt, \tag{9}$$

since $y'(t)/h(y'(t))$ is continuous and positive. Therefore

$$4M^2 \leqq \beta \int_{t_1}^{t_2} \frac{dt}{r(t)} \int_{t_1}^{t_2} [r(t)h(y'(t))] y'(t)\, dt,$$

since $0 < y'(t)/h(y'(t)) \leqq \beta$.

Integrating the second integral of the right-hand side by parts, we have

$$\frac{4M^2}{\beta \int_{t_1}^{t_2} \frac{dt}{r(t)}} \leqq -\int_{t_1}^{t_2} y(t)[r(t)h(y'(t))]'\, dt, \tag{10}$$

since $y(t_1) = y(t_2) = 0$. It follows, from (10), that

$$\frac{4M^2}{\beta \int_{t_1}^{t_2} \frac{dt}{r(t)}} \leqq \int_{t_1}^{t_2} |y(t)|\, |[r(t)h(y'(t))]'|\, dt. \tag{11}$$

From (6) and (11),

$$\frac{4M}{\beta \int_{t_1}^{t_2} \frac{dt}{r(t)}} \leqq \int_{t_1}^{t_2} |[r(t)h(y'(t))]'|\, dt. \tag{12}$$

From (5) and (12), we have

$$\frac{4M}{\beta\int_{t_1}^{t_2}\frac{dt}{r(t)}} \leqq \int_{t_1}^{t_2}\int_{s}^{\alpha_1}\frac{(x-s)^{n-3}}{(n-3)!}|a(x)||y(x)|f(L)\,dx\,ds$$
$$+\int_{t_1}^{t_2}\int_{s}^{\alpha_1}\frac{(x-s)^{n-3}}{(n-3)!}|b(x)|\,dx\,ds. \tag{13}$$

Dividing by M and noting that $t_2 > \alpha_1$, we have, from (13),

$$\frac{4}{\beta\int_{t_1}^{t_2}\frac{dt}{r(t)}} \leqq f(L)\int_{t_1}^{t_2}\int_{s}^{t_2}\frac{(x-s)^{n-3}}{(n-3)!}|a(x)|\,dx\,ds$$
$$+\frac{1}{M}\int_{t_1}^{t_2}\int_{s}^{t_2}\frac{(x-s)^{n-3}}{(n-3)!}|b(x)|\,dx\,ds. \tag{14}$$

From (14), we have

$$4 \leqq \beta\int_{t_1}^{t_2}\frac{dt}{r(t)}\left\{f(L)\int_{t_1}^{t_2}\frac{(s-t_1)^{n-2}}{(n-2)!}|a(s)|\,ds+\frac{1}{M}\int_{t_1}^{t_2}\frac{(s-t_1)^{n-2}}{(n-2)!}|b(s)|\,ds\right\},$$

and the proof is complete.

REMARK 1. Eliason [2] has discussed the lemma in the case $n=2$, $h(y'(t)) \equiv y'(t)$, $b(t) \equiv 0$ and $f(y(t-\tau(t))) \equiv f(y(t))$. Dahiya-Singh [4] has discussed it in the case $n=2$, $b(t) \equiv 0$, and Singh [5] also has discussed it in the case $r(t) \equiv 1$, $b(t) \equiv 0$, $h(y'(t)) \equiv y'(t)$ and $f(y(t-\tau(t))) \equiv 1$.

2. Theorems. We now give a generalization of Lyapunov inequality for the equation

$$y^{(n)}(t)+a(t)y(t)=0. \tag{15}$$

THEOREM 1. *Assume that $r_2 > r_3 > \dots > r_{n-1}$ are zeros of $y''(t)$, $y'''(t), \dots, y^{(n-1)}(t)$ respectively, where $y(t)$ is a solution of equation* (15). *Let $t_1 < r_{n-1}$ and $t_2 > r_2$ be zeros of $y(t)$. Then*

$$\frac{4}{t_2-t_1} \leqq \int_{t_1}^{t_2}\frac{(t-t_1)^{n-2}}{(n-2)!}|a(t)|\,dt. \tag{16}$$

Proof. In the lemma, we put

$$r(t) \equiv 1, \quad h(y'(t)) \equiv y'(t), \quad f(y(t-\tau(t))) \equiv 1 \quad \text{and} \quad b(t) \equiv 0,$$

and the conclusion follows.

THEOREM 2. *Assume that $f(x)$ is bounded and*

$$\int^{\infty} t^{n-2}|a(t)|\,dt < \infty, \tag{17}$$

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$$\int^{\infty} t^{n-2}|b(t)|\,dt < \infty, \tag{18}$$

and

$$\int^{\infty} \frac{dt}{r(t)} < \infty. \tag{19}$$

Let $y(t)$ be an oscillatory solution of equation (1). *Then*

$$\lim_{t\to\infty} y(t) = 0.$$

Proof. Let

$$M_1 = \sup_{0 \leqq x < \infty} f(x). \tag{20}$$

Suppose to the contrary that $\lim_{t\to\infty} y(t) \neq 0$. Then

$$\liminf_{t\to\infty} |y(t)| = 0, \tag{21}$$

and for some positive d,

$$\limsup_{t\to\infty} |y(t)| > 2d. \tag{22}$$

Due to the oscillatory nature of $y(t)$, $[r(t)h(y'(t))]^{(n-2)}$ must be oscillatory. In fact if $[r(t)h(y'(t))]^{(n-2)}$ is non-oscillatory, then $r(t)h(y'(t))$ assumes one sign eventually. Since $r(t) > 0$, $\operatorname{sgn} h(y'(t)) = \operatorname{sgn} y'(t)$, $h(y'(t))$ is continuous and odd, $y'(t)$ becomes non-oscillatory which in turn forces $y(t)$ to be non-oscillatory, which is a contradiction. Hence $[r(t)h(y'(t))]^{(n-2)}$ is oscillatory. Similarly

$$[r(t)h(y'(t))]^{(n-3)}, [r(t)h(y'(t))]^{(n-4)}, \ldots, [r(t)h(y'(t))]'$$

are all oscillatory. Let T be large enough so that

$$\int_T^{\infty} t^{n-2}|a(t)|\,dt < \frac{1}{M_1}, \tag{23}$$

$$\int_T^{\infty} t^{n-2}|b(t)|\,dt < d \tag{24}$$

and

$$\int_T^{\infty} \frac{dt}{r(t)} < \frac{1}{\beta}. \tag{25}$$

Let $T < t_1 < \alpha_{n-2} < \ldots < \alpha_3 < \alpha_2 < \alpha_1 < T_0$ be points such that

$$y(t_1) = 0, \tag{26}$$

$$[r(\alpha_i)h(y'(\alpha_i))]^{(i)} = 0, \quad i = 1, 2, \ldots, n-2, \tag{27}$$

and

$$M = \sup_{t_1 \leqq t \leqq T_0} |y(t)| > d. \tag{28}$$

Let $t_2 > T_0$ be another zero of $y(t)$. Let

$$M_0 = \sup_{t_1 \leqq t \leqq t_2} |y(t)|.$$

Then $M_0 > d$. From the conclusion of the lemma, we have

$$4 \leqq \beta \int_{t_1}^{t_2} \frac{dt}{r(t)} \left\{ M_1 \int_{t_1}^{t_2} \frac{(s-t_1)^{n-2}}{(n-2)!} |a(s)| ds + \frac{1}{M_0} \int_{t_1}^{t_2} \frac{(s-t_1)^{n-2}}{(n-2)!} |b(s)| ds \right\}. \tag{29}$$

From (23), (24), (25), the fact that $M_0 > d$ and (29), we have

$$4 \leqq 1 + (d/d) = 2. \tag{30}$$

This contradiction proves the theorem.

REMARK 2. For the case $h(y'(t)) \equiv y'(t), f(y(t-\tau(t))) \equiv 1$, our Theorem 2 coincides with Theorem 1 of Singh [5].

THEOREM 3. *Suppose that* (17) *and* (19) *are satisfied, and that* $f(x)$ *is bounded. Then every solution of* (2) *is non-oscillatory.*

Proof. Following the proof of Theorem 2, we arrive at conclusion (29). From (29), we get

$$4 \leqq \beta \int_{t_1}^{t_2} \frac{dt}{r(t)} \left\{ M_1 \int_{t_1}^{t_2} \frac{(s-t_1)^{n-2}}{(n-2)!} |a(s)| ds \right\} \leqq 1, \tag{31}$$

using (23) and (25). This contradiction proves the theorem.

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