10

Back to Banach Space Theory

Supper's ready. After the display of quasilinear and homological techniques we have presented so far, we are ready to return to the place the journey started: classical Banach space theory. Much in the spirit, we hope, of Eliot, 'We shall not cease from exploration, and the end of all our exploring will be to arrive where we started and know the place for the first time'. But the twist is that we can now provide solutions for, or at least a better understanding of, a number of open problems. Among the topics covered, the reader will encounter vector-valued forms of Sobczyk's theorem, isomorphically polyhedral $\mathscr{L}_\infty$-spaces, Lipschitz and uniformly homeomorphic $\mathscr{L}_\infty$-spaces, properties of kernels of quotient operators from $\mathscr{L}_1$-spaces, sophisticated 3-space problems, the extension of $\mathscr{L}_\infty$-valued operators, Kadec spaces, Kalton–Peck spaces and, at last, the space Z_2. All these topics can be easily considered as part of classical Banach space theory, even if the techniques we employ involve most of the machinery developed throughout the book.

10.1 Vector-Valued Versions of Sobczyk's Theorem

We are ready to tackle the question of what exactly a 'vector-valued Sobczyk's theorem' should mean. Probably the answer should be a result along one of the following lines. Let X be a separable quasi-Banach space and let (E_n) be quasi-Banach spaces.

(u) If $\operatorname{Ext}(X, E_n) = 0$ uniformly on n then $\operatorname{Ext}(X, c_0(\mathbb{N}, E_n)) = 0$.

(s) Let Y be a subspace of a quasi-Banach space Z. Let $\tau\colon Y \longrightarrow c_0(\mathbb{N}, E_n)$ be an operator. If there is a λ such that each component $\pi_n\tau\colon Y \longrightarrow E_n$ admits a λ-extension to Z then τ admits an $f(\lambda)$-extension to Z for some function f.

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(w) Variations of these, maybe under additional hypotheses on the spaces involved.

Here (u) stands for 'uniform', (s) for 'single' and (w) for 'who knows'. We already know from 5.2.5 and 2.14.8 that (u) and (s) hold for Banach spaces X that have the BAP; we also know from the Pełczyński–Lusky sequence that (s) is false when X fails the BAP. Let us show that (u) is extremely false for p-Banach spaces if X fails the BAP.

10.1.1 A p-Banach space oddity *For each $p \in (0,1)$ there is a non-trivial exact sequence of p-Banach spaces* $0 \longrightarrow c_0(\mathbb{N}, \ell_p^n) \longrightarrow \cdot \longrightarrow L_p \longrightarrow 0$.

Proof Write $L_p = \overline{\bigcup_n F_n}$ with $F_n \approx \ell_p^n$ and $F_n \subset F_{n+1}$. The Pełczyński–Lusky sequence $0 \longrightarrow c_0(\mathbb{N}, \ell_p^n) \longrightarrow c(\mathbb{N}, F_n) \longrightarrow L_p \longrightarrow 0$ does not split since actually there is no non-zero operator $L_p \longrightarrow c(\mathbb{N}, F_n)$. □

Therefore, even if $\operatorname{Ext}_{p\mathbf{B}}(L_p, c_0) = 0$ and $\operatorname{Ext}_{p\mathbf{B}}(L_p, \ell_p^n) = 0$ uniformly on n (because $\operatorname{Ext}_{p\mathbf{B}}(L_p, \ell_p) = 0$), still $\operatorname{Ext}_{p\mathbf{B}}(L_p, c_0(\ell_p)) \neq 0$. We do not know if something similar can occur in a Banach space ambient. Thus, the role of the BAP seems to be central in this matter, even if the true truth underneath is that a vector-valued Sobczyk's theorem, whatever it is, requires a balance between properties of the spaces E_n and properties of the space X. And this balance even determines which operators (one, all, some, ...) can be extended. The equilibrium can be achieved in different ways: in the classical scalar case of Sobczyk's theorem, in which all the spaces $E_n = \mathbb{K}$ are 1-injective, the '$\operatorname{Ext}(X, E_n) = 0$ uniformly on n' condition requires nothing from X beyond its separability. On the other hand, asking nothing on the E_n side requires asking for the BAP on X, or at least something near to that. All versions known or presented so far fit into this schema one way or another. Thus, let us enter into the non-locally-convex zone, where M-ideals do not dare to thread. If E_n is a sequence of quasi-Banach spaces then $\ell_\infty(\mathbb{N}, E_n)$ or $c_0(\mathbb{N}, E_n)$ is a quasi-Banach space if and only if the moduli of concavity of the spaces E_n are uniformly bounded, in which case they can be given uniformly equivalent r-norms for some $r \in (0,1]$. If $\Phi\colon X \longrightarrow c_0(\mathbb{N}, E_n)$ is quasilinear, its components $\Phi_n\colon X \longrightarrow E_n$ form a sequence of quasilinear maps such that $\lim \|\Phi_n x\| = 0$ for all $x \in X$ and $Q(\Phi_n) \leq Q(\Phi)$.

Lemma 10.1.2 *Let $\Phi_n\colon X \longrightarrow Y$ be a sequence of p-linear maps vanishing on a fixed Hamel basis of X. If* $\sup_n Q^{(p)}(\Phi_n) < \infty$ *and Φ_n is pointwise null, then* $\lim_n Q^{(p)}(\Phi_n) = 0$. *If, moreover, X is finite-dimensional,* $\lim_n \|\Phi_n\| = 0$.

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Proof The first assertion is part of the completeness Theorem 3.6.3. The second part is straightforward after that. □

Proposition 10.1.3 *Let E_n and X be p-Banach spaces and let $\Phi\colon X \longrightarrow c_0(\mathbb{N}, E_n)$ be a p-linear map whose components $\Phi_k\colon X \longrightarrow E_k$ are μ-trivial. If X is separable and has the λ-AP, then Φ is trivial. More precisely, for each $\varepsilon > 0$, there is a dense subspace $X_\varepsilon \subset X$ and a linear map $\Lambda\colon X_\varepsilon \longrightarrow c_0(\mathbb{N}, E_n)$ such that $\|\Phi - \Lambda\| < \mu(1 + \lambda^p)^{1/p} + \varepsilon$ on X_ε.*

Proof Fix $\varepsilon \in (0, 1)$, a chain of finite-dimensional subspaces (X_k) whose union is dense in X and operators $B_k \in \mathfrak{F}(X)$ such that $B_k|_{X_k} = \mathbf{1}_{X_k}$, $B_k[X] = X_{k+1}$ and $\|B_k\| < \lambda + \varepsilon$ for all k. Set $X_\varepsilon = \bigcup X_n$. Fix a Hamel basis $\mathscr{H}$ of X that contains a basis for each of the X_k and assume without loss of generality that Φ vanishes on $\mathscr{H}$, so that each Φ_n vanishes on $\mathscr{H}$. For each j, choose a natural number $N(j)$ such that $\|\Phi_n|_{X_{j+1}}\| \le \varepsilon^j$ for $n \ge N(j)$, with $N(j + 1) > N(j)$ for all j. We are ready to construct a linear map $\Lambda\colon X \longrightarrow c_0(\mathbb{N}, E_n)$ at a finite distance from Φ as follows: since Φ_n is μ-trivial, we pick a linear map L_n such that $\|\Phi_n - L_n\| \le \mu$ and set

$$\Lambda(x)(n) = \begin{cases} L_n(x) & \text{for } n < N(1), \\ (L_n - L_n B_j)(x) & \text{for } N(j) \le n < N(j+1). \end{cases}$$

It is clear that $\Lambda(x) \in c_0(\mathbb{N}, E_n)$ for $x \in \bigcup_{n\ge1} X_n$. Indeed, if $x \in X_k$, then for $n \ge N(k)$, we have $\Lambda(x)(n) = 0$ since $B_j(x) = x$ for $j \ge k$. The argument concludes by taking into account that for $n \ge N(j)$ we have

$$\|L_n|_{X_{j+1}}\|^p \le \|\Phi_n|_{X_{j+1}} - L_n|_{X_{j+1}}\|^p + \|\Phi_n|_{X_{j+1}}\|^p \le \mu^p + \varepsilon^{jp}.$$

Finally, let us estimate $\|\Phi - \Lambda\| = \sup_n \|\Phi_n - \Lambda_n\|$, where $\Lambda_n(x) = \Lambda(x)(n)$. For $n < N(1)$, we have $\|\Phi_n - \Lambda_n\| = \|\Phi_n - L_n\| \le \mu$, while for $N(j) \le n < N(j + 1)$, the number $\|\Phi_n - \Lambda_n\|^p$ is at most

$$\|\Phi_n - L_n + L_n B_j\|^P \le \|\Phi_n - L_n\|^p + \|L_n B_j\|^p \le \mu^p + (\mu^p + \varepsilon^{jp})(\lambda + \varepsilon)^p. \quad \square$$

We translate this into an extension result

Proposition 10.1.4 *Let Z and E_n be p-Banach spaces. Let Y be subspace of Z such that Z/Y is separable and has the λ-AP. If $\tau\colon Y \longrightarrow c_0(\mathbb{N}, E_n)$ is an operator such that every component $\tau_n\colon Y \longrightarrow E_n$ admits a μ-extension to Z then, for every $\varepsilon > 0$, τ admits a $(\lambda^p + \mu^p 2^{1-1/p}\lambda^p)^{1/p} 2^{1/p} + \varepsilon$ -extension to Z.*

Proof Put $X = Z/Y$, take $\varepsilon > 0$ and use Corollary 3.3.8 to obtain a quasilinear map $\Phi\colon X \longrightarrow Y$ and an isomorphism $u\colon Y \oplus_\Phi X \longrightarrow Z$ with $\|u\| < 2^{1/p-1} + \varepsilon$ and $\|u^{-1}\| < 2^{1/p} + \varepsilon$. Now, each τ_n admits μ-extensions to Z, hence they admit

$(\mu 2^{1/p-1} + \varepsilon)$-extensions to $Y \oplus_\Phi X$ and thus, by Lemma 3.5.4, each $\tau_n \circ \Phi$ is $(\mu 2^{1/p-1}+\varepsilon)$-trivial. From here, Lemma 3.5.4 yields that $\tau\circ\Phi\colon X \longrightarrow c_0(\mathbb{N}, E_n)$ is $(\mu 2^{1/p-1} + \varepsilon)((1 + \lambda^p)^{1/p} + \varepsilon)$-trivial on a certain dense subspace $X_\varepsilon \subset X$. Lemma 3.5.4 once again says that τ admits a $(\mu 2^{1/p-1} + \varepsilon)((1 + \lambda^p)^{1/p} + \varepsilon)$-extension to $Y \oplus_\Phi X_\varepsilon$ (which is dense in $Y \oplus_\Phi X$) and so a $(\mu 2^{1/p-1} + \varepsilon)((1 + \lambda^p)^{1/p} + \varepsilon)(2^{2/p-1} + \varepsilon)$-extension to the corresponding subspace of Z. Since p-norms are continuous, the proof is done. □

10.2 Polyhedral $\mathscr{L}_\infty$-Spaces

A Banach space is said to be *polyhedral* if the unit ball of every finite-dimensional subspace is a polyhedron. Since this a geometrical notion – c_0 is polyhedral, while c is not – we will consider the isomorphism version: a space is said to be *isomorphically polyhedral* if it can be renormed to be polyhedral. The space $C(\alpha)$ and all of its subspaces are isomorphically polyhedral for every ordinal α. The wicked ways of polyhedral spaces were exhausted by Fonf in results that are ‘most enjoyable to encounter, to lecture on, and to write about. Plainly speaking, they are too clever by half’ (Diestel [152, p. 172]). Among them, a fundamental result [178, Theorem 6.21]: infinite-dimensional polyhedral spaces are c_0-saturated (closed infinite-dimensional subspaces contain a copy of c_0). Polyhedral spaces are baffling objects: there exist polyhedral spaces that admit ℓ_p, $1 < p < \infty$, as a quotient [185]; other polyhedral spaces are even more exotic [341]. Polyhedral $\mathscr{L}_\infty$ or Lindenstrauss spaces are baffling too. A clean presentation of the connections between polyhedral *and* Lindenstrauss spaces is in [178, Section 6]. We have:

- There are polyhedral spaces that are not $\mathscr{L}_\infty$-spaces, such as the Schreier space, or any subspace of c_0 not isomorphic to c_0, since subspaces of $c_0(I)$ are $\mathscr{L}_\infty$-spaces if and only if they are isomorphic to some $c_0(J)$ by 1.6.3 (b).
- There are Lindenstrauss spaces with no polyhedral renorming: $C(\Delta)$ is one.
- A Banach space whose dual is isometric to ℓ_1 has a polyhedral renorming for which the dual space is still isometric to ℓ_1 [177]. This result it optimal in view of the next two items.
- The Bourgain–Delbaen (second) space is not isomorphically polyhedral since it does not contain c_0, while its dual is isomorphic to ℓ_1.
- Isometric preduals of $\ell_1(I)$ with I uncountable are not necessarily isomorphically polyhedral. All known counterexamples are $C(K)$-spaces with K

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scattered (so that $C(K) = \ell_1(K)$) that for some reason have no polyhedral renorming. This was first established for Kunen's compactum K: under [CH], the space $C(\mathsf{K})$ has the rare property that every uncountable set of elements contains one that belongs to the closed convex hull of the others. This property was used by Jiménez and Moreno [222] to show that every renorming of $C(\mathsf{K})$ has a countable boundary, which forbids equivalent polyhedral renormings. Later on, it was proved that spaces of continuous functions on some tree spaces do the same in [ZFC]; see [179].

The following counterexample was requested by Fonf and appears, along with variations, in [120]. Somehow it shows that some conjectures one might come up with are false:

Proposition 10.2.1 *There exist separable polyhedral $\mathscr{L}_\infty$-spaces that lack Pełczyński's property* (V)*; in particular, they are not isomorphic to any Lindenstrauss space.*

Proof Recall from Lemma 9.3.14 that for each N there is an operator $T_N\colon c_0 \longrightarrow \ell_\infty(\omega^N)$ with $\operatorname{dist}(T_N(x), C(\omega^N)) \le \|x\|$ for all $x \in c_0$ and such that $\|T_N - L\| \ge \rho_{2N}(c_0)$ for every linear map $L\colon c_0 \longrightarrow C(\omega^N)$. Taking into account the comment after Proposition 8.6.8, we can simply assume $\|T_N - L\| \ge N$. We shall use the spaces $Z[T_N]$ introduced during the proof of Proposition 9.3.15 and the isometrically exact sequences

$$0 \longrightarrow C(\omega^N) \longrightarrow Z[T_N] \xrightarrow{Q_N} c_0 \longrightarrow 0 \tag{10.1}$$

with inclusion $f \longmapsto (f, 0)$ and quotient map $(f, x) \longmapsto x$. The twisted sum space $Z[T_N]$ is isomorphic in this occasion to c_0, thus it has a polyhedral norm. The main result in [146] asserts that in a separable isomorphically polyhedral space the polyhedral norms are dense; thus, let $\|\cdot\|_N$ be a polyhedral norm in $d_{T_N}(C(\omega^N, c_0))$ that is 2-equivalent to $\|\cdot\|_{T_N}$. The sequences (10.1) split but they are increasingly far from trivial by the choice of T_N; thus, their c_0-sum cannot split and neither can the c_0-sum (here $Q = (Q_N)$):

$$0 \longrightarrow c_0(\mathbb{N}, C(\omega^N)) \longrightarrow c_0(\mathbb{N}, Z[T_N]) \xrightarrow{Q} c_0(c_0) \longrightarrow 0$$

The space $c_0(\mathbb{N}, Z[T_N])$ is polyhedral since any c_0-sum of polyhedral spaces is polyhedral [208]. Define the 'diagonal' operator $\tau\colon c_0 \longrightarrow c_0(c_0)$ by $\tau(x) = (N^{-1/2}x)_N$ and form the pullback diagram

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$$\begin{tikzcd}
0 \arrow[r] & c_0(\mathbb{N}, C(\omega^N)) \arrow[r] & c_0(\mathbb{N}, Z[T_N]) \arrow[r, "Q"] & c_0(c_0) \arrow[r] & 0 \\
0 \arrow[r] & c_0(\mathbb{N}, C(\omega^N)) \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r, "\underline{Q}"] \arrow[u, "\underline{\tau}"] & c_0(c_0) \arrow[r] \arrow[u, "\tau"'] & 0
\end{tikzcd}$$

The plan is to show that $\underline{Q}$ is strictly singular, which prevents PB from having property (V), and thus prevents it from being Lindenstrauss under any equivalent norm. Assume that $\underline{Q}$ is not strictly singular, and find an infinite-dimensional subspace $E \subset c_0$ and a lifting $T\colon E \longrightarrow \mathrm{PB}$ such that $\underline{Q}T = \tau|_E$. By the c_0 standard saturation and distortion properties of $c_0 = c_0(c_0)$, there is no loss of generality assuming that E is a 2-isomorphic copy of c_0. Since $Q\underline{\tau}T = \tau|_E$, we get $Q_N\underline{\tau}T(e) = N^{-1/2}e$ for all $e \in E$, which in particular means that $\underline{\tau}T$ has the form $(L_N e, N^{-1/2}e)_N$ where $L_N\colon E \longrightarrow C(\omega^N)$ is a certain linear map. Therefore, there is a constant M such that $\|(L_N e, N^{-1/2}e)\| \le M\|e\|$, which means $\|L_N e - T_N N^{-1/2}e\| \le M\|e\|$, yielding a contradiction:

$$2N \le \rho_{2N}(c_0) \le 2\rho_{2N}(E) \le 2\|N^{1/2}L_N - T_N\| \le 2MN^{1/2}.$$

To conclude, the space PB is a subspace of $c_0(\mathbb{N}, Z[T_N]) \oplus_\infty c_0$, and thus it is polyhedral. □

10.3 Lipschitz and Uniformly Homeomorphic $\mathscr{L}_\infty$-Spaces

Non-linear geometry of Banach spaces has seen spectacular advances since the Benyamini–Lindenstrauss book [42]; [275; 192] are good surveys on the matter. The part we are interested in here is the collision between three ways of classifying Banach spaces: isomorphic, Lipschitz and uniformly homeomorphic. Our entrance is:

Lemma 10.3.1 *Let* $0 \longrightarrow A \longrightarrow B \xrightarrow{\pi} C \longrightarrow 0$ *be an exact sequence of Banach spaces. If* π *admits a Lipschitz / uniformly continuous section then* $A \times C$ *and* B *are Lipschitz / uniformly homeomorphic.*

Proof Let $s\colon C \longrightarrow B$ be a Lipschitz / uniformly continuous section of π. The Lipschitz / uniformly continuous map $f\colon A \times C \longrightarrow B$ given by $f(a, c) = a + s(c)$ has Lipschitz / uniformly continuous inverse $f^{-1}(b) = (b - s\pi(b), \pi(b))$. □

Thus, if one is able to find good reasons to prevent $B \simeq A \times C$, the game is over. Let us say that a Banach space is determined by its Lipschitz

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/ uniform structure if it is linearly isomorphic to every Banach space to which it is Lipschitz / uniformly homeomorphic. Next we consider the Lipschitz and uniform structure of $\mathscr{L}_\infty$-spaces. To tackle the problem, it would help us, as always, to know the behaviour of Lipschitz / uniform structures under homological manipulations:

Lemma 10.3.2 *If (the quotient map of) a short exact sequence has a Lipschitz / uniformly continuous section then so does any pushout or pullback sequence.*

Proof Indeed, given a pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r, "\overline{\pi}"] & C \arrow[r] & 0 \end{tikzcd}$$

if s is a Lipschitz / uniformly continuous section of π then $\beta \circ s$ is a Lipschitz / uniformly continuous section of $\overline{\pi}$. In a pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r, "\pi"] & C \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r, "\underline{\pi}"] \arrow[u, "\underline{\gamma}"] & C' \arrow[r] \arrow[u, "\gamma"'] & 0 \end{tikzcd}$$

the map $(s\gamma, \mathbf{1}_{C'})$ is a Lipschitz / uniformly continuous section of $\underline{\pi}$. □

The pullback part is due to Kalton [276] and the pushout part is due to Suárez [446]. We are ready for (counter) examples and surprises. We know from Proposition 8.3.1 that the sequence $0 \longrightarrow c_0 \longrightarrow \ell_\infty \longrightarrow \ell_\infty/c_0 \longrightarrow 0$ admits a Lipschitz projection (retraction sounds better in this context), which raises the question of whether a Lipschitz or at least a uniformly continuous section can be found. Kalton shows in [276] that the quotient map $\ell_\infty \longrightarrow \ell_\infty/c_0$ has no uniformly continuous section, which is simultaneously unexpected and surprising. Unexpected because it makes Lemma 2.1.7, the basic fuel for most homological arguments, contain empty calories in the non-linear world: the existence of Lipschitz retraction and Lipschitz section are no longer equivalent! And surprising because Aharoni and Lindenstrauss [1] had already shown:

Lemma 10.3.3 *The Nakamura–Kakutani sequences admit Lipschitz sections.*

Proof We show that the quotient map $\pi\colon C_0(\wedge_{\mathcal{M}}) \longrightarrow c_0(\mathcal{M})$ admits a Lipschitz selector on the finitely supported elements of $c_0(\mathcal{M})$. Write

$$x = \sum_{n=1}^{N} a_n e_{\gamma_n} - \sum_{n=1}^{M} b_m e_{\mu_m}$$

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with $a_1 \geq a_2 \geq \cdots \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq 0$. This is a representation of x as a difference $x = x^+ - x^-$ of two disjointly supported positive elements. Set

$$\gamma_n^* = \gamma_n \setminus \bigcup_{1\leq i<n} \gamma_i \qquad \mu_m^* = \mu_m \setminus \bigcup_{1\leq j<m} \mu_j$$

for $1 \leq n \leq N$ and $1 \leq m \leq M$, and define

$$s(x) = \sum\nolimits_{n=1}^{N} a_n 1_{\gamma_n^*} - \sum\nolimits_{n=1}^{M} b_m 1_{\mu_m^*}.$$

Clearly, $\pi(s(x)) = x$ and s is Lipschitz because

$$s(x)(k) = \operatorname{dist}(x^+, [e_\gamma : k \notin \gamma]) - \operatorname{dist}(x^-, [e_\gamma : k \notin \gamma]). \qquad \square$$

Thus, $C(\triangle_{\mathcal{M}})$ is Lipschitz but not linearly homeomorphic to $c_0(\mathcal{M}) = c_0 \times c_0(\mathcal{M})$. The same is true for the spaces JL_p and CC obtained via the pullback diagrams (2.38) and (2.39) thanks to Lemma 10.3.2:

Corollary 10.3.4 *The Johnson–Lindenstrauss space* JL_p *is Lipschitz homeomorphic but not linearly homeomorphic to* $c_0 \times \ell_p(\mathfrak{c})$. *The space* CC *is Lipschitz homeomorphic but not linearly homeomorphic to* $c_0 \times \ell_\infty$.

Therefore, there is no doubt that the linear structure of non-separable $\mathscr{L}_\infty$- or even $\mathscr{C}$-spaces is not determined by their Lipschitz structure. Aharoni and Lindenstrauss [1] asked whether a similar result holds in the separable setting. A partial answer was provided by Johnson, Lindenstrauss and Schechtman [229, Corollary 3.2]: if a $\mathscr{C}$-space is uniformly homeomorphic to c_0 then it is linearly homeomorphic to c_0. This result raises the question [229, Problem (d)] of whether every separable $\mathscr{L}_\infty$-space is determined by its uniform structure. Again, the answer is no, as shown by Suárez in [446]. Let X be a Banach space. Let $\omega\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the function $\omega(t) = \sqrt{t}$ if $t \leq 1$ and $\omega(t) = 1 + \frac{1}{2}(t-1)$ if $t \geq 1$. It is clear that ω is concave, hence subadditive, and therefore, the function $d(x,y) = \omega(\|x-y\|)$ is a metric on X. Let $X(\omega)$ be the metric space obtained by endowing X with d. The formal identity $X(\omega) \longrightarrow X$ is a 2-Lipschitz map with uniformly continuous inverse. Form the exact sequence involving the Lipschitz-free space of Section 4.6.1 (e):

$$0 \longrightarrow \ker\beta \longrightarrow \mathcal{F}(X(\omega)) \xrightarrow{\beta} X \longrightarrow 0$$

The quotient map β admits a uniformly continuous section, namely the composition of the identity $X \longrightarrow X(\omega)$ with the natural Lipschitz map $\delta\colon X(\omega) \longrightarrow \mathcal{F}(X(\omega))$. The space $\mathcal{F}(X(\omega))$ is Schur [270, Theorem 4.6], and thus it cannot be linearly homeomorphic to $\ker\beta \times X$ when X fails the

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Schur property, even if they *are* uniformly homeomorphic. To obtain an $\mathscr{L}_\infty$ counterexample, pick $X = c_0$ and form the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \ker\beta \arrow[r] \arrow[d] & \mathcal{F}(c_0(\omega)) \arrow[r, "\beta"] \arrow[d] & c_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{L}_\infty^{\mathsf{BP}}(\ker\beta) \arrow[r] \arrow[d] & \mathrm{PO} \arrow[r, "\overline{\beta}"] \arrow[d] & c_0 \arrow[r] & 0 \\
 & S \arrow[r, equal] \arrow[d] & S \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \qquad (10.2)$$

Since β admits a uniformly continuous selection, $\overline{\beta}$ does as well, and thus PO is a separable $\mathscr{L}_\infty$-space that is uniformly homeomorphic to $\mathscr{L}_\infty^{\mathsf{BP}}(\ker\beta) \times c_0$. By the 3-space property of Schur spaces, PO must be Schur, thus it cannot be isomorphic to $\mathscr{L}_\infty^{\mathsf{BP}}(\ker\beta) \times c_0$. In conclusion:

Proposition 10.3.5 *There are separable $\mathscr{L}_\infty$-spaces that are uniformly homeomorphic but not linearly homeomorphic.*

10.4 Properties of Kernels of Quotient Maps on $\mathscr{L}_1$ Spaces

Throughout this section, $Q\colon L_1(\mu) \longrightarrow X$ will denote a quotient map, and we consider the sequence

$$0 \longrightarrow \ker Q \longrightarrow L_1(\mu) \xrightarrow{\ Q\ } X \longrightarrow 0$$

Our aim is to connect properties of X with those of $\ker Q$. We have already treated some aspects of the problem: both the BAP and the UAP pass from X to $\ker Q$; see Proposition 5.3.4 and its Corollary 5.3.12. Most questions about kernels of quotient maps on $L_1(\mu)$ are simply too difficult; we will see good examples in this and the following two sections. Bourgain's paper [48] concludes with the following remarks: 'Let Δ be the Cantor group and define E as the subspace of $L_1(\Delta)$ generated by the Walsh functions w_S for

$|S| \geq 2$. Obviously E is uncomplemented. What about the following: Is E an $\mathscr{L}_1$-space? Is E isomorphic to $L_1(\Delta)$?' Bourgain attributes these questions to Pisier. To understand what is being asked, write $\Delta = \{\pm 1\}^{\mathbb{N}}$ and consider the characters $\chi_n : \Delta \longrightarrow \{\pm 1\}$ given by $\chi_n(x) = x_n$. Note that (χ_n) is the Rademacher sequence in disguise. Then $E = \{f \in L_1(\Delta) \colon \int_\Delta f\chi_n = 0\, \forall n \in \mathbb{N}\}$, where the integral is taken with respect to the Haar measure on Δ. The space E is the kernel of the operator $\chi \colon L_1(\Delta) \longrightarrow \ell_\infty$ given by $\chi(f) = (\int_\Delta f\chi_n)_{n\geq 1}$. By the Riemann–Lebesgue lemma, χ takes values in c_0, and basic harmonic folklore implies that χ is onto c_0. Thus, the questions of Bourgain and Pisier refer to the kernel of a well-behaved quotient map $L_1 \longrightarrow c_0$. As Johnson [224] says; 'at any rate, both of them as well as Kisliakov, Zippin, Schechtman, and I thought about them around that time'. Again following Johnson, the underlying problem seems to be whether there is a 'natural', uncomplemented $\mathscr{L}_1$-subspace of an $L_1(\mu)$-space, in particular an invariant one in $L_1(G)$, where G is a locally compact Abelian group. The narration continues with Johnson mentioning that 'while lecturing on their results in 1995 and 1996, the authors of [284] asked whether such an E could have local unconditional structure'. To show how ugly the subspace E can be, we will first show that it is not an ultrasummand (so it is not an $L_1(\mu)$-space) and its bidual is not complemented in a Banach lattice (so it is not even an $\mathscr{L}_1$-space).

When Is $\ker Q$ an Ultrasummand?

Our use of the Radon–Nikodým property (RNP) is merely pragmatic. Anyway, everything there is to know about it can be found in [155, Chapter III]. What we need is to know that a Banach space X has the RNP if, for every finite measure μ, every operator $T \colon L_1(\mu) \longrightarrow X$ is *representable*, that is, has the form $Tf = \int fg d\mu$ for some $g \in L_\infty(\mu, X)$. This implies that for any measure λ, finite or not, every operator $L_1(\lambda) \longrightarrow X$ factorises through some $\ell_1(I)$. Clearly, the RNP is inherited by subspaces. Separable dual spaces and subspaces of spaces with RNP have RNP, whereas c_0 and L_1 do not.

Lemma 10.4.1 *If X is an ultrasummand with the RNP then the kernel of every quotient map $L_1(\mu) \longrightarrow X$ is an ultrasummand.*

Proof The second and third rows in the following diagram are equivalent (see Section 2.10; also compare with Diagrams (2.28) and (2.29)):

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$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\delta_K"] & L_1(\mu) \arrow[r, "Q"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 & \text{(z)} \\
0 \arrow[r] & K^{**} \arrow[r] \arrow[d, equal] & \mathrm{PO} \arrow[r] & X \arrow[r] \arrow[d, equal] & 0 & \\
0 \arrow[r] & K^{**} \arrow[r] \arrow[d, equal] & \mathrm{PB} \arrow[r, "\underline{Q^{**}}"] \arrow[d, "\underline{\delta_X}"] & X \arrow[r] \arrow[d, "\delta_X"] & 0 & \\
0 \arrow[r] & K^{**} \arrow[r] & L_1(\mu)^{**} \arrow[r, "Q"] & X \arrow[r] & 0 & \text{(z}^{**}\text{)}
\end{tikzcd}$$

The existence of a projection $P\colon X^{**} \longrightarrow X$ and the fact that $Q^{**}\underline{\delta_X} = \delta_X\underline{Q^{**}}$ imply $PQ^{**}\underline{\delta_X} = P\delta_X\underline{Q^{**}} = \underline{Q^{**}}$; thus, since X has the RNP and $L_1(\mu)^{**}$ is isometric to some $L_1(\lambda)$-space, $\underline{Q^{**}}$ factorises through some $\ell_1(I)$. Therefore $[\mathsf{z}\,\underline{Q^{**}}] = 0$. Moreover, quite obviously, $[\mathsf{z}^{**}\,\delta_X Q] = 0$. The diagonal principle implies that $K \times \mathrm{PB} \simeq K^{**} \times L_1(\mu)$, thus K is an ultrasummand. □

One cannot replace $L_1(\mu)$ by an arbitrary $\mathscr{L}_1$-space: after all, not every $\mathscr{L}_1$-space is an ultrasummand. Replacing L_1 with ℓ_1 allows us to present a local version of the result above:

Proposition 10.4.2 *If X is a Banach space without the RNP then $\kappa(X)$ is not an ultrasummand.*

Proof Since X lacks the RNP, there is a finite measure μ and an operator $\phi\colon L_1(\mu) \longrightarrow X$ that is not representable. Take a projective presentation of X and form the pullback with ϕ:

$$\begin{tikzcd}
0 \arrow[r] & \kappa(X) \arrow[r] & \ell_1(I) \arrow[r, "\pi"] & X \arrow[r] & 0 \\
0 \arrow[r] & \kappa(X) \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & L_1(\mu) \arrow[r] \arrow[u, "\phi"] & 0
\end{tikzcd}$$

The lower sequence does not split since ϕ cannot be lifted to $\ell_1(I)$ and Lindenstrauss' lifting tells us that $\kappa(X)$ is not an ultrasummand. □

From this it immediately follows:

Corollary 10.4.3 *If X contains c_0 then $\kappa(X)$ is not complemented in its bidual.*

We supply an alternative proof for this corollary. Let X be a space containing c_0 and consider the situation

$$\begin{tikzcd}
0 \arrow[r] & \kappa(X) \arrow[r] & \ell_1(I) \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & \kappa(X) \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & c_0 \arrow[r] \arrow[u, "\jmath"'] & 0 \\
0 \arrow[r] & \kappa(X) \arrow[r] \arrow[u, equal] & \mathrm{PB}' \arrow[r] \arrow[u] & \ell_2 \arrow[r] \arrow[u, "\imath"'] & 0
\end{tikzcd}$$

in which $\jmath\colon c_0 \longrightarrow X$ is the embedding and $\imath\colon \ell_2 \longrightarrow c_0$ is the natural inclusion. An appeal to the surprising sequence (5.1) implies that $\kappa(X)$ cannot be an ultrasummand: otherwise the lower pullback sequence would split and $\jmath\imath$ could be lifted to an operator $\ell_2 \longrightarrow \ell_1(I)$, which would be compact, something that $\jmath\imath$ is not. □

Now, we pass to the general situation [284, Proposition 2.2]:

Proposition 10.4.4 *If μ is a finite measure, X is a Banach space containing c_0 and $Q\colon L_1(\mu) \longrightarrow X$ is a quotient map then* $\ker Q$ *is not an ultrasummand.*

Proof Since μ is finite, the space X must be WCG, and thus all copies of c_0 it contains must be complemented. The hard work of [284] is a delicate analysis, which we will omit, showing the existence of a copy $\jmath\colon c_0 \longrightarrow X$ that is complemented via a projection $P\colon X \longrightarrow c_0$ such that PQ is representable. Once this is done, the rest is easy: since c_0 lacks the RNP, there is an operator $\tau\colon L_1 \longrightarrow c_0$ that is not representable. This means that the composition $\jmath\tau$ cannot be lifted to an operator $T\colon L_1 \longrightarrow L_1(\mu)$ since otherwise $\tau = P\jmath\tau = PQT$ would be representable. By Lindenstrauss' lifting, if $\jmath\tau$ cannot be lifted then $\ker Q$ cannot be complemented in its bidual. □

If the measure is not σ-finite then the kernel can be an ultrasummand: consider a quotient map $Q\colon L_1(\mu) \longrightarrow c_0$ and the double adjoint $Q^{**}\colon L_1(\mu)^{**} \longrightarrow \ell_\infty$. Then $L_1(\mu)^{**} = L_1(\lambda)$ for some (very large) λ, and $\ker Q^{**} = (\ker Q)^{**}$ is an ultrasummand.

When Is $\ker Q$ an $\mathscr{L}_1$-Space?

The results so far suggest that $\ker Q$ 'tends not to be an $L_1(\mu)$ space'. Could it be at least an $\mathscr{L}_1$-space? The answer to this question is a resounding no.

Proposition 10.4.5 *Let X be a Banach space containing ℓ_∞^n uniformly, and let $Q\colon \mathscr{L}_1 \longrightarrow X$ be a quotient map. Then* $\ker Q$ *is not an $\mathscr{L}_1$-space.*

Proof By Proposition 5.2.22 there is a non-trivial sequence of Banach spaces $0 \longrightarrow \ell_2 \longrightarrow E \longrightarrow X \longrightarrow 0$. By Lindenstrauss' lifting, Q lifts to an operator $\widehat{Q}\colon \mathscr{L}_1 \longrightarrow E$, and one gets the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker Q \arrow[r] \arrow[d, "\widehat{Q}|_{\ker Q}"'] & \mathscr{L}_1 \arrow[r, "Q"] \arrow[d, "\widehat{Q}"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \ell_2 \arrow[r] & E \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

The restriction $\widehat{Q}|_{\ker Q}$ cannot extend to $\mathscr{L}_1$ since the lower sequence is not trivial, which means that $\widehat{Q}|_{\ker Q}$ is not 2-summing, and thus $\ker Q$ cannot be an $\mathscr{L}_1$-space. □

When Does $\ker Q$ Have l.u.st.?

A Banach space X has local unconditional structure (l.u.st.) if there is a constant Λ such that for every finite-dimensional subspace $E \subset X$, there is another finite-dimensional subspace $F \subset X$ containing E and admitting a Λ-unconditional basis.

Lemma 10.4.6 *A Banach space X with l.u.st. embeds as a locally complemented subspace of a Banach lattice L which is (crudely) finitely representable in X. If X is separable then L can be taken separable.*

Proof Let $\mathfrak{U}$ be an ultrafilter refining the order filter on $\mathscr{F}(X)$. For every $E \in \mathscr{F}(X)$, we select $F_E \in \mathscr{F}(X)$ containing E with a normalised Λ-unconditional basis, and we renorm it such that the new constant of the basis is 1. If we denote this renorming of F_E by $\tilde{E}$, then it is clear that $\|x\| \le \|x\|_{\tilde{E}} \le \Lambda\|x\|$ for every $x \in F$. Consider the operators

$$X \xrightarrow{\text{`inclusion'}} [E]_{\mathfrak{U}} \xrightarrow{\text{inclusion}} [F_E]_{\mathfrak{U}} \xrightarrow{\text{identity}} [\tilde{E}]_{\mathfrak{U}} \xrightarrow{\text{inclusion}} X_{\mathfrak{U}}$$

Here, the first arrow sends each $x \in X$ into the class of $(x1_E(x))_E$, and the others are the obvious arrows. The ultraproduct $[\tilde{E}]_{\mathfrak{U}}$ is a Banach lattice where X sits locally complemented because the composition of all the arrows is the diagonal embedding of X into $X_{\mathfrak{U}}$. Also, $[\tilde{E}]_{\mathfrak{U}}$ is finitely representable in X because it embeds into $X_{\mathfrak{U}}$. Finally, if X is separable, then so is $L(X)$, the (closed) sublattice generated by X in $[\tilde{E}]_{\mathfrak{U}}$, where X is again locally complemented. □

Thus we get:

Proposition 10.4.7 *If $Q : L_1 \longrightarrow c_0$ is a quotient map then* $\ker Q$ *does not have l.u.s.t.*

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Proof Assume then that ker Q has l.u.st. and let $L(\ker Q)$ be a separable lattice as in the lemma. Form the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \ker Q \arrow[r] \arrow[d] & L_1 \arrow[r, "Q"] \arrow[d] & c_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L(\ker Q) \arrow[r] \arrow[d] & \mathrm{PO} \arrow[r] \arrow[d] & c_0 \arrow[r] & 0 \\
 & V \arrow[r, equal] \arrow[d] & V \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The local splitting of the left column has two immediate consequences. First is that V is (crudely) finitely representable in $L(\ker Q)$, hence also in $\ker Q$, and so also in L_1. Second is that the middle column, which is a pushout of the left column, also locally splits. This entails that PO is (crudely) finitely representable in $V \times L_1$, hence also in L_1. Using [329], we obtain that PO embeds into L_1, which was our goal. Now form the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & L(\ker Q) \arrow[r, equal] \arrow[d] & L(\ker Q) \arrow[d] & & \\
0 \arrow[r] & \mathrm{PO} \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

and observe that N contains c_0. Proposition 10.4.4 yields that $L(\ker Q)$ cannot be an ultrasummand, something that it actually *is*: $L(\ker Q)$ does not contain c_0, and any Banach lattice that does not contain c_0 is an ultrasummand [335, Theorem 1.c.4]. □

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In the Appendix of [224] Johnson manages to prove that if L embeds into a Banach lattice that does not contain ℓ_∞^n uniformly and $Q\colon L \longrightarrow X$ is a quotient map whose kernel has Gordon–Lewis local unconditional structure (GL-l.u.st.), then X does not contain ℓ_∞^n uniformly. Even if this is just a result of [174], let us take as a definition that a Banach space Y has GL-l.u.st. if Y^{**} is complemented in a Banach lattice or, equivalently, if Y is locally complemented in a Banach lattice. See [153, p. 348] for a proof (bearing in mind Johnson's warning [224]: 'but keep in mind that in [153] GL-l.u.st. is called l.u.st. while l.u.st. is called DPR-l.u.st').

When Does $\ker Q$ Have the Dunford–Pettis Property?

Obtaining spaces with the DPP is always a challenge. Let us consider now the question of when kernels of quotient maps on $\mathscr{L}_1$-spaces have the DPP. The classical Lohman's lifting yields that if Z has the DPP and Y is a subspace of Z that does not contain ℓ_1 then Z/Y has the DPP [151]. Thus, quotients of an $\mathscr{L}_\infty$-space by a reflexive subspace have the DPP, as well as all their higher duals [294], and consequently, the same occurs with the kernels of quotient maps from an $\mathscr{L}_1$-space onto a reflexive space. If we, however, relax the conditions and simply ask for this quotient to be a separable dual then $(\ker Q)^*$ could fail the DPP, as the following example shows. The kernel in the sequence $0 \longrightarrow \ell_1(\mathbb{N}, \ell_2^n) \longrightarrow \ell_1(\mathbb{N}, \ell_1^{2^n}) \longrightarrow \ell_1(\mathbb{N}, \ell_1^{2^n}/\ell_2^n) \longrightarrow 0$ is a Schur space and thus it enjoys the DPP, but its dual $\ell_\infty(\mathbb{N}, \ell_2^n)$ does not because it contains complemented copies of ℓ_2: just take a free ultrafilter on $\mathbb{N}$ and lift (the identity of) any infinite-dimensional separable subspace of $[\ell_2^n]_{\mathfrak{U}}$ to $\ell_\infty(\mathbb{N}, \ell_2^n)$. However, $\ker Q$ itself has the DPP:

Proposition 10.4.8 *Let U be an ultrasummand with the RNP and let $Q\colon \mathscr{L}_1 \longrightarrow U$ be a quotient map. Then* $\ker Q$ *has the DPP.*

Proof Let us observe the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \kappa(U) \arrow[r] \arrow[d] & \ell_1(I) \arrow[r] \arrow[d] & U \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker Q \arrow[r] & \mathscr{L}_1 \arrow[r, "Q"] & U \arrow[r] & 0
\end{tikzcd}$$

The space $\kappa(U)$ is an ultrasummand by Lemma 10.4.1. Thus, $\operatorname{Ext}(\mathscr{L}_1, \kappa(U)) = 0$ by Lindenstrauss' lifting. The Diagonal principle then yields $\kappa(U) \times \mathscr{L}_1 \simeq \ker Q \times \ell_1(I)$. The Schur space $\kappa(U)$ has the DPP, and thus $\ker Q \times \ell_1(I)$ also has the DPP, as well as $\ker Q$. $\square$

Just make a sideways step to spaces far from 'ultrasummands with the RNP' and you will encounter a problem treated in [284]: does the kernel of a quotient map $Q\colon L_1(\mu) \longrightarrow c_0(I)$ have the DPP? In particular, does the kernel of a quotient map $Q\colon L_1(\mu) \longrightarrow c_0$ have the DPP? Does the kernel of a quotient map $Q\colon X \longrightarrow c_0$ have the DPP when X has the DPP (a question from Pełczyński)? Regarding these problems, Kalton and Pełczyński observe that if S is a Sidon set in a locally compact abelian group G then the 'truncated' Fourier transform $Q_S : L_1(G) \longrightarrow c_0(S)$ given by $Q_S(f) = (\hat{f}(\gamma))_{\gamma\in S}$ is onto (this can be taken as the definition of a Sidon set, if one wants) and prove, swallowed with a good draught of hard analysis, that $\ker Q_S$ does have the DPP. Of course that $\kappa(c_0)$ has the DPP (it is Schur), but the hard questions are whether either $\kappa(c_0)^* = \ell_\infty/\ell_1$ [100, Question 1 (c)] and [109, Problem B] or $\kappa(c_0)^{**}$ [109, below Problem B] has the DPP. Both questions are treated next.

When Does $\mathscr{L}_\infty/X$ Have the Dunford–Pettis Property?

Related to questions of when $\ker Q$ has the DPP, and somewhat dual, are questions of when quotients of $\mathscr{L}_\infty$-spaces have the DPP. It is clear that the DPP is stable under products and passes to complemented subspaces. In fact, it passes to locally complemented subspaces because weakly compact operators extend to weakly compact operators from locally complemented subspaces; see Proposition 5.1.9. The DPP is not a 3-space property, even in locally trivial sequences: there exist Schur spaces S and non-trivial sequences $0 \longrightarrow S \longrightarrow \ell_1 \times \ell_2 \longrightarrow L_1 \longrightarrow 0$ (go to Proposition 2.12.5 and set $X = \ell_2$ and L_1 in the place of $C(K)$). However, if X does not contain ℓ_1 and X^{**}/X has the DPP then X^{**} also has the DPP [101]. A Banach space is called Asplund if all its separable subspaces have separable duals; equivalently, if its dual has the RNP. An Asplund space cannot contain ℓ_1, and thus every quotient of an $\mathscr{L}_\infty$-space by an Asplund subspace has the DPP. Moreover:

Proposition 10.4.9 *The dual of every quotient of an $\mathscr{L}_\infty$ space by an Asplund space has the DPP.*

Proof If A is an Asplund space, A^* has the RNP, and Proposition 10.4.8 applies to the dual sequence $0 \longrightarrow (\mathscr{L}_\infty/A)^* \longrightarrow \mathscr{L}_1 \longrightarrow A^* \longrightarrow 0$. □

The bidual of $\mathscr{L}_\infty/A$ can, however, fail the DPP, as the sequence $0 \longrightarrow c_0(\mathbb{N}, K_n) \longrightarrow c_0(\mathbb{N}, \ell_\infty^{2^n}) \longrightarrow c_0(\mathbb{N}, \ell_2^n) \longrightarrow 0$ shows. Probably the least Asplund space in sight is ℓ_1, and thus deciding whether $\ell_\infty/\ell_1 = \kappa(c_0)^*$ has DPP would round off the situation. Recall that the Lindenstrauss–Rosenthal theorem makes the space ℓ_∞/ℓ_1 well defined and, since $C(\Delta)$ is ℓ_1-automorphic (Proposition 7.4.15 plus Theorem 8.5.4), so is $C(\Delta)/\ell_1$. We have:

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10.4.10 *ℓ_∞/ℓ_1 has the DPP $\iff$ $C(\Delta)/\ell_1$ has the DPP.*

Proof $\Longrightarrow$ is a consequence of the following general fact: if E is a subspace of an $\mathscr{L}_\infty$-space $\mathscr{L}'_\infty$, and in turn a subspace of another $\mathscr{L}_\infty$-space $\mathscr{L}_\infty$ such that $\mathscr{L}_\infty/E$ has the DPP, then $\mathscr{L}'_\infty/E$ also has the DPP. Just have a look at the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E \arrow[r, equal] \arrow[d] & E \arrow[d] & & \\
0 \arrow[r] & \mathscr{L}'_\infty \arrow[r] \arrow[d] & \mathscr{L}_\infty \arrow[r] \arrow[d] & \mathscr{L}_\infty/\mathscr{L}'_\infty \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{L}'_\infty/E \arrow[r] \arrow[d] & \mathscr{L}_\infty/E \arrow[r] \arrow[d] & \mathscr{L}_\infty/\mathscr{L}'_\infty \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since the middle row splits locally, the same is true of the lower sequence, and thus the DPP passes from $\mathscr{L}_\infty/E$ to $\mathscr{L}'_\infty/E$.

To prove $\Longleftarrow$ it is clearly enough to show that if $C(\Delta)/\ell_1$ had the DPP, every separable subspace of ℓ_∞/ℓ_1 would be contained in some subspace with the DPP. Let $E \subset \ell_\infty/\ell_1$ be separable and take a separable $F \subset \ell_\infty$ such that $\pi[F] \supset E$ (here π is the quotient map). Now, every separable subspace $E \subset \ell_\infty$ is contained in a copy G of $C(\Delta)$ inside ℓ_∞. The rest is easy: pick $G \subset \ell_\infty$ isomorphic to $C(\Delta)$ and containing both F and the relevant copy of ℓ_1. Then $\pi[G]$ contains E and is isomorphic to $C(\Delta)/\ell_1$. $\square$

The space ℓ_∞ can be replaced in the proposition by any injective space. The general problem of which quotients of an $\mathscr{L}_\infty$-space have the DPP is wide open, but the previous discussion suggests another question [109, Problem B and Conjecture C]: If X, Y are isomorphic subspaces of $C(\Delta)$, is it true that $C(\Delta)/X$ has the DPP if and only if $C(\Delta)/Y$ has the DPP?

10.5 3-Space Problems

Three-space problems that only require a direct application of basic homological techniques were treated in Section 2.12. Here we will consider 3-space problems that require either more sophisticated applications of the basic techniques or more sophisticated tools altogether.

Pełczyński's and Rosenthal's Property (V)

The 3-space problem for Pełczyński's property (V) was solved in the negative using an involved construction of Ghoussoub and Johnson that leads to a strictly singular surjection onto c_0 whose kernel has property (V). The construction can be found in [102, Section 6.9]. Other clean examples follow from Corollary 9.3.8 or Proposition 9.3.15, in which singular sequences $0 \longrightarrow C(K) \longrightarrow \cdot \longrightarrow c_0 \longrightarrow 0$ appear for either $K = \Delta$ or $K = \omega^\omega$. The singularity of such sequences implies that the middle space cannot have property (V). The role of c_0 cannot be reversed:

Proposition 10.5.1 *Let* $0 \longrightarrow c_0(I) \xrightarrow{\imath} Z \xrightarrow{\rho} X \longrightarrow 0$ *be an exact sequence of Banach spaces. If X has Pełczyński's property* (V) *then so does Z.*

Proof Let $\phi\colon Z \longrightarrow E$ be an operator. If the restriction $\phi\imath$ is an isomorphism on some copy of c_0 then the so is ϕ. Otherwise, $\phi\imath$ is weakly compact and admits a weakly compact extension $\varphi\colon Z \longrightarrow E$. As $\phi - \varphi$ vanishes on $\ker\rho$, we have $\phi - \varphi = \psi\rho$ for some $\psi \in \mathfrak{L}(X, E)$. If ψ is weakly compact then so is $\phi = \varphi + \psi\rho$. Otherwise, there is a subspace of X isomorphic to c_0 on which ψ is an isomorphism. Let $\jmath\colon c_0 \longrightarrow X$ be the corresponding embedding

$$\begin{tikzcd}
0 \arrow[r] & c_0(I) \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \\
& & & c_0 \arrow[u, "\jmath"'] &
\end{tikzcd}$$

If $J\colon c_0 \longrightarrow Z$ is a lifting of $\jmath$, the existence of which is clear from $\operatorname{Ext}_{\mathbf{B}}(c_0, c_0(I)) = 0$, the composition ϕJ cannot be strictly singular since $\phi J = (\varphi + \psi\rho)J = \varphi J + \psi\jmath$, where φJ is strictly singular and $\psi\jmath$ is an isomorphism onto its range. □

A counterexample can also be supplied for the 3-space problem for Rosenthal's property (V): use Proposition 5.2.20 to get a non-trivial element of $\operatorname{Ext}(\ell_\infty, \ell_2)$, and then Lemma 9.3.1 to obtain an exact sequence $0 \longrightarrow \ell_\infty(I, \ell_2) \longrightarrow \diamond \longrightarrow \ell_\infty \longrightarrow 0$ whose quotient map is not an isomorphism on any copy of ℓ_∞, so that $\diamond$ fails Rosenthal's property (V). However, $\ell_\infty(I, \ell_2)$ has property (V) because Rosenthal's property (V) obviously passes to quotients and $\ell_\infty(I, \ell_2)$ is a quotient of $\ell_\infty(I, \ell_\infty) \approx \ell_\infty(I)$.

Universal Separable Injectivity

Injectivity and separable injectivity are classical notions that describe the behaviour of ℓ_∞ and c_0, respectively. Universal separable injectivity, defined next, reflects the behaviour of ℓ_∞/c_0. A thorough study of all these variations of injectivity can be found in [22].

Definition 10.5.2 A Banach space U is said to be universally separably injective (USI) if every operator $\tau\colon X \longrightarrow U$ with separable range can be extended anywhere.

Replacing 'separable range' by 'separable domain' does not affect the definition. Important examples of USI spaces are provided by the following result:

Proposition 10.5.3 *The following Banach spaces are USI:*

(a) $\ell_\infty/c_0 = C(\mathbb{N}^*)$ *and, more generally, the quotient of any USI space by a separably injective subspace.*
(b) *All ultraproducts of families of $\mathscr{L}_{\infty,\lambda}$-spaces following countably incomplete ultrafilters.*

Proof (a) Assume Z is injective and $Y \subset Z$ is separably injective. Let X be a separable Banach space and $\tau\colon X \longrightarrow Z/Y$ be an operator. It is clear that τ lifts to Z: just consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & Z \arrow[r, "\pi"] & Z/Y \arrow[r] & 0 \\ & & & X \arrow[u, "\tau"'] & \end{tikzcd}$$

Let $L\colon X \longrightarrow Z$ be a lifting of τ and $T \in \mathfrak{L}(X, Z)$ be an extension of L. Then $\pi T\colon X \longrightarrow Z/Y$ is the required extension of τ.

(b) Let us check that $[X_i]_\mathcal{U}$ is USI if X_i are $\mathscr{L}_{\infty,\lambda}$-spaces and $\mathcal{U}$ is countably incomplete. It clearly suffices to show that every separable subspace S of $[X_i]_\mathcal{U}$ is contained in a USI subspace. Let $(E^n)_{n\geq 1}$ be a chain of finite-dimensional subspaces whose union is dense in S. For each n we may take a lifting L^n of $\mathbf{1}_{E_n}$ to $\ell_\infty(I, X_i)$ with norm at most $1 + 1/n$ (use Theorem 2.14.5 or just do it by hand). Write $L^n = (L^n_i)_{i\in I}$ and put $E^n_i = L^n_i[E^n]$. Since X_i is an $\mathscr{L}_{\infty,\lambda}$-space, one can pick a finite-dimensional $F^n_i \subset X_i$ containing E^n_i and λ-isomorphic to some ℓ^k_∞. Now use the hypothesis on $\mathcal{U}$ to 'diagonalise': pick $n\colon I \longrightarrow \mathbb{N}$ such that

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$n(i) \to \infty$ along $\mathcal{U}$ and consider the family $(F_i^{n(i)})_{i \in I}$. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & c_0^{\mathcal{U}}(I, X_i) \arrow[r] & \ell_\infty(I, X_i) \arrow[r] & [X_i]_{\mathcal{U}} \arrow[r] & 0 \\ 0 \arrow[r] & c_0^{\mathcal{U}}(I, F_i^{n(i)}) \arrow[r] \arrow[u] & \ell_\infty(I, F_i^{n(i)}) \arrow[r] \arrow[u] & [F_i^{n(i)}]_{\mathcal{U}} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

in which the vertical arrows are plain inclusions and $[F_i^{n(i)}]_{\mathcal{U}}$ contains S. But since $c_0^{\mathcal{U}}(I, F_i^{n(i)})$ is an M-ideal in $\ell_\infty(I, F_i^{n(i)}) \simeq \ell_\infty(I, \ell_\infty^{k(i)})$, which is injective, and $[F_i^{n(i)}]_{\mathcal{U}}$ obviously has the BAP, we conclude that $[F_i^{n(i)}]_{\mathcal{U}}$ is USI in much the same way as before. □

Injectivity and separable injectivity are 3-space properties. Universal separable injectivity is not, at least under CH, even if it admits the following clean characterisation: a Banach space U is USI if and only if every separable subspace $S \subset U$ is contained in another subspace $V \subset U$ isomorphic to ℓ_∞; see [22, Definition 2.25 and Theorem 2.26]. Although it might throw the reader for a loop, we begin our treatment of the 3-space problem for USI spaces with the following:

Proposition 10.5.4 *No ultrapower of the Foiaş–Singer sequence splits.*

We adhere to the notation of Section 2.2. Take the sequence

$$0 \longrightarrow C(\Delta) \xrightarrow{\text{inclusion}} D \xrightarrow{J} c_0(\Delta_0) \longrightarrow 0$$

and let $\mathcal{U}$ be an ultrafilter on I and form the ultrapower sequence

$$0 \longrightarrow C(\Delta)_{\mathcal{U}} \xrightarrow{\text{inclusion}} D_{\mathcal{U}} \xrightarrow{J_{\mathcal{U}}} c_0(\Delta_0)_{\mathcal{U}} \longrightarrow 0 \tag{10.3}$$

To show that this sequence does not split, we will prove that the quotient space contains a copy of $c_0(\mathbb{N}^{\mathcal{U}})$ that cannot be lifted to $D_{\mathcal{U}}$. To see this, observe that if (q_i) is a family of points of Δ_0 indexed by I, then the class of (e_{q_i}) in the ultrapower $c_0(\Delta_0)_{\mathcal{U}}$ depends only on the class of (q_i) in the set-theoretic ultrapower $\Delta_0^{\mathcal{U}}$. Thus, given $q \in \Delta_0^{\mathcal{U}}$, let us write $e_q = [(e_{q_i})]$, where $\langle (q_i) \rangle = q$. Clearly, if $q^1, \ldots, q^n$ are different points of $\Delta_0^{\mathcal{U}}$, then

$$\left\| \sum_{k=1}^n \lambda_k e_{q^k} \right\|_{c_0(\Delta_0)_{\mathcal{U}}} = \max_{1 \le k \le n} |\lambda_k|.$$

In this way, we may consider $c_0(\Delta_0^{\mathcal{U}})$ as a closed subspace of $c_0(\Delta_0)_{\mathcal{U}}$. We now prove that the pullback sequence

$$0 \longrightarrow C(\Delta)_{\mathcal{U}} \xrightarrow{\text{inclusion}} J_{\mathcal{U}}^{-1}[c_0(\Delta_0^{\mathcal{U}})] \xrightarrow{J_{\mathcal{U}}} c_0(\Delta_0^{\mathcal{U}}) \longrightarrow 0 \tag{10.4}$$

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does not split. The proof consists of 'interpreting' what we did in Lemma 2.2.3 in the ultrapower structure, so let us give heartfelt homage to Larry Tesler:

Lemma 10.5.5 *Let (f_q) be any family in $D_{\mathcal{U}}$ such that $J_{\mathcal{U}}(f_q) = e_q$ for every $q \in \Delta_0^{\mathcal{U}}$. Then, given $\lambda^1, \dots, \lambda^n \in \mathbb{R}; q^1, \dots, q^n \in \Delta_0^{\mathcal{U}}$ and $\varepsilon > 0$, there exist $q \in \Delta_0^{\mathcal{U}} \setminus \{q^1, \dots, q^n\}$ and $\lambda = \pm 1$ such that*

$$\left\| \lambda f_q + \sum_{k=1}^n \lambda^k f_{q^k} \right\|_{D_{\mathcal{U}}} \geq 1 + \left\| \sum_{k=1}^n \lambda^k f_{q^k} \right\|_{D_{\mathcal{U}}} - \varepsilon.$$

Proof Notice that if $f \in D_{\mathcal{U}}$ and $t \in \Delta^{\mathcal{U}}$, then the 'value of f at t' is given by $f(t) = \lim_{\mathcal{U}(i)} f_i(t_i)$, where $[f_i]$ is any representative of f and (t_i) is any representative of t. Clearly, $\|f\|_{D_{\mathcal{U}}} = \sup_{q \in \Delta_0^{\mathcal{U}}} |f(q)|$. Also, note that for each $q \in \Delta_0^{\mathcal{U}}$, we can define the 'right limit' $f(q^+) = \lim_{\mathcal{U}(i)} f_i(q_i^+)$, where (f_i) is any representative of f and (q_i) is any representative of q. Now, assume that there is $q \in \Delta_0^{\mathcal{U}} \setminus \{q^1, \dots, q^n\}$ such that

$$\sum_{k=1}^n \lambda^k f_{q^k}(q) > \left\| \sum_{k=1}^n \lambda^k f_{q^k} \right\|_{D_{\mathcal{U}}} - \varepsilon.$$

Clearly, $f_{q^k}(q^+) = f_{q^k}(q)$ for $1 \leq k \leq n$. Now, since $J_{\mathcal{U}}(f_q) = e_q$, if $f_q(q) \geq -1$, $f_q(q^+) \geq 1$ and

$$\left\| f_q + \sum_{k=1}^n \lambda^k f_{q^k} \right\|_{D_{\mathcal{U}}} \geq \left(f_q + \sum_{k=1}^n \lambda^k f_{q^k} \right)(q^+) \geq 1 + \left\| \sum_{k=1}^n \lambda^k f_{q^k} \right\|_{D_{\mathcal{U}}} - \varepsilon.$$

And if $f_q(q) < -1$ then

$$\left\| -f_q + \sum_{k=1}^n \lambda^k f_{q^k} \right\|_{D_{\mathcal{U}}} \geq \left(-f_q(q) + \sum_{k=1}^n \lambda^k f_{q^k}(q) \right) > 1 + \left\| \sum_{k=1}^n \lambda^k f_{q^k} \right\|_{D_{\mathcal{U}}} - \varepsilon.$$

This is enough to conclude the argument. □

Theorem 10.5.6 *Let $\mathcal{U}$ be a countably incomplete ultrafilter.*

(a) $\mathrm{Ext}(X, C(\Delta)_{\mathcal{U}}) \neq 0$ *for* $X = C(\Delta)_{\mathcal{U}}, (c_0)_{\mathcal{U}}, c_0(\mathbb{N}^{\mathcal{U}})$.
(b) [CH] $\mathrm{Ext}(X, C(\mathbb{N}^*)) \neq 0$ *for* $X = c_0(\aleph_1), C(\mathbb{N}^*), \ell_\infty$.
(c) [CH] *Universal separable injectivity is not a* 3*-space property.*

Proof (a) The exact sequence (10.4) is not trivial, so $\mathrm{Ext}(c_0(\Delta_0^{\mathcal{U}}), C(\Delta)_{\mathcal{U}}) \neq 0$ and, of course, $c_0(\Delta_0^{\mathcal{U}}) \approx c_0(\mathbb{N}^{\mathcal{U}})$. Since (10.4) is a pullback of (10.3), one also has $\mathrm{Ext}(c_0(\Delta_0)_{\mathcal{U}}), C(\Delta)_{\mathcal{U}}) \neq 0$. As c_0 is complemented in $C(\Delta)$, $C(\Delta)_{\mathcal{U}}$ also contains a complemented copy of $(c_0)_{\mathcal{U}}$, and the result follows.

(b) Let us explain the role of CH here. If $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, then the ultrapower algebra $C(\Delta)_{\mathcal{U}}$ is isometric to a $C(K)$, where K is a compactum having the properties that characterise $\mathbb{N}^*$ in Section 1.6 (6): it is a totally disconnected F-space without isolated points of weight $\mathfrak{c}$ and such that non-empty G_δ subsets have non-empty interior; see [22, Proposition 4.12] for a proof. Thus, by Parovičenko's theorem, under CH it follows that K is homeomorphic to $\mathbb{N}^*$ and the case $X = c_0(\aleph_1)$ of (b) follows from (a). The case $X = \ell_\infty$ follows from $\operatorname{Ext}(C(\mathbb{N}^*), C(\mathbb{N}^*)) \neq 0$. Indeed, starting with a non-trivial self-extension of $C(\mathbb{N}^*)$, the middle sequence in the pullback diagram

$$\begin{tikzcd} & & 0 & 0 & \\ 0 \arrow[r] & C(\mathbb{N}^*) \arrow[r] & \diamond \arrow[u] \arrow[r, "\rho"] & C(\mathbb{N}^*) \arrow[u] \arrow[r] & 0 \\ 0 \arrow[r] & C(\mathbb{N}^*) \arrow[u, equal] \arrow[r] & \mathrm{PB} \arrow[u] \arrow[r] & \ell_\infty \arrow[u, "\pi"] \arrow[r] & 0 \\ & & c_0 \arrow[u] \arrow[r, equal] & c_0 \arrow[u] & \\ & & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd} \tag{10.5}$$

cannot split since otherwise π would admit a lifting $L\colon \ell_\infty \longrightarrow \diamond$ and one would have a pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & c_0 \arrow[r] \arrow[d, "L|_{c_0}"] & \ell_\infty \arrow[r, "\pi"] \arrow[d, "L"] & C(\mathbb{N}^*) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C(\mathbb{N}^*) \arrow[r, "\jmath"] & \diamond \arrow[r, "\rho"] & C(\mathbb{N}^*) \arrow[r] & 0 \end{tikzcd}$$

which that cannot be: the USI property of $C(\mathbb{N}^*)$ would then allow to extend $L|_{c_0}$, and the lower sequence should split.

(c) Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, consider the ultrapower sequence (10.3), and multiply it by a complement C of $(c_0)_\mathcal{U}$ in $C(\Delta)_\mathcal{U}$ so as to obtain a sequence $0 \longrightarrow C(\Delta)_\mathcal{U} \longrightarrow D_\mathcal{U} \times C \longrightarrow C(\Delta)_\mathcal{U} \longrightarrow 0$ which, under CH, takes the form

$$0 \longrightarrow C(\Delta)_\mathcal{U} \longrightarrow \diamond \xrightarrow{\ \rho\ } C(\mathbb{N}^*) \longrightarrow 0$$

Keep in mind that the quotient map ρ is not invertible on a certain copy of $c_0(\mathfrak{c})$ inside $C(\Delta)_\mathcal{U}$. Place the sequence as the upper row in Diagram (10.5). We claim that the pullback space PB fails to be USI because the inclusion $c_0 \longrightarrow \mathrm{PB}$ does not extend to ℓ_∞. If it did, the two vertical sequences in that diagram

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would be semi-equivalent, and then the Parallel lines principle 2.11.5 would make the two horizontal sequences in that same diagram semi-equivalent too, which results in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C(\Delta)_{\mathfrak{U}} \arrow[r] & \mathrm{PB} \arrow[r] & \ell_\infty \arrow[r] & 0 \\
0 \arrow[r] & C(\Delta)_{\mathfrak{U}} \arrow[r] \arrow[u, equal] & \diamond \arrow[r, "\rho"] \arrow[u] & C(\mathbb{N}^*) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

But this is impossible: as we said, ρ is not invertible on a certain copy of $c_0(\mathfrak{c})$, and thus the new pullback sequence

$$\begin{tikzcd}
0 \arrow[r] & C(\Delta)_{\mathfrak{U}} \arrow[r] & \mathrm{PB} \arrow[r] & \ell_\infty \arrow[r] & 0 \\
0 \arrow[r] & C(\Delta)_{\mathfrak{U}} \arrow[r] \arrow[u, equal] & \diamond \arrow[r, "\rho"] \arrow[u] & C(\mathbb{N}^*) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & C(\Delta)_{\mathfrak{U}} \arrow[r] \arrow[u, equal] & \mathrm{PPB} \arrow[r] \arrow[u] & c_0(\mathfrak{c}) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

does not split. But since every operator $c_0(\mathfrak{c}) \longrightarrow \ell_\infty$ has separable range and $C(\Delta)_{\mathfrak{U}}$ is separably injective, the operator lifts to PB, and the lower pullback sequence splits. $\square$

Proposition 10.5.4 and Theorem 10.5.6 are taken from [23]. A previous analysis of the 3-space problem for universal separable injectivity can be found in [22, Section 6.2], where the interested reader will find a more systematic study of the injectivity properties of $\mathscr{C}$-spaces and ultraproducts. The assertion $\operatorname{Ext}(C(\mathbb{N}^*), C(\mathbb{N}^*)) \neq 0$ provides an improvement on the assertion '$C(\mathbb{N}^*)$ contains an uncomplemented copy of itself' that can be found in [121, Proposition 5.3]. The paper [64] contains some results on 'abstract' sequences of Foiaş–Singer type, explains why the quotient space tends to be $c_0(\Gamma)$ and shows, among other things, that $\operatorname{Ext}(c_0(\mathfrak{c}), C(\mathbb{N}^*)) \neq 0$ in the Cohen standard model, a model for set theory in which the first thing we do is renege on CH.

Vogt's Duality Problem in Focus

Recall Vogt's problem from Proposition 2.12.3: *must an exact sequence* $0 \longrightarrow A^* \longrightarrow Z \longrightarrow B^* \longrightarrow 0$ *be the dual sequence of another exact sequence? Must* Z *be a dual space?* A counterexample was already presented in Proposition 2.12.3.

Now we will present an exposition of duality issues in Banach spaces that is lush with detail and buoyed by simpatico and that produces, in the end, an

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optimal concrete counterexample to Vogt's problem. The shimmering details of a quasilinear version of what follows can be found in [65]. Let us begin by elucidating exactly what a dual space and dual exact sequence are.

Lemma 10.5.7 *Let P and D be Banach spaces. The following are equivalent:*

(i) *D is isomorphic to the dual of P.*

(ii) *There is an embedding $\jmath\colon P \longrightarrow D^*$ such that $\jmath^*\delta_D\colon D \longrightarrow D^{**} \longrightarrow P^*$ is an isomorphism.*

Proof Only the implication (i) $\implies$ (ii) needs a proof. Assume that we have an isomorphism $\phi\colon D \longrightarrow P^*$. Then $\phi^*\colon P^{**} \longrightarrow D^*$ is an isomorphism, $\phi^*\delta_P\colon P \longrightarrow D^*$ is an embedding and D is the dual of $\phi^*\delta_P[P]$ through the restriction of the duality between D^* and D: $\langle\phi(d), p\rangle = \langle\phi^*(\delta_P(p)), d\rangle$. □

If P and D are as in (i), we say that P is an isomorphic predual of D; the advantage of (ii) is that one can always find a copy of each isomorphic predual of D in D^* acting through the restriction of the duality between D^* and D. Simply out of curiosity, Dixmier a long time ago characterised the corresponding subspaces of D^* as those closed subspaces that are total over D and minimal with respect to the property of being total over D. From now on, when referring to a predual of D, we tacitly assume it lies in D^*. A Banach space can have many preduals, some even isometric and others not even isomorphic: if K is a metrisable scattered compactum, then $C(K)^* = \ell_1(K) \approx \ell_1$. Furthermore, both ℓ_1 and L_1 are preduals of ℓ_∞. Each exact sequence of Banach spaces $0 \longrightarrow A \xrightarrow{\imath} B \xrightarrow{\pi} C \longrightarrow 0$ has an *adjoint* sequence, namely

$$0 \longrightarrow C^* \xrightarrow{\pi^*} B^* \xrightarrow{\imath^*} A^* \longrightarrow 0$$

The fact that Banach spaces can have many different preduals compels us to make the following definition:

Definition 10.5.8 An exact sequence is said to be a dual sequence if it is isomorphic to an adjoint sequence.

In other words, the sequence (z) in the next diagram is a dual sequence if there exists some exact sequence $0 \longrightarrow A \xrightarrow{\imath} B \xrightarrow{\pi} C \longrightarrow 0$ and a commutative diagram

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$$\begin{tikzcd} 0 \arrow[r] & C^* \arrow[r, "\pi^*"] & B^* \arrow[r, "\iota^*"] & A^* \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r, "\jmath"] \arrow[u, "\gamma"] & Z \arrow[r, "\rho"] \arrow[u, "\beta"] & X \arrow[r] \arrow[u, "\alpha"] & 0 \end{tikzcd} \tag{z}$$

in which α, β, γ are isomorphisms. This means, in particular, that A, B, C are preduals of X, Z, Y, respectively, with embeddings given by $\alpha^*\delta_A, \beta^*\delta_B, \gamma^*\delta_C$. Every dual sequence is itself the adjoint of a sequence formed with suitably chosen subspaces of the duals of the spaces occurring in it. Indeed, taking adjoints in the previous diagram one gets the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\delta_A"] & B \arrow[r, "\pi"] \arrow[d, "\delta_B"] & C \arrow[r] \arrow[d, "\delta_C"] & 0 \\ 0 \arrow[r] & A^{**} \arrow[r, "\iota^{**}"] \arrow[d, "\alpha^*"] & B^{**} \arrow[r, "\pi^{**}"] \arrow[d, "\beta^*"] & C^{**} \arrow[r] \arrow[d, "\gamma^*"] & 0 \\ 0 \arrow[r] & X^* \arrow[r, "\rho^*"] & Z^* \arrow[r, "\jmath^*"] & Y^* \arrow[r] & 0 \end{tikzcd}$$

Set $X_* = \alpha^*\delta_A[A]$, $Z_* = \beta^*\delta_B[B]$ and $Y_* = \gamma^*\delta_C[C]$. Then $(X_*)^* = X, (Z_*)^* = Z$ and $(Y_*)^* = Y$ under the obvious dualities. Moreover, $\rho^*[X_*] \subset Z_*$ and $\jmath^*[Z_*] = Y_*$, so (z) is the adjoint of the exact sequence

$$0 \longrightarrow X_* \xrightarrow{\rho^*|_{X_*}} Z_* \xrightarrow{\jmath^*|_{Z_*}} Y_* \longrightarrow 0$$

We now characterise dual sequences:

Proposition 10.5.9 *An exact sequence* $0 \longrightarrow Y \xrightarrow{\jmath} Z \xrightarrow{\rho} X \longrightarrow 0$ *is a dual sequence if and only if there are preduals* $X_* \subset X^*$ *of* X *and* $Y_* \subset Y^*$ *of* Y*, an exact sequence* $0 \longrightarrow X_* \xrightarrow{\iota} V \xrightarrow{\pi} Y_* \longrightarrow 0$ *and a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & X_* \arrow[r, "\iota"] \arrow[d, "\text{inclusion}"'] & V \arrow[r, "\pi"] \arrow[d, "v"] & Y_* \arrow[r] \arrow[d, "\text{inclusion}"] & 0 \\ 0 \arrow[r] & X^* \arrow[r, "\rho^*"] & Z^* \arrow[r, "\jmath^*"] & Y^* \arrow[r] & 0 \end{tikzcd} \tag{10.6}$$

Proof The ‘only if’ part is obvious from the definition. As for the ‘if’, taking adjoints in the hypothesised diagram and splicing with it the starting sequence, we obtain

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$$\begin{tikzcd}
0 \arrow[r] & Y_*^* \arrow[r, "\pi^*"] & V^* \arrow[r, "\imath^*"] & X_*^* \arrow[r] & 0 \\
0 \arrow[r] & Y^{**} \arrow[r, "\rho^*"] \arrow[u, "\text{restriction}"] & Z^{**} \arrow[r, "\jmath^*"] \arrow[u, "v^*"'] & X^{**} \arrow[r] \arrow[u, "\text{restriction}"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "\jmath"] \arrow[u, "\delta_Y"] & Z \arrow[r, "\rho"] \arrow[u, "\delta_Z"] & X \arrow[r] \arrow[u, "\delta_X"] & 0
\end{tikzcd}$$

By the 3-lemma, $v^*\delta_Z$ is an isomorphism: this shows that the starting sequence is isomorphic to the adjoint of $0 \longrightarrow X_* \xrightarrow{\imath} V \xrightarrow{\pi} Y_* \longrightarrow 0$ and also that $v[V]$ is a predual of Z. □

The following two propositions yield especially remarkable examples of dual sequences:

Proposition 10.5.10 *If* $\mathrm{Ext}(X, Y^{**}/Y) = 0$ *then every exact sequence* $0 \longrightarrow X^* \longrightarrow Z \longrightarrow Y^* \longrightarrow 0$ *is the adjoint of a sequence* $0 \longrightarrow Y \longrightarrow Z_* \longrightarrow X \longrightarrow 0$ *for some* $Z_* \subset Z^*$*, which is necessarily a predual of* Z*.*

Proof Taking adjoints and forming the pullback we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y^{**} \arrow[r] & Z^* \arrow[r] & X^{**} \arrow[r] & 0 \\
0 \arrow[r] & Y^{**} \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "\delta_X"] & 0
\end{tikzcd}$$

The hypothesis implies that the lower sequence becomes trivial after forming the pushout with the quotient map $Y^{**} \longrightarrow Y^{**}/Y$. Therefore, there is a pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & Y^{**} \arrow[r] & \mathrm{PB} = \mathrm{PO} \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, "\delta_Y"] & Z_* \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Now assemble the two diagrams and apply the preceding proposition. □

An unrefined version of this result appears in [150, proposition 3]. The following statement is in some sense dual to it.

Proposition 10.5.11 $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ *is a dual sequence if and only if there is a predual* Z_* *of* Z *for which* Y *is weak*-closed in* Z*.*

Proof If Y is weak*-closed then $Y = (Z_*/Y_\perp)^*$, where $Y_\perp = \{f \in Z_* : \langle f, y\rangle = 0\, \forall y \in Y\}$ by the bipolar theorem, thus $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ is

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the adjoint of $0 \longrightarrow Y_\perp \longrightarrow Z_* \longrightarrow Z/Y_\perp \longrightarrow 0$. The other implication is obvious. □

We are ready for the promised counterexample.

10.5.12 A counterexample to Vogt's duality problem *There is a separable dual W^* and an exact sequence $0 \longrightarrow \ell_2 \longrightarrow P \longrightarrow W^* \longrightarrow 0$ that is not a dual sequence. In particular, the space P is not isomorphic to a dual space.*

Proof The basic idea is simple: we start with a Banach space W complemented in its bidual and write $W^{**} = W \oplus A$, where A is a complement of W in W^{**}. Now we take a non-trivial extension $0 \longrightarrow A \longrightarrow E \longrightarrow R \longrightarrow 0$ where R is reflexive. Multiplying on the left by W, we obtain

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\imath_A"] & E \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 & (\mathtt{e}) \\
0 \arrow[r] & W^{**} = W \oplus A \arrow[r] & W \oplus E \arrow[r] & R \arrow[r] & 0 & (\imath_A\mathtt{e})
\end{tikzcd}$$

By Proposition 10.5.10, $(\imath_A\mathtt{e})$ is the adjoint of a sequence

$$0 \longrightarrow R^* \longrightarrow P \longrightarrow W^* \longrightarrow 0 \qquad (\mathtt{p})$$

This sequence cannot be the adjoint of a sequence $0 \longrightarrow W \longrightarrow F \longrightarrow R \longrightarrow 0$ because, if it were, one could form the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] \arrow[d, "\imath_W"'] & F \arrow[r] \arrow[d, "v"] & R \arrow[r] \arrow[d, equal] & 0 & (\mathtt{f}) \\
0 \arrow[r] & W^{**} = W \oplus A \arrow[r] \arrow[d, "\pi_A"'] & W \oplus E \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 & (\imath_A\mathtt{e}) \\
0 \arrow[r] & A \arrow[r] & \mathrm{PO} \arrow[r] & R \arrow[r] & 0 &
\end{tikzcd}$$

But this cannot be unless $(\mathtt{e})$ is trivial: on one hand, the class of the lower row is $[\pi_A\imath_W\mathtt{f}] = 0$, since $\pi_A\imath_W = 0$ while, on the other hand, it is $[\pi_A\imath_A\mathtt{e}] = [\mathtt{e}]$, since $\pi_A\imath_A = \mathbf{1}_A$. This shows that sequence $(\mathtt{p})$ cannot be the adjoint of a sequence in which the subspace is W. To complete the proof, we must prove the same for all possible preduals of W^*, and the idea is to choose W such that any other predual of W^* is in 'essentially the same position' as W in W^{**}. Let W be a separable Banach space such that $W^{**}/W \simeq \ell_1$. Such a W exists by a result of Lindenstrauss [5, Section 15.1]: every separable Banach space U can be represented as $U \simeq W^{**}/W$ for some separable space W. As ℓ_1 is projective, we can write $W^{**} = W \oplus A$, where $A \simeq \ell_1$ is a fixed subspace of W^{**}. Brown and Ito proved in [54] that if $V \subset W^{**}$ is another predual of W^*

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then there is a decomposition $W^{**} = V \oplus B$, where $B \cap A$ has finite codimension in A. Now we can start with a non-trivial element of $\operatorname{Ext}(\ell_2, \ell_1)$ to be used as the sequence (e) and whose existence is guaranteed by Proposition 5.2.20, and conclude the proof as follows: if (p) is a dual sequence there is an exact sequence $0 \longrightarrow V \longrightarrow G \longrightarrow R^* \longrightarrow 0$ where $V \subset W^{**}$ is a predual of W^* and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "\imath_V"'] & G \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 & (\mathsf{g}) \\
0 \arrow[r] & W^{**} = V \oplus B \arrow[r] \arrow[d, "\pi_B"'] & W \oplus E \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 & (\imath_A \mathsf{e}) \\
0 \arrow[r] & B \arrow[r] & \mathrm{PO} \arrow[r] & R \arrow[r] & 0 & (\pi_B \imath_A \mathsf{e})
\end{tikzcd}$$

It follows that $[\pi_B \imath_A \mathsf{e}] = [\pi_B \imath_B \mathsf{g}] = 0$. Since $\pi_A \imath_A - \pi_B \imath_A$ has finite rank, $[\mathsf{e}] = 0$, which completes the proof of the first part. To conclude, the space P is not isomorphic to any dual space: otherwise, R^* would be weak*-closed in P and Proposition 10.5.11 would imply that (p) is a dual sequence. □

We can modify the counterexample $0 \longrightarrow \ell_2 \longrightarrow P \longrightarrow W^* \longrightarrow 0$ to obtain one whose quotient space is a bidual: take a separable Banach space V such that $V^{**}/V = c_0$ and set $W = V^*$ in the construction above. The final subtleties in the previous proof cannot be avoided. If J is James quasireflexive space such that $\mathsf{J}^{**}/\mathsf{J}$ has dimension 1 then $\ell_2(\mathsf{J})^{**} = \ell_2(\mathsf{J}) \oplus H$, where H is a separable Hilbert space. Take the Kalton–Peck sequence $0 \longrightarrow \ell_2 \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0$ and multiply it by $\ell_2(\mathsf{J})$ to get the non-trivial sequence $0 \longrightarrow \ell_2(\mathsf{J}) \times \ell_2 \longrightarrow \ell_2(\mathsf{J}) \times Z_2 \longrightarrow \ell_2 \longrightarrow 0$; this sequence can be identified with one of the form $0 \longrightarrow \ell_2(\mathsf{J})^{**} \longrightarrow \diamond \longrightarrow \ell_2(\mathsf{J})^* \longrightarrow 0$, which must be the transpose of some sequence $0 \longrightarrow \ell_2^* \longrightarrow \diamond_* \longrightarrow \ell_2(\mathsf{J})^* \longrightarrow 0$. This sequence cannot be the adjoint of one of the form $0 \longrightarrow \ell_2(\mathsf{J}) \longrightarrow \diamond_{**} \longrightarrow \ell_2 \longrightarrow 0$ by the same reasoning as in the first part of the proof of the counterexample. However, the sequence is the adjoint of a sequence in which the subspace is another predual of $\ell_2(\mathsf{J})^*$ inside $\ell_2(\mathsf{J})^{**}$ because any hyperplane of J^{**} is a predual of J^*.

10.6 Extension of $\mathscr{L}_\infty$-Valued Operators

Do the $\mathscr{C}$-valued extension results of Chapter 8 remain valid for $\mathscr{L}_\infty$-valued operators? Some obviously do, such as the Johnson–Zippin theorem 8.6.2. This question for the Lindenstrauss–Pełczyński theorem is posed by Zippin as Problem 6.15 in [466], and the answer is a strong no:

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10.6.1 Example *Let H be a subspace of c_0 such that $c_0/H \not\simeq c_0$. The Bourgain–Pisier embedding $\imath\colon H \longrightarrow \mathscr{L}_\infty^{\mathsf{BP}}(H)$ cannot be extended to c_0.*

Proof Observe the two exact sequences

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "\jmath"] \arrow[d, equal] & c_0 \arrow[r] & c_0/H \arrow[r] & 0 \\ 0 \arrow[r] & H \arrow[r, "\imath"] & \mathscr{L}_\infty^{\mathsf{BP}}(H) \arrow[r] & S \arrow[r] & 0 \end{tikzcd}$$

Since $\mathscr{L}_\infty^{\mathsf{BP}}(H)$ is separable, Sobczyk's theorem provides an extension of $\jmath$ through $\imath$. If $\imath$ would also extend through $\jmath$, the two sequences would be semi - equivalent, and then the diagonal principles yield $\mathscr{L}_\infty^{\mathsf{BP}}(H) \times c_0/H \simeq c_0 \times S$. In particular, c_0/H is a complemented subspace of $c_0 \times S$. Since S and c_0 are totally incomparable by the Schur property of S, we can apply the

10.6.2 Edelstein–Wojtaszczyk decomposition *Let X and Y be Banach spaces such that every operator from Y into X is strictly singular. Let P be a projection of $X \times Y$ onto an infinite-dimensional subspace E. Then there exists an automorphism τ_0 of $X \oplus Y$ and complemented subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that $\tau_0[E] = X_0 \times Y_0$*

(see [334, 2.c.13]) to obtain that c_0/H is isomorphic to some $A \times B$ with A complemented in c_0 and B complemented in S. Since c_0/H is a subspace of c_0, the space B must be finite-dimensional, hence $c_0/H \simeq c_0$, against the hypothesis. $\square$

The Kalton extendability Theorem 8.5.4 also is not valid for ℓ_1:

10.6.3 Example *Let $\jmath\colon \ell_1 \longrightarrow C(\Delta)$ be any embedding. The Bourgain–Pisier embedding $\imath\colon \ell_1 \longrightarrow \mathscr{L}_\infty^{\mathsf{BP}}(\ell_1)$ cannot be extended to $C(\Delta)$.*

Proof An extension of $\imath$ through $\jmath$ yields a pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & \ell_1 \arrow[r, "\jmath"] \arrow[d, equal] & C(\Delta) \arrow[r] \arrow[d, "J"] & B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ell_1 \arrow[r, "\imath"] & \mathscr{L}_\infty^{\mathsf{BP}}(\ell_1) \arrow[r] & S \arrow[r] & 0 \end{tikzcd}$$

Since $\mathscr{L}_\infty^{\mathsf{BP}}(\ell_1)$ is a Schur space, it does not contain c_0, and therefore J must be weakly compact. But, then, its restriction $J|_{\ell_1} = \imath$ should be compact. $\square$

On the other hand, a live agenda item in Lindenstrauss' memoir [323] is:

Proposition 10.6.4 *Compact $\mathscr{L}_{\infty,\lambda}$-valued operators admit compact λ-extensions anywhere.*

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This can be proved by approximation since finite-rank operators have finite-rank λ-extensions. However, even weakly compact $\mathscr{C}$-valued operators do not necessarily admit extensions (weakly compact or otherwise), while weakly compact operators defined *on* $\mathscr{L}_\infty$-spaces admit weakly compact extensions. Thus, there are juicy classes of operators for which $\mathscr{L}_\infty$-valued operator extensions exist, even though Lindenstrauss–Pelczynski's and Kalton's theorems do not hold. Ok, worse things happen at sea.

Lindenstrauss–Pełczyński Spaces

Definition 10.6.5 A Banach space E is said to be a Lindenstrauss–Pełczyński space (LP) if all operators from subspaces of c_0 into E can be extended to c_0. When every operator $\tau\colon H \longrightarrow E$ admits a λ-extension, we shall say that E is an LP_λ space.

Each LP space is clearly an LP_λ space for some λ. As c_0 contains almost isometric copies of every finite-dimensional space, we see each LP_λ space is λ^+-locally injective and hence an $\mathscr{L}_\infty$-space. It therefore makes sense to ask; which $\mathscr{L}_\infty$-spaces are LP spaces?

Proposition 10.6.6 *The Banach spaces in* (a)–(e) *are* LP *spaces:*

(a) *$\mathscr{L}_\infty$-spaces not containing c_0,*
(b) *complemented subspaces of Lindenstrauss spaces,*
(c) *separably injective space,*
(d) *every quotient of an* LP *space by a separably injective subspace,*
(e) *the c_0-sum (in particular, the product) of* LP_λ *spaces.*
(f) *To be an* LP *space is not a* 3*-space property.*

Proof In what follows, H is always a subspace of c_0. To prove (a), observe that when a Banach space X contains no copy of c_0, every operator $H \longrightarrow X$ must be compact, and thus Proposition 10.6.4 applies.

Assertion (b) follows from the theorem in Section 8.8.2, and (c) is obvious.

To prove (d), let E be a separably injective space, let $0 \longrightarrow E \longrightarrow \mathsf{LP} \xrightarrow{\rho} X \longrightarrow 0$ be an exact sequence and let $\tau\colon H \longrightarrow X$ be an operator. Since $\operatorname{Ext}(H, E) = 0$, τ can be lifted through ρ to an operator $H \longrightarrow \mathsf{LP}$ which, in turn, can be extended to an operator $T\colon c_0 \longrightarrow \mathsf{LP}$. The operator $\rho T\colon c_0 \longrightarrow X$ is the desired extension of τ.

(e) Keep in a handful of quietness the balance we mentioned after 10.1.1, the observation that one can easily consider all E_n equal; say, E, and the fact that since LP_λ spaces are $\mathscr{L}_{\infty,\mu}$-spaces for some μ, $\ell_\infty(E)$ is an $\mathscr{L}_{\infty,\mu}$-space, as well as their quotient, who must, therefore, have the BAP. Pick X a quotient of c_0 (with or without the BAP – as it could well be the case: see the space Z_∞ in [448, p. 276]). Then, apply Corollary 2.14.7 setting $Z = c_0$, $Z/Y = X$, $J = c_0(E)$ and $A = \ell_\infty(E)$.

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(f) Combine a singular sequence $0 \longrightarrow C(K) \longrightarrow \diamond \longrightarrow c_0 \longrightarrow 0$ with Bourgain's $0 \longrightarrow \mathcal{B} \longrightarrow c_0 \longrightarrow c_0 \longrightarrow 0$ in a pullback diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & C(K) \arrow[r] & \diamond \arrow[u] \arrow[r, "\rho"] & c_0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & C(K) \arrow[u, equal] \arrow[r] & \mathrm{PB} \arrow[u] \arrow[r] & c_0 \arrow[u, "\pi"'] \arrow[r] & 0 \\
 & & \mathcal{B} \arrow[u, "\jmath"] \arrow[r, equal] & \mathcal{B} \arrow[u, "\imath"'] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and show that PB cannot be an LP space. If $\jmath$ could be extended to c_0 through $\imath$, there would be an operator $\bar{\jmath}\colon c_0 \longrightarrow \diamond$ yielding a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{B} \arrow[r] & c_0 \arrow[r, "\pi"] & c_0 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{B} \arrow[u, equal] \arrow[r] & \mathrm{PB} \arrow[u] \arrow[r] & \diamond \arrow[u, "\rho"'] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{B} \arrow[u, equal] \arrow[r] & c_0 \arrow[u] \arrow[r, "\pi"] & c_0 \arrow[u, "\bar{\jmath}"'] \arrow[r] & 0
\end{tikzcd}$$

Since ρ is strictly singular, $\rho\bar{\jmath}$ is also strictly singular and therefore compact. It follows from Lemma 4.3.3 that the diagram above is impossible. □

It is clear how to use (b) to obtain concrete examples of LP spaces, while (c) and (d) can even be used to obtain non-separable LP spaces. On the other hand, at least three types of $\mathscr{L}_\infty$-spaces that do not contain c_0 appear in the literature:

- the Bourgain–Pisier space $\mathscr{L}_\infty^{\mathsf{BP}}(X)$ when X does not contain c_0, since not containing c_0 is a 3-space property [102, Theorem 3.2.e];
- the Bourgain–Delbaen isomorphic preduals of ℓ_1 without copies of c_0 [51];
- H.I. $\mathscr{L}_\infty$-spaces [17].

LP spaces enjoy additional properties: cheating a bit, they are those $\mathscr{L}_\infty$-spaces having all subspaces of c_0 placed in a unique position. That is not completely correct since the LP space $c_0 \times \ell_\infty$ contains complemented and uncomplemented copies of c_0 and thus it cannot be c_0-automorphic. Put precisely:

Proposition 10.6.7

(a) *A Banach space that contains c_0 and is automorphic for all subspaces of c_0 is an LP space.*

(b) *A separable $\mathscr{L}_\infty$-space is an LP space if and only if it is automorphic for all subspaces H of c_0.*

Proof (a) Let X be automorphic for all subspaces of c_0 and assume that there is an embedding $\jmath\colon c_0 \longrightarrow X$. Assume there is a subspace $H \subset c_0$ and an operator $T\colon H \longrightarrow X$ that cannot be extended to c_0. It turns out that, for small $\varepsilon > 0$, the operator $\jmath|_H + \varepsilon T\colon H \longrightarrow X$ is an embedding that cannot be extended to an operator $R \in \mathfrak{L}(X)$ through $\jmath|_H$; if, otherwise, $R\jmath|_H = \jmath|_H + \varepsilon T$ then $\varepsilon^{-1}(R\jmath - \jmath)$ would be an extension of T.

(b) The 'if' part is contained in (a) for spaces that contain c_0 and in Proposition 10.6.6(a) for those that do not. Let us show the other implication. Let X be a separable LP space. If X does not contain c_0 then the result is (vacuously) true. So let $\imath\colon H \longrightarrow X$ be an embedding where H is a subspace of c_0 and let $\jmath\colon H \longrightarrow c_0$ be the inclusion map. We can assume that $\imath$ has infinite-dimensional cokernel and that H is uncomplemented in c_0. Otherwise, the result follows directly from Sobczyk's theorem. The extension $J\colon c_0 \longrightarrow X$ of $\imath$ that exists because X is an LP space yields the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "\jmath"] \arrow[d, equal] & c_0 \arrow[r, "\rho"] \arrow[d, "J"] & c_0/H \arrow[r] \arrow[d, "J'"] & 0 \\ 0 \arrow[r] & H \arrow[r, "\imath"] & X \arrow[r, "\pi"] & X/\imath[H] \arrow[r] & 0 \end{tikzcd} \tag{10.7}$$

which, in combination with Sobczyk's theorem, makes those two sequence semi-equivalent. The diagonal principles yield that the sequences

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "{(\imath,0)}"] \arrow[d, equal] & X \times c_0 \arrow[r] & X/\imath[H] \times c_0 \arrow[r] & 0 \\ 0 \arrow[r] & H \arrow[r, "{(\jmath,0)}"] & c_0 \times X \arrow[r] & c_0/H \times X \arrow[r] & 0 \end{tikzcd} \tag{10.8}$$

are isomorphic. We now show that the operator $\pi J = J'\rho$ is not weakly compact. Otherwise, it would be compact, and thus J' would also be compact. Since X is separable, $\imath$ can be extended to c_0, which yields a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "\imath"] \arrow[d, equal] & X \arrow[r, "\pi"] \arrow[d, "I"] & X/\imath[H] \arrow[r] \arrow[d, "I'"] & 0 \\ 0 \arrow[r] & H \arrow[r, "\jmath"] & c_0 \arrow[r, "\rho"] & c_0/H \arrow[r] & 0 \end{tikzcd} \tag{10.9}$$

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Putting Diagrams (10.7) and (10.9) together, we get

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "J"] \arrow[d, equal] & c_0 \arrow[r] \arrow[d, "IJ"] & c_0/H \arrow[r] \arrow[d, "I'J'"] & 0 \\
0 \arrow[r] & H \arrow[r, "J"] & c_0 \arrow[r, "\rho"] & c_0/H \arrow[r] & 0
\end{tikzcd}$$

in which $I'J'$ is compact. Lemma 4.3.3 shows that this is impossible. Since $\mathscr{C}$-spaces have property (V) and πJ is not weakly compact, it must be an isomorphism on a subspace isomorphic to c_0, as well as π. This last copy of c_0 on which π is an isomorphism will necessarily be complemented in both $X/\imath[H]$ and X, which means that the sequences

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "\imath"] \arrow[d, equal] & X \arrow[r] & X/\imath[H] \arrow[r] & 0 \\
0 \arrow[r] & H \arrow[r, "(\imath,0)"] & X \times c_0 \arrow[r] & X/\imath[H] \times c_0 \arrow[r] & 0
\end{tikzcd} \tag{10.10}$$

are isomorphic. Matching (10.8) with (10.10), we get that the sequences

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "\imath"] \arrow[d, equal] & X \arrow[r] & X/\imath[H] \arrow[r] & 0 \\
0 \arrow[r] & H \arrow[r, "(\jmath,0)"] & c_0 \times H \arrow[r] & c_0/H \times X \arrow[r] & 0
\end{tikzcd}$$

must be isomorphic too. Since the same is true starting with a different embedding $H \longrightarrow X$ with infinite-dimensional cokernel, the proof of (b) is done. □

$\mathscr{L}_\infty$-Envelopes

The natural embedding $X \longrightarrow C(B_X^*)$ enjoys the universal property that every $\mathscr{C}$-valued operator on X admits a 1-extension to $C(B_X^*)$. In other words, it is a $\mathscr{C}$-envelope in the following sense:

Definition 10.6.8 Let $\mathscr{A}$ be a class of Banach spaces. An $\mathscr{A}$-envelope of X is a space A in $\mathscr{A}$, together with an isometry $\delta\colon X \longrightarrow A$ such that, for every B in $\mathscr{A}$, every operator $\tau\colon X \longrightarrow B$ admits a 1-extension through δ.

Do similar envelopes exist for other classes of $\mathscr{L}_\infty$-spaces? For instance, any embedding of a Banach space X into a λ-injective space yields a λ-injective envelope of X, and Proposition 7.3.2, which yields the construction of an (α,β)-universal disposition envelope, can be easily modified to yield an (α,β)-injective envelope. We now focus on the construction of Lindenstrauss

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envelopes of separable Banach spaces. Since G contains 1-complemented copies of all separable Lindenstrauss spaces, every separable Banach space X has an isometry $\delta\colon X \longrightarrow \mathsf{G}$ that acts as a Lindenstrauss envelope. This can be arranged from the 'Fraïssé' construction of G simply by taking a chain of finite-dimensional subspaces $(X_n)_{n\geq 1}$ whose union is dense in X and considering a Fraïssé class of isometries containing the inclusions $X_n \longrightarrow X_m$ for $n \geq m$. Let us take a different approach based on the Bourgain–Pisier construction and which will presumably lead to a smaller envelope. Note that since $\mathscr{L}_\infty^{\mathsf{BP}}(X)/X$ is Schur, $\mathscr{L}_\infty^{\mathsf{BP}}(X)$ cannot be a Lindenstrauss space. On the other hand, as was shown in [129] and will be proved here soon, Lindenstrauss-valued operators on X admit 1^+-extensions to $\mathscr{L}_\infty^{\mathsf{BP}}(X)$, so we are close to the goal.

Proposition 10.6.9 *Every separable Banach space admits a Lindenstrauss envelope.*

Proof A drawing might help the reader understand the extension schema we will follow:

$$
\begin{tikzcd}
S_1 \arrow[r, "s_1"] \arrow[d, "u_1"] & \ell_\infty^{a(1)} \arrow[d] \arrow[ddddddr, dotted] & & & & & \\
X_1 \arrow[r] \arrow[dr] \arrow[dddddrr] & \mathrm{PO}_1 \arrow[dr] & S_2 \arrow[r, "s_2"] \arrow[d, "u_2"] & \ell_\infty^{a(2)} \arrow[d] \arrow[dddddl, dotted] & & & \\
& X_2 \arrow[r] \arrow[dr] \arrow[ddddr] & \mathrm{PO}_1' \arrow[r] & \mathrm{PO}_2 \arrow[dr] & S_3 \arrow[r, "s_3"] \arrow[d, "u_3"] & \ell_\infty^{a(3)} \arrow[d] \arrow[ddddlll, dotted] & \\
& & X_3 \arrow[rr] \arrow[dr] \arrow[ddd] & & \mathrm{PO}_2' \arrow[r] & \mathrm{PO}_3 & \\
& & & \cdot \arrow[dr, dotted] & & & \\
& & & & X \arrow[rr] \arrow[dll, "\phi"] & & X_\omega \arrow[dllll, "\Phi"] \\
& & \mathcal{L} & & & &
\end{tikzcd}
$$

Keep the construction of $\mathscr{L}_\infty^{\mathsf{BP}}(X)$ in mind, especially Diagram (2.43). A subtle change is needed to make the resulting space Lindenstrauss: instead of fixing the parameters λ, η, choose sequences $(\lambda_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 1}$ with

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$\lambda_n^{-1} < \eta_n < 1$ and $\lim \lambda_n = 1$, and use λ_n, η_n at step n. The relevant property of the sequence of 'almost isometries' $(u_n)_{n\geq 1}$ is the estimate $\lambda_n^{-1}\|s\| \leq \|u_n(s)\| \leq \eta_n\|s\|$, for $s \in S_n$. And thus, PO_n is λ_n-isomorphic to $\ell_\infty^{a(n)}$, and this makes X_ω a Lindenstrauss space. The isometry $X \longrightarrow X_\omega$ remains unchanged. We prove the extension property: let $\mathcal{L}$ be a Lindenstrauss space and $\phi\colon X \longrightarrow \mathcal{L}$ be a norm 1 operator. The composition ϕu_1 is a finite-rank operator with norm at most $\eta_1 < 1$, and since $\mathcal{L}$ is locally 1^+-injective, there is a contractive extension $\ell_\infty^{a(1)} \longrightarrow \mathcal{L}$. Now use the pushout properties to get a contractive extension $\Phi_1 : \mathrm{PO}_1 \longrightarrow \mathcal{L}$; applying the universal property of PO_1' to this Φ_1 and $\phi|_{X_2} : X_2 \longrightarrow \mathcal{L}$ yields a new contractive extension $\phi_1 : \mathrm{PO}_1' \longrightarrow \mathcal{L}$, and the process can be iterated, now using $\phi_1 u_2$. The desired extension of ϕ is then defined locally by $\Phi(x) = \Phi_n(x)$ if $x \in \mathrm{PO}_n$. □

The original Bourgain–Pisier space $\mathscr{L}_\infty^{\mathsf{BP}}(X)$ corresponds to the choice $\lambda_n = \lambda$ and $\eta_n = \eta$, thus

Corollary 10.6.10 *Every Lindenstrauss-valued operator defined on a separable Banach space X admits a 1^+-extension to $\mathscr{L}_\infty^{\mathsf{BP}}(X)$.*

To deal with $\mathscr{L}_{\infty,\lambda}$-valued operators, we have to make a few tricky adjustments to simultaneously get a bigger $\mathscr{L}_{\infty,\lambda}$-superspace and the equal norm extension of $\mathscr{L}_{\infty,\lambda}$-valued operators on X.

Proposition 10.6.11 *For fixed $1 < \lambda < \infty$, every separable Banach space admits an $\mathscr{L}_{\infty,\lambda}$-envelope.*

Proof Set $\eta = \lambda^{-1}$ and simplify everything by choosing isometric embeddings $S_n \longrightarrow C(\Delta)$. All the rest goes as before. The resulting space X_ω is a separable $\mathscr{L}_{\infty,\lambda}$-space since it is the inductive limit of the spaces PO_n, all of them λ-isomorphic to $C(\Delta)$. □

More than 9/10 of the authors of this book are convinced that any reader arriving at this point will be able to construct an LP_λ-envelope and other similar envelopes without difficulties. A different matter is how to construct $\mathscr{L}_\infty$-envelopes. But, seriously: *Do $\mathscr{L}_\infty$-envelopes exist*?

10.7 Kadec Spaces

The p-Kadec space treated in Chapter 6 is the unique separable p-Banach space of AUCD with a skeleton. Proposition 6.3.13 provides different separable p-Banach spaces with property [Ↄ]. Moreover, Note 6.5.1 explained that any attempt to set $\varepsilon = 0$ and obtain a space of UCD (defined right below) directly expels it to the density of the continuum outside. Let us explore that outside.

Definition 10.7.1 A p-Banach space U is of universal complemented disposition (UCD) if, for all 1-pairs $u\colon\ E \rightleftarrows U$ and $v\colon\ E \rightleftarrows F$, where F is a finite-dimensional p-normed space, there exists a 1-pair $w\colon\ F \rightleftarrows U$ such that $u = w \circ v$.

The funny thing is that unearthing spaces of UCD is simpler than it was in the *almost* case. We just need a less rambunctious use of the Device:

Proposition 10.7.2 *Every p-Banach space can be isometrically embedded as a* 1*-complemented subspace of a p-Banach space of UCD.*

Proof Let's prepare a `recipe` with the following ingredients:

- the ordinal ω_1 and your favourite p-Banach space X;
- a set $\mathscr{F}^{(p)}$ containing exactly one isometric copy of each finite-dimensional p-Banach space;
- we work in the category of p-Banach spaces and 1-pairs and will construct an inductive system $(X_\alpha)_{0\le\alpha<\omega_1}$ starting with $X_0 = X$;
- assuming all X_α have been defined for $\alpha < \beta$, if β is a limit ordinal then $X_\beta = \overline{\bigcup_{\alpha<\beta} X_\alpha}$ being $\imath_{\alpha,\beta}\colon\ X_\alpha \rightleftarrows X_\beta$ the obvious pairs;
- to obtain $X_{\alpha+1}$, let $\mathfrak{J}$ be the set of 1-pairs between the spaces in $\mathscr{F}^{(p)}$ and let $\mathfrak{L}_\alpha$ be the set of 1-pairs $u = \langle u^\flat, u^\sharp\rangle$ with domain in $\mathscr{F}^{(p)}$ and codomain X_α, excluding – this caution is crucial – those for which $u^\flat = \imath_{\eta,\alpha} v^\flat$ for some 1-pair v with codomain in X_η for some $\eta < \alpha$. Then consider the set $I_\alpha = \{(u, v) \in \mathfrak{L}_\alpha \times \mathfrak{J}\colon\ \operatorname{dom} u = \operatorname{dom} v\}$ and apply Lemma 6.3.15 to obtain a p-Banach space $X_{\alpha+1}$ and a 1-pair $\imath_{\alpha,\alpha+1}\colon\ X_\alpha \rightleftarrows X_{\alpha+1}$ with the corresponding pushout property. Furthermore, for every 1-pairs $v = \langle v^\flat, v^\sharp\rangle\colon\ E \rightleftarrows F$ in $\mathfrak{J}$ and $u = \langle u^\flat, u^\sharp\rangle\colon\ E \rightleftarrows X_\alpha$, regardless of whether u has been excluded, the construction miraculously provides a 1-pair $w\colon\ F \rightleftarrows X_{\alpha+1}$ such that $\imath_{\alpha,\alpha+1} \circ u = w \circ v$.

Let us prove that the space X_{ω_1} is of UCD. Consider a 1-pair $v \in \mathfrak{J}$ with domain E and codomain F and any 1-pair $u\colon\ E \rightleftarrows X_{\omega_1}$. Pick $\alpha < \omega_1$ such that $u^\flat[E]$ is contained in X_α. For each $\beta > \alpha$, the pair $\langle \imath_{\alpha,\beta} u^\flat,\ u^\sharp|_{X_\beta}\rangle$: $E \rightleftarrows X_\beta$ is one of the elements excluded from $\mathfrak{L}_\beta$, so the extending (through v) 1-pair $w_\beta\colon\ F \rightleftarrows X_{\beta+1}$, which exists, has $w_\beta^\flat = \imath_{\alpha,\beta+1} u^\flat$. Denoting the canonical pair by $\imath_\alpha\colon\ X_\alpha \rightleftarrows X_{\omega_1}$, we are ready to construct the desired 1-pair $w\colon\ F \rightleftarrows X_{\omega_1}$, that extends u. To do this, set $w^\flat = \imath_\alpha^\flat u^\flat$ and $w^\sharp x = \lim_{\mathfrak{U}} w_\alpha^\sharp \imath_{\alpha+1}^\sharp x$ using some ultrafilter refining the order filter on ω_1. □

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There is no control on the size of the output in this construction because even with a 1-dimensional seed $X_0 = \mathbb{K}$, the dimension of the first space X_1 in the chain is already $\mathfrak{c}$. Skeleton issues mark the difference here. An *ω-skeleton* of a p-Banach space Y is a continuous ω_1-chain $(Y_\alpha)_{\alpha<\omega_1}$ of separable subspaces in which each Y_α is 1-complemented in $Y_{\alpha+1}$ and whose union is dense in Y. Here, *continuous* means that $Y_\beta = \overline{\bigcup_{\alpha<\beta} Y_\alpha}$ for all limit ordinals $\beta < \omega_1$. Kubiś proved [309, Lemma 6.1] that if $(Y_\alpha)_{\alpha<\omega_1}$ is an ω-skeleton of a Banach space Y (the proof works for p-Banach spaces as well) then Y admits a PRI $(P_\alpha)_{\omega\leq\alpha\leq\omega_1}$ such that $P_\alpha[Y] = Y_\alpha$.

No serious trouble is caused by our using ω_1 as index set for the skeleton and $[\omega, \omega_1)$ for the indices of the PRI: assume, if you like, that $Y_\alpha = Y_0$ for $0 \leq \alpha \leq \omega$.

Proposition 10.7.3 [CH] *A p-Banach space with ω-skeleton can be isometrically embedded as a* 1*-complemented subspace of a p-Banach space of UCD with ω-skeleton.*

Proof We need to perform a scrupulously ordered version of the previous recipe. By doing so, we construct the directed system of 1-pairs $X_\alpha \rightleftarrows X_\beta$ for $0 \leq \alpha \leq \beta < \omega_1$ with X_β separable for all $\beta < \omega_1$ in such a way that its limit X_{ω_1} is of UCD and contains a 1-complemented copy of Y. Assume Y has an ω-skeleton $(Y_\alpha)_{0\leq\alpha<\omega_1}$. We start with $X_0 = Y_0$ (not with Y!). As in the previous recipe, fix the set $\mathfrak{J}$ of 1-pairs with domain and codomain in $\mathscr{F}^{(p)}$ and let $\mathfrak{L}_0$ be the set of 1-pairs with domain in $\mathscr{F}^{(p)}$ and codomain X_0. Since, under CH, a set of size $\mathfrak{c}$ can be written as an increasing union of ω_1 countable sets and $|\mathfrak{L}_0 \times \mathfrak{J}| = \mathfrak{c}$, form $I_0 = \{(u, v) \in \mathfrak{L}_0 \times \mathfrak{J} \colon \operatorname{dom} u = \operatorname{dom} v\}$ and write $I_0 = \bigcup_{\alpha<\omega_1} I_0^\alpha$ with each I_0^α countable. Apply Lemma 6.3.15 using only I_0^1 to obtain a separable space X_1' and a 1-pair $Y_0 \rightleftarrows X_1'$. Forming the pushout

$$\begin{array}{ccc} Y_0 & \longrightarrow & X_1' \\ \big\downarrow & & \big\downarrow \\ Y_1 & \longrightarrow & X_1 \end{array}$$

produces our first separable space X_1 and 1-pair $\jmath_1 \colon \; Y_1 \rightleftarrows X_1$. Now assume that the separable space X_β has already been obtained. Let $\mathfrak{L}_\beta$ be the set of 1-pairs with domain in $\mathscr{F}^{(p)}$ and codomain X_β, from which we exclude those 1-pairs u for which $u^\flat = \imath_{\eta,\beta} v^\flat$, for some $\eta < \beta$ and some 1-pair v, which have

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already been used in the construction of X_β. Let $I_\beta = \{(u,v) \in \mathfrak{L}_\beta \times \mathfrak{J} \colon \operatorname{dom} u = \operatorname{dom} v\}$ and write $I_\beta = \bigcup_{\alpha<\omega_1} I_\beta^\alpha$ with each I_β^α countable. Apply Lemma 6.3.15 using only I_β^β to obtain a separable space $X'_{\beta+1}$ and a 1-pair $Y_\beta \rightleftarrows X'_{\beta+1}$. Then form a new pushout

$$\begin{tikzcd} Y_\beta \arrow[r] \arrow[d] & X'_{\beta+1} \arrow[d] \\ Y_{\beta+1} \arrow[r] & X_{\beta+1} \end{tikzcd}$$

to obtain the 1-pair $\jmath_{\beta+1} \colon \; Y_{\beta+1} \rightleftarrows X_{\beta+1}$ with $X_{\alpha+1}$ separable. Set $X_\beta = \overline{\bigcup_{\alpha<\beta} X_\alpha}$ when β is a limit ordinal. The space X_{ω_1} has an ω-skeleton formed by $(X_\alpha)_{\alpha<\omega_1}$ and is of UCD, by a proof similar to that of Proposition 10.7.2. The space X_{ω_1} contains an isometric 1-complemented copy of $\bigcup_{\alpha<\omega_1} Y_\alpha$ with isometry $\jmath^\flat y = \imath_\alpha^\flat \jmath_\alpha^\flat y$ for $y \in Y_\alpha$ and projection $\jmath^\sharp x = \jmath_\alpha^\sharp \imath_\alpha^\sharp x$ for $x \in X_\alpha$. $\square$

When the resulting space X_{ω_1} has ω-skeleton or the BAP then the same is true for Y and in this way we obtain the existence of spaces of UCD with or without ω-skeleton and without the BAP. To obtain spaces of UCD with the BAP, start with a space Y with the BAP and proceed methodically to show that each X_α can be obtained with a skeleton. Indeed, if X_α has a skeleton then so does $X_{\alpha+1}$: simply observe that the required countable pushout can also be formed via a countable number of pushouts performed with a finite number of operators each. Finally, if each X_α has the BAP then so does X_{ω_1} since any of its finite-dimensional subspaces is contained in some X_α. Finding different kinds of UCD spaces is an open problem: for instance, it is anybody's guess whether an ultrapower of a space of AUCD is of UCD (or has property [$\Game$]).

10.8 The Kalton–Peck Spaces

$\ell_p(\varphi)$ as a Fenchel–Orlicz Space

The Fenchel–Orlicz spaces (Section 1.8.2) are exactly those modular (hence Banach) spaces built over a Young funcion. Our purpose here is to show that $\ell_p(\varphi)$ is the Fenchel–Orlicz space associated with a certain function on $\mathbb{K}^2$. The general argument contains an elementary proof that the spaces $\ell_p(\varphi)$ are Banach spaces for $p > 1$.

Theorem 10.8.1 *For every $1 < p < \infty$ and every $\varphi \in \operatorname{Lip}_0(\mathbb{R}^+)$, there is a Young function $\Phi \colon \mathbb{K}^2 \longrightarrow \mathbb{R}^+$ such that $\ell_p(\varphi) = h(\Phi)$, with equivalent quasinorms.*

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Proof The natural candidate to be the Young function for the space

$$\ell_p(\varphi) = \left\{ (y, x) \in \mathbb{K}^{\mathbb{N}} \times \mathbb{K}^{\mathbb{N}} : \sum_{n=1}^{\infty} \left| y(n) - x(n)\, \varphi\left(\log \frac{\|x\|}{|x(n)|} \right) \right|^p + |x(n)|^p < \infty \right\}$$

is $\Phi_p(y, x) = \left| y - x\, \varphi(-\log|x|) \right|^p + |x|^p$, which, unfortunately, is not convex. Let us circumvent this difficulty. A function $\phi\colon V \longrightarrow \mathbb{R}$ defined on a convex subset V of a linear space is said to be *quasiconvex* if there is some $C > 0$ such that, for all $v_1, v_2 \in V$ and all $t \in [0, 1]$, one has $\phi(tv_1 + (1-t)v_2) \le C(t\phi(v_1) + (1-t)\phi(v_2))$.

Claim The function $\Phi_p\colon \mathbb{K}^2 \longrightarrow \mathbb{R}^+$ above is quasiconvex for $1 < p < \infty$.

Proof of the claim The proof is based on Lemma 3.2.5, the convexity of the function $x \longmapsto |x|^p$ and the following simple inequalities:

- $|x + y|^p \le 2^{1-1/p}(|x|^p + |y|^p)$, for every $x, y \in \mathbb{C}$,
- $|t \log t|^p \le t$ for $t \in (0, 1]$.

So, for $i = 1, 2$, choose $(y_i, x_i) \in \mathbb{K}^2$ and $t_i \in [0, 1]$ such that $t_1 + t_2 = 1$. Then,

$$\Phi_p\Big(\sum t_i(y_i, x_i) \Big) = \overbrace{\left| \sum_i t_i y_i - \Big(\sum_i t_i x_i \Big) \varphi\left(-\log \Big| \sum_i t_i x_i \Big| \right) \right|^p}^{\text{(a)}} + \overbrace{\Big| \sum_i t_i x_i \Big|^p}^{\text{(b)}},$$
$$\sum t_i \Phi_p(y_i, x_i) = \underbrace{\sum_i t_i \big| y_i - x_i \varphi(-\log|x_i|) \big|^p}_{\text{(c)}} + \underbrace{\sum_i t_i |x_i|^p}_{\text{(d)}},$$

We have (b) $\le$ (d), by convexity. Let us focus on (a). It will be helpful to recall the function $\omega_\varphi(x) = x\varphi(-\log|x|)$ from Section 3.2:

$$\begin{aligned}
\text{(a)} &= \left| \sum_i t_i y_i - \omega_\varphi\Big(\sum_i t_i x_i \Big) \right|^p \\
&\le 2\left(\left| \sum_i (t_i y_i - \omega_\varphi(t_i x_i)) \right|^p + \left| \omega_\varphi\Big(\sum_i t_i x_i \Big) - \sum_i \omega_\varphi(t_i x_i) \right|^p \right).
\end{aligned}$$

Thanks to Lemma 3.2.5, the second summand can be bounded by

$$\left| \omega_\varphi\Big(\sum_i t_i x_i \Big) - \sum_i \omega_\varphi(t_i x_i) \right|^p \le \left(\frac{2\,\mathrm{Lip}(\varphi)}{e} \right)^p \left(\sum_i |t_i x_i| \right)^p \le C \sum_i t_i |x_i|^p.$$

As for the first part, we treat each summand separately as

$$\left| t_i y_i - \omega_\varphi(t_i x_i) \right|^p \le 2\left(\left| t_i y_i - t_i \omega_\varphi(x_i) \right|^p + \left| t_i \omega_\varphi(x_i) - \omega_\varphi(t_i x_i) \right|^p \right).$$

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Its first chunk is dominated by (c), while its second chunk satisfies

$$\left|t_i\omega_\varphi(x_i) - \omega_\varphi(t_ix_i)\right|^p = \left|t_ix_i\varphi(-\log|x_i|) - t_ix_i\varphi(-\log|t_ix_i|)\right|^p$$
$$\leq \left|t_ix_i \operatorname{Lip}(\varphi)\log t_i\right|^p \leq \operatorname{Lip}(\varphi)t_i|x_i|^p. \qquad \square$$

Quasiconvex functions cannot be used as Young functions in principle, but Φ_p is equivalent to a convex function, in the same way that the usual quasinorm of $\ell_p(\varphi)$ is not a norm while being equivalent to a norm. Not every quasiconvex function is equivalent to a convex function: after all, every quasinorm is quasiconvex. The situation is more favourable in finite dimensions:

Lemma 10.8.2 *Every quasiconvex function defined on a finite-dimensional convex set is equivalent to some convex function.*

Proof This is a very standard proof based on Carathéodory's theorem [164, p. 34]: every point v of the convex hull of a subset W of a real k-dimensional space can be written as $v = \sum_{1\leq i\leq k+1} t_iv_i$, where $v_i \in W, t_i \in [0,1]$ and $\sum_{1\leq i\leq k+1} t_i = 1$. To be fussy, if W is connected, k points suffice. Let $\phi\colon V \longrightarrow \mathbb{R}^+$ be any non-negative function defined on a convex set. The greatest convex minorant of ϕ is the function $\operatorname{gcm}(\phi)\colon V \longrightarrow \mathbb{R}^+$ given by

$$\operatorname{gcm}(\phi)(v) = \inf\left\{\sum_{i=1}^m t_i\phi(v_i) : m \in \mathbb{N}, v = \sum_{i=1}^m t_iv_i, 1 = \sum_{i=1}^m t_i, t_i \in [0,1]\right\}.$$

It is clear that $\operatorname{gcm}(\phi)$ is convex, that $\operatorname{gcm}(\phi) \leq \phi$ and that $\operatorname{gcm}(\phi)(v)$ equals the infimum of those $t \in \mathbb{R}^+$ for which the point (v,t) belongs to the convex hull of the set $W = \{(u,\phi(u)) \in V \times \mathbb{R}^+ : u \in V\}$. If V is n-dimensional then W has dimension at most $n+1$, and every point (v,t) in the convex hull of W can be written as a convex combination of $n+2$ or fewer points in W so that $v = \sum_{i=1}^m t_iv_i$ and $t = \sum_{i=1}^m t_i\phi(v_i)$ with $m \leq n+2$ and, in the end,

$$\operatorname{gcm}(\phi)(v) = \inf\left\{\sum_{i=1}^m t_i\phi(v_i) : m \leq n+2, v = \sum_{i=1}^m t_iv_i, 1 = \sum_{i=1}^m t_i, t_i \in [0,1]\right\}.$$

Finally, if ϕ is quasiconvex then, for each $m \geq 2$, there is c_m such that for every $v_1,\ldots,v_m \in V$ and every $t_i \in [0,1]$ satisfying $\sum_{i=1}^m t_i = 1$, we have

$$\phi\left(\sum_{i=1}^m t_iv_i\right) \leq c_m \sum_{i=1}^m t_i\phi(v_i).$$

Hence $\operatorname{gcm}(\phi) \leq \phi \leq c_{n+2} \operatorname{gcm}(\phi)$. That is, ϕ is equivalent to $\operatorname{gcm}(\phi)$. $\square$

Let V be a finite-dimensional vector space, and let $\Phi\colon V \longrightarrow \mathbb{R}^+$ be a quasiconvex and even function such that $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(tx) = \infty$ for all non-zero $x \in V$. Then $\operatorname{gcm}(\Phi)$ is a Young function. With a slight abuse of notation, we can consider the space

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$$h(\Phi) = \left\{ v \in V^{\mathbb{N}} : \sum_{k=1}^{\infty} \Phi(tv(k)) < \infty \text{ for all } t > 0 \right\},$$

which agrees with the straight modular space $h(\text{gcm}(\Phi))$, and the functional

$$\|v\|_{\Phi} = \inf \left\{ t > 0 : \sum_{k=1}^{\infty} \Phi(v(k)/t) \le 1 \right\},$$

which is a quasinorm equivalent to the Luxemburg norm on $h(\text{gcm}(\Phi))$. It being obvious after Claim 1 that these considerations apply to Φ_p, we are ready to conclude the proof that $\ell_p(\varphi) = h(\Phi_p)$ with equivalent quasinorms. This is done with the aid of the Kalton–Peck map $\mathsf{KP}_{\varphi}(x) = x\varphi\left(\frac{\|x\|}{|x|}\right)$ and its non-homogeneous version $\mathsf{kp}_{\varphi} \colon \ell_p \longrightarrow \mathbb{K}^{\mathbb{N}}$ defined as $\mathsf{kp}_{\varphi}(x) = x\,\varphi(-\log|x|)$ in Proposition 3.12.5 and the inequality $\|\mathsf{kp}_{\varphi}(x) - \mathsf{KP}_{\varphi}(x)\| \le \text{Lip}(\varphi)\big|\|x\| \log \|x\|\big|$ obtained there. Letting

$$|(y,x)|_{\mathsf{kp}_{\varphi}} = \|y - \mathsf{kp}_{\varphi}(x)\|^p + \|x\|^p = \sum_{k=1}^{\infty} \Phi_p(y(k), x(k)),$$

it should be obvious that $\ell_p(\varphi) = \{(y,x) \in \mathbb{K}^{\mathbb{N}} \times \mathbb{K}^{\mathbb{N}} \colon |(y,x)|_{\mathsf{kp}_{\varphi}} < \infty\}$ and also that $|(\cdot,\cdot)|_{\mathsf{kp}_{\varphi}}$ is 'coarsely equivalent' to $\|(\cdot,\cdot)\|_{\mathsf{KP}_{\varphi}}$ in the sense that

$$|(y,x)|_{\mathsf{kp}_{\varphi}} \le f(\|(y,x)\|_{\mathsf{KP}_{\varphi}}) \qquad \text{and} \qquad \|(y,x)\|_{\mathsf{KP}} \le g(|(y,x)|_{\mathsf{kp}_{\varphi}}),$$

for $f(t) = 2\,\text{Lip}(\varphi)\max(t^p, |t \log t|^p)$ and $g(s) = 2\,\text{Lip}(\varphi)\max(s^{1/p}, |s^{1/p} \log s^{1/p}|)$. In particular,

$$0 < r = \inf\{\|(y,x)\|_{\mathsf{KP}_{\varphi}} \colon |(y,x)|_{\mathsf{kp}_{\varphi}} \ge 1\} \le \sup\{\|(y,x)\|_{\mathsf{KP}} \colon |(y,x)|_{\mathsf{kp}_{\varphi}} \le 1\} = R < \infty.$$

Assuming $\|(y,x)\|_{\Phi_p} < 1$, by the very definition, there is an $s > 1$ such that

$$\sum_{k \ge 1} \Phi_p(s(y(k), x(k))) = |(sy, sx)|_{\mathsf{kp}_{\varphi}} \le 1 \implies \|(y,x)\|_{\mathsf{KP}} \le \|(sy, sx)\|_{\mathsf{KP}_{\varphi}} \le R.$$

The other inclusion is also easy: if $\|(y,x)\|_{\mathsf{KP}} < r$ then $|(y,x)|_{\mathsf{kp}_{\varphi}} \le 1$, and so $\|(y,x)\|_{\Phi_p} \le 1$. □

Orlicz Subspaces of $\ell_p(\varphi)$

Let $0 < p < \infty$. The subspace $D = \{x \in \mathbb{K}^{\mathbb{N}} \colon (0,x) \in \ell_p(\varphi)\}$ of $\ell_p(\varphi)$ will be called the domain of the centralizer $\mathsf{KP}_{\varphi} \colon \ell_p \longrightarrow \mathbb{K}^{\mathbb{N}}$ because $D = \{x \in \ell_p \colon \mathsf{KP}_{\varphi}(x) \in \ell_p\}$. It is clear from Lemma 3.12.4 that D is an unconditional sequence space: if $x \in D$ and $a \in \ell_{\infty}$, then $ax \in D$ and

$$\|ax\|_D = \|(0, ax)\|_{\mathsf{KP}_{\varphi}} \le C\|a\|_{\infty}\|(0,x)\|_{\mathsf{KP}_{\varphi}} = C\|a\|_{\infty}\|x\|_D,$$

where C is a constant depending only on $\mathrm{Lip}(\varphi)$ and p. If we put $\phi_p(t) = \Phi_p(0,t) = |t\varphi(-\log|t|)|^p + |t|^p$ then ϕ_p is an Orlicz function, and it should be obvious from the proof of the preceding claim that

- $D = \ell_{\phi_p}$ and the quasinorm of D is equivalent to the Luxemburg quasinorm of ℓ_{ϕ_p} given by $\|x\|_{\phi_p} = \inf\{r > 0 \colon \sum_k \phi_p(x(k)/r) \le 1\}$;
- if $p > 1$ then ϕ_p is equivalent to a convex Orlicz function, hence to $\mathrm{gcm}(\phi_p)$, and so $D = \ell_{\phi_p} = h(\mathrm{gcm}(\phi_p))$. This also follows from the fact that $\phi_p(t)/t$ is increasing near zero along with classical material on Orlicz functions.

Lemma 10.8.3 *Let $(v_n)_{n\ge1}$ be a normalised block sequence in ℓ_p. If $w_n = (\mathsf{KP}_\varphi(v_n), v_n)$ then*

(a) *$(w_n)_{n\ge1}$ is an unconditional basic sequence in $\ell_p(\varphi)$;*
(b) *if $\varphi'(t)$ decreases to zero as $t \longrightarrow \infty$, then $(w_n)_{n\ge1}$ has a subsequence equivalent to the unit basis of either ℓ_{ϕ_p} or ℓ_p;*
(c) *if φ is the identity on $\mathbb{R}^+$, then (w_n) is equivalent to the unit basis of ℓ_{ϕ_p}.*

Proof Part (a) is clear from the centralizer property of the Kalton–Peck maps. Let $(c_n)_{n\ge1}$ be a sequence of scalars for which the series $\sum_n c_n w_n$ converges in $\ell_p(\varphi)$, and assume that $|d_n| \le |c_n|$ for all n. We set

$$w = \sum_n c_n w_n, \qquad v = \sum_n c_n v_n, \qquad u = \sum_n c_n \mathsf{KP}_\varphi(v_n),$$

$$\tilde{w} = \sum_n d_n w_n, \qquad \tilde{v} = \sum_n d_n v_n, \qquad \tilde{u} = \sum_n d_n \mathsf{KP}_\varphi(v_n),$$

where the series defining w converges in $\ell_p(\varphi)$, those of v and $\tilde{v}$ converge in ℓ_p and the other three are just pointwise sums. We must show that $\tilde{w} \in \ell_p(\varphi)$ and $\|\tilde{w}\| \le C\|w\|$ for some C independent on (c_n). Define $a\colon \mathbb{N} \longrightarrow \mathbb{K}$ by taking $a(k) = d_n(k)/c_n(k)$ for $k \in \operatorname{supp} v_n$ and filling with zeros. Then $\|a\|_\infty \le 1$ and

$$w = (u, v), \qquad \tilde{w} = (\tilde{u}, \tilde{v}), \qquad \tilde{u} = au, \qquad \tilde{v} = av,$$

hence $\tilde{w} \in \ell_p(\varphi)$ with $\|\tilde{w}\|_{\mathsf{KP}} \le \max(\Delta C(\mathsf{KP}_\varphi), 1)\|w\|_{\mathsf{KP}}$ by Lemma 3.12.4, where Δ is the modulus of concavity of ℓ_p. The proof of (b) is simpler after realising that the 'coordinate functionals' $(y, x) \longmapsto y(k)$ and $(y, x) \longmapsto x(k)$ are (uniformly) bounded on $\ell_p(\varphi)$, which follows trivially from the centralizer property of KP_φ. Indeed, for each $k \in \mathbb{N}$ and all $(y, x) \in \ell_p(\varphi)$, we have

$$\begin{aligned}|y(k)| + |x(k)| &= \|e_k y - \mathsf{KP}(e_k x)\| + \|e_k x\| \\ &= \|(e_k y, e_k x)\|_{\mathsf{KP}_\varphi} \le \max(\Delta C(\mathsf{KP}_\varphi), 1)\|(y, x)\|_{\mathsf{KP}_\varphi}.\end{aligned}$$

The immediate consequence is that convergence in $\ell_p(\varphi)$ implies convergence in every coordinate of $\mathbb{K}^{\mathbb{N}} \times \mathbb{K}^{\mathbb{N}}$, which provides a manageable criterion for the convergence of a series of the form $\sum_{n\ge1} t_n w_n$.

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Claim The series $\sum_{n\geq 1} t_n w_n$ converges in $\ell_p(\varphi)$ if and only if the pointwise sum $w = \left(\sum_{n=1}^{\infty} t_n \mathsf{KP}_\varphi(v_n), \sum_{n=1}^{\infty} t_n v_n\right)$ belongs to $\ell_p(\varphi)$.

Proof of the claim The 'only if' direction is clear. To prove the converse, assume $w \in \ell_p(\varphi)$ and let us prove that the 'remainder'

$$r_N = w - \sum_{n\leq N}(t_n \mathsf{KP}_\varphi(v_n), t_n v_n) = \left(\sum_{n>N} t_n \mathsf{KP}_\varphi(v_n), \sum_{n>N} t_n v_n\right)$$

converges to zero in $\ell_p(\varphi)$. While this leads to serious difficulties if we insist on using the quasinorm of $\ell_p(\varphi)$, it becomes transparent if we use the 'equivalent' modular $|(y,x)|_{\mathsf{kp}_\varphi} = \|y - \mathsf{kp}_\varphi(x)\|^p + \|x\|^p$ instead, taking advantage of the fact that kp_φ and $\|\cdot\|^p$ are additive on disjoint families. Now,

$$\begin{aligned}
|w|_{\mathsf{kp}_\varphi} &= \left\|\sum_{n=1}^{\infty} t_n v_n \left(\varphi\left(\log\frac{1}{|v_n|}\right) - \varphi\left(\log\frac{1}{|t_n||v_n|}\right)\right)\right\|^p + \left\|\sum_{n=1}^{\infty} t_n v_n\right\|^p \\
&= \sum_{n=1}^{\infty} |t_n|^p \left\|v_n \left(\varphi\left(\log\frac{1}{|v_n|}\right) - \varphi\left(\log\frac{1}{|t_n||v_n|}\right)\right)\right\|^p + \sum_{n=1}^{\infty} |t_n|^p, \\
|r_N|_{\mathsf{kp}_\varphi} &= \sum_{n>N} |t_n|^p \left\|v_n \left(\varphi\left(\log\frac{1}{|v_n|}\right) - \varphi\left(\log\frac{1}{|t_n||v_n|}\right)\right)\right\|^p + \sum_{n>N} |t_n|^p.
\end{aligned}$$

So, certainly, $|r_N|_{\mathsf{kp}} \to 0$ as $N \to \infty$ if $|w|_{\mathsf{kp}} < \infty$. □

In more computable terms, the series $\sum_{n\geq 1} t_n w_n$ converges if and only if the numerical series $\sum_n |t_n|^p$ and

$$\sum_{n=1}^{\infty} \sum_{k\in \operatorname{supp} v_n} |t_n|^p |v_n(k)|^p \underbrace{\left|\varphi\left(\log\frac{1}{|v_n(k)|}\right) - \varphi\left(\log\frac{1}{|t_n||v_n(k)|}\right)\right|^p}_{(\star)} \tag{10.11}$$

are convergent. Now, if $\varphi'(t)$ is decreasing, with limit 0, then φ is increasing and concave, and for every n and k,

$$(\star) \leq \left|\varphi(0) - \varphi\left(\log\frac{1}{|t_n|}\right)\right| = \left|\varphi\left(\log\frac{1}{|t_n|}\right)\right|.$$

It follows that if $(t_n) \in \ell_{\phi_p}$ then $\sum_n t_n w_n \in \ell_p(\varphi)$ and

$$\left|\sum_n t_n w_n\right|_{\mathsf{kp}} \leq \sum_n \phi_p(t_n).$$

We now need to distinguish two cases, depending on the behaviour of the norms $\|v_n\|_\infty$.

• If $\|v_n\|_\infty \geq \varepsilon$ for some $\varepsilon > 0$ and all n, then (w_n) is equivalent to the unit basis of ℓ_{ϕ_p}.

Indeed, selecting for each n some $k \in \operatorname{supp} v_n$ such that $|v_n(k)| \geq \varepsilon$, we get

$$\begin{aligned} |w|_{\mathsf{kp}} &\geq \sum_{n=1}^{\infty} |t_n|^p + \sum_{n=1}^{\infty} |t_n|^p |v_n(k)|^p \left| \varphi\left(\log \frac{1}{|v_n(k)|}\right) - \varphi\left(\log \frac{1}{|t_n||v_n(k)|}\right) \right|^p \\ &\geq \sum_{n=1}^{\infty} |t_n|^p + \varepsilon^p \sum_{n=1}^{\infty} |t_n|^p \left| \varphi\left(\log \frac{1}{|t_n|}\right) \right|^p \geq \min(1, \varepsilon^p) \sum_{n=1}^{\infty} \phi_p(t_n). \end{aligned}$$

• If, otherwise, $\liminf_n \|v_n\|_\infty = 0$, then we may assume, passing to a subsequence if necessary, that $|\varphi'(t)| \leq 2^{-n}$ for $t \geq \log(1/\|v_n\|_\infty)$. In this case, (w_n) is equivalent to the unit basis of ℓ_p. Assume (t_n) is a sequence in the unit ball of ℓ_p and consider the pointwise sum $\sum_n t_n w_n$.

Let us estimate $(\star)$. By the mean value theorem, for each n and every $k \in \operatorname{supp} v_n$, there is $t \in (-\log|v_n(k)|, -\log|t_n||v_n(k)|)$ such that

$$\left| \varphi\left(\log \frac{1}{|v_n(k)|}\right) - \varphi\left(\log \frac{1}{|t_n||v_n(k)|}\right) \right| = |\varphi'(t)| \log \frac{1}{|t_n|} \leq 2^{-n} \log \frac{1}{|t_n|}$$

since $-\log|v_n(k)| \geq -\log|v_n|_\infty$. Recalling once again that $|t \log t| \leq e^{-1}$ for $0 \leq t \leq 1$, we have

$$\begin{aligned} \left| \sum_n t_n w_n \right|_{\mathsf{kp}} &\leq \sum_n |t_n|^p + \sum_{n=1}^{\infty} \sum_{k \in \operatorname{supp} v_n} |t_n|^p |v_n(k)|^p 2^{-pn} \big| \log|t_n| \big|^p \\ &\leq 1 + \sum_{n=1}^{\infty} \sum_{k \in \operatorname{supp} v_n} e^{-p} 2^{-pn} |v_n(k)|^p = 1 + \frac{e^{-p} 2^{-p}}{1 - 2^{-p}}. \end{aligned}$$ □

Theorem 10.8.4 *Suppose that either $\varphi'(t)$ decreases to zero or φ is the identity on $\mathbb{R}^+$. Then every normalised pointwise null basic sequence in $\ell_p(\varphi)$ has a subsequence equivalent to the unit basis of either ℓ_p or ℓ_{ϕ_p}.*

Proof Let $w_n = (u_n, v_n)$ be such a sequence. If $\|v_n\| \to 0$, then $\tilde{w}_n = w_n - (\mathsf{KP}_\varphi v_n, v_n)$ belongs to $\imath[\ell_p]$ and $\|\tilde{w}_n - w_n\| \to 0$. It follows from the customary argument on perturbation of bases that (w_n) has a subsequence equivalent to the unit basis of ℓ_p. If (v_n) is not null, we may directly assume that $\|v_n\| \geq \varepsilon$ for some $\varepsilon > 0$ and all n. The hypothesis implies that (v_n) is pointwise null in ℓ_p and, passing to a subsequence, we may assume that there is a block basic sequence $(x_n)_{n \geq 1}$ in ℓ_p such that $\|v_n - x_n\| \leq 2^{-n}$. By applying Lemma 10.8.3 and relabelling, we can assume that $z_n = (\mathsf{KP}_\varphi x_n, x_n)$ is equivalent to one of the two bases under consideration. Now we distinguish two cases:

• If $\liminf_n \|w_n - z_n\| = 0$, then (w_n) and (z_n) have equivalent subsequences, and we are done.

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• Otherwise, assume that (a subsequence of) $(w_n - z_n)$ is a basic sequence. But $w_n - z_n = (u_n - \mathsf{KP}_\varphi(x_n), v_n - x_n)$, and since $\|v_n - x_n\| \longrightarrow 0$ in ℓ_p, the first paragraph of the proof shows that $(w_n - z_n)$ has a subsequence equivalent to the unit basis of ℓ_p. It turns out that (w_n) and (z_n) are equivalent bases: if $\sum_n t_n w_n$ converges then $\sum_n t_n v_n$ and $\sum_n t_n x_n$ converge, hence $\sum_n |t_n|^p < \infty$ and $\sum_n t_n(w_n - z_n)$ converge, and so does $\sum_n t_n z_n$. The argument is reversible. □

Corollary 10.8.5 *Given* $0 < r \leq 1$, *define* $\varphi_r(t) = \min(t, t^r)$. *Then, for each fixed* $0 < p < \infty$, *the spaces* $\ell_p(\varphi_r)$ *are mutually non-isomorphic.*

Proof In fact, none of the spaces $\ell_p(\varphi_r)$ can be embedded into any other. Assume $T\colon \ell_p(\varphi_r) \longrightarrow \ell_p(\varphi_s)$ is an embedding. Then $f_n = T(0, e_n)$ is a basic sequence in $\ell_p(\varphi_s)$. If (f_n) is pointwise null then some subsequence would be equivalent to the unit basis of ℓ_p or to the unit basis of the Orlicz space associated with the function $t^p(1 + \log^{sp}(1/t))$, which cannot be, since (e_n) is equivalent to the unit basis of the Orlicz space associated with the function $t^p(1 + \log^{rp}(1/t))$. Otherwise, some subsequence of (f_n), which we do not relabel, is pointwise convergent in $\mathbb{K}^\mathbb{N} \times \mathbb{K}^\mathbb{N}$, and the preceding argument applies to $(e_{2n-1} - e_{2n})_{n\geq 1}$. □

Many other examples can be created, for instance, by introducing a second parameter and considering the family of functions $\varphi(t) = t^r \log^s(1/t)$ for $0 \leq r < 1, 0 < s < \infty$, and so on.

Z_p Obtained by Complex Interpolation

We briefly describe the complex interpolation method for pairs, following [43; 278]. Let $\mathbb{S}$ denote the open strip $\{z \in \mathbb{C}\colon 0 < \mathrm{Re}(z) < 1\}$ in the complex plane, and let $\overline{\mathbb{S}}$ be its closure. A pair (X_0, X_1) of complex Banach spaces will form an admissible or interpolation pairs if there exist injective operators $X_0 \longrightarrow \Sigma$ and $X_1 \longrightarrow \Sigma$ into some Banach space Σ. We will identify both X_0, X_1 with their continuous images in Σ without further mention. The Calderón space $\mathcal{H} = \mathcal{H}(X_0, X_1)$ is the space of continuous bounded functions $G\colon \overline{\mathbb{S}} \longrightarrow \Sigma$ that are holomorphic on $\mathbb{S}$ and satisfy the following boundary condition:

- for $k = 0, 1$, $G(k + it) \in X_k$ for each $t \in \mathbb{R}$ and $\sup_t \|G(k + it)\|_{X_k} < \infty$.

The space $\mathcal{H}$ is complete under the norm $\|G\| = \sup\{\|G(k + it)\|_{X_k}\colon k = 0, 1; t \in \mathbb{R}\}$. The evaluation map $\delta_z\colon \mathcal{H} \longrightarrow \Sigma$ is continuous for all $z \in \overline{\mathbb{S}}$. Given $\theta \in (0, 1)$, one defines the *interpolation* space as

$$X_\theta = \{x \in \Sigma\colon\ x = G(\theta)\ \text{ for some } G \in \mathcal{H}\}$$

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endowed with the norm $\|x\|_{X_\theta} = \inf\{\|G\| : x = G(\theta), G \in \mathcal{H}\}$. This space is isometric to the quotient $\mathcal{H}/\ker\delta_\theta$, hence it is a Banach space. Now, if $z \in \mathbb{S}$ then the map $\delta'_\theta\colon \mathcal{H} \longrightarrow \Sigma$ given by evaluation of the derivative at θ, being the pointwise limit of a sequence of operators, is also bounded by the uniform boundedness principle. The connection between complex interpolation and twisted sums is provided by the following lemma:

Lemma 10.8.6 *$\delta'_\theta\colon \ker\delta_\theta \longrightarrow X_\theta$ is bounded and onto for $0 < \theta < 1$.*

Proof The crucial property of $\mathcal{H}$ in this bussiness is that if $\phi\colon \mathbb{S} \longrightarrow \mathbb{D}$ is a conformal equivalence vanishing at θ then $\ker\delta_\theta = \phi \cdot \mathcal{H}$, in the sense that every $F \in \mathcal{H}$ vanishing at θ has a factorisation $F = \phi \cdot G$, with $G \in \mathcal{H}$ and $\|G\| = \|F\|$. A conformal equivalence vanishing at θ is

$$\phi(z) = \frac{\exp(i\pi z) - \exp(i\pi\theta)}{\exp(i\pi z) - \exp(-i\pi\theta)}$$

(any other has the form $u\phi$ with $|u| = 1$). Now, if $F \in \ker\delta_\theta$, writing $F = \phi \cdot G$, we have $F' = \phi' G + \phi G'$, and so $\delta'_\theta(F) = \phi'(\theta)\delta_\theta(G)$, hence $\|\delta'_\theta\colon \ker\delta_\theta \longrightarrow X_\theta\| \le |\phi'(\theta)|$. That δ'_θ maps $\ker\delta_\theta$ onto X_θ is also clear: if $x \in X_\theta$, then $x = G(\theta)$ for some $G \in \mathcal{H}$, and x is the derivative of $\phi(\theta)^{-1}\phi \cdot G$ at θ. □

Thus, for each $\theta \in (0, 1)$, there is a pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker\delta_\theta \arrow[r] \arrow[d, "\delta'_\theta"] & \mathcal{H}(X_0, X_1) \arrow[r, "\delta_\theta"] \arrow[d, "\overline{\delta'_\theta}"] & X_\theta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X_\theta \arrow[r, "\jmath"] & \mathrm{PO} \arrow[r, "\rho"] & X_\theta \arrow[r] & 0 \end{tikzcd} \tag{10.12}$$

where the lower row is a self-extension of X_θ. The twisted sum space PO, which is a quotient of $\mathcal{H}$, admits a nice description as a certain subspace of $\Sigma \times \Sigma$, as we now see. We call $dX_\theta = \{(f'(\theta), f(\theta))\colon f \in \mathcal{H}\}$ equipped with the quotient norm $\|(y, x)\|_{dX_\theta} = \inf\{\|F\|\colon y = F'(\theta), x = F(\theta))\}$ the *derived* space. Let $Q\colon \mathcal{H} \longrightarrow dX_\theta$ be the natural quotient map. To prove that $dX_\theta \simeq \mathrm{PO}$, we show that the pushout sequence in (10.12) is equivalent to

$$0 \longrightarrow X_\theta \xrightarrow{\ \imath\ } dX_\theta \xrightarrow{\ \pi\ } X_\theta \longrightarrow 0 \tag{10.13}$$

with $\imath(y) = (y, 0)$ and $\pi(y, x) = x$. The operator π is correctly defined and maps the (open) unit ball of dX_θ onto that of X_θ. The kernel of π consists of those points $(y, 0) \in \Sigma \times \Sigma$ where y is the value at θ of the derivative of some function in $\mathcal{H}$ vanishing at θ; the previous lemma not only tells us that this is exactly X_θ

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but also that $\|(y,0)\|_{dX_\theta} = |\phi'(\theta)|\,\|y\|_{X_\theta}$. Thus, $\imath$ is continuous, and the sequence is exact. We show that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X_\theta \arrow[r, "\jmath"] \arrow[d, equal] & \mathrm{PO} \arrow[r, "\rho"] \arrow[d, "\gamma"] & X_\theta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X_\theta \arrow[r, "\imath"] & dX_\theta \arrow[r, "\pi"] & X_\theta \arrow[r] & 0 \end{tikzcd} \tag{10.14}$$

The operator γ is defined by the universal property of the pushout applied to the commutative square

$$\begin{tikzcd} \ker\delta_\theta \arrow[r] \arrow[d, "\delta'_\theta"'] & \mathcal{H} \arrow[d, "Q"] \\ X_\theta \arrow[r, "\imath"] & dX_\theta \end{tikzcd}$$

which produces the unique operator $\gamma\colon \mathrm{PO} \longrightarrow dX_\theta$ such that $\gamma\jmath = \imath$ and $\gamma\overline{\delta'_\theta} = Q$. This makes the left square of (10.14) commutative. The commutativity of the right square is also clear: $\rho\colon \mathrm{PO} \longrightarrow X_\theta$ is the only operator satisfying $\rho\jmath = 0$ and $\delta_\theta = \rho\overline{\delta'_\theta}$, and $\pi\gamma$ does the same. Of course, the sequence (10.13) can be described by a quasilinear map. To see which one, fix $\varepsilon > 0$ and, for each $x \in X_\theta$, (homogeneously) select $F_x \in \mathcal{H}(X_0, X_1)$ such that $x = F_x(\theta)$ and $\|F_x\| \le (1+\varepsilon)\|x\|_{X_\theta}$. Define $\Omega\colon X_\theta \longrightarrow \Sigma$ by $\Omega(x) = F'_x(\theta)$. With the notation of Section 3.12, we have:

Lemma 10.8.7 *Ω is quasilinear from X_θ to X_θ and $dX_\theta = X_\theta \oplus_\Omega X_\theta$, with equivalent quasinorms.*

Proof Pick $x, y \in X_\theta$ and let $F_x, F_y, F_{x+y} \in \mathcal{H}(X_0, X_1)$ be the corresponding extremals. One has $\Omega(x+y) - \Omega(x) - \Omega(y) = \delta'_\theta(F_{x+y} - F_x - F_y) \in X_\theta$, and since $F_{x+y} - F_x - F_y \in \ker\delta_\theta$, Lemma 10.8.6 applies to yield $\Omega(x+y) - \Omega(x) - \Omega(y) \in X_\theta$. Moreover,

$$\begin{aligned} \|\Omega(x+y) - \Omega(x) - \Omega(y)\|_{X_\theta} &= \|\delta'_\theta(F_{x+y} - F_x - F_y)\|_{X_\theta} \\ &\le \|\delta'_\theta : \ker\delta_\theta \longrightarrow X\|\Big(\|F_{x+y}\| + \|F_x\| + \|F_y\|\Big) \\ &\le \|\delta'_\theta\|(1+\varepsilon)\Big(\|x+y\|_{X_\theta} + \|x\|_{X_\theta} + \|y\|_{X_\theta}\Big) \\ &\le 2(1+\varepsilon)\|\delta'_\theta\|\Big(\|x\|_{X_\theta} + \|y\|_{X_\theta}\Big). \end{aligned}$$

We now check that $dX_\theta = X_\theta \oplus_\Omega X_\theta$ with equivalent norms. First note that $f'(\theta) - \Omega(f(\theta)) \in X_\theta$ for every $f \in \mathcal{H}$. Indeed, since $f - F_{f(\theta)} \in \ker\delta_\theta$, we have $f'(\theta) - \Omega(f(\theta)) = f'(\theta) - F'_{f(\theta)}(\theta)) = (f - F_{f(\theta)})'(\theta)) \in X_\theta$. One thus obtains the containment $dX_\theta \subset X_\theta \oplus_\Omega X_\theta$. Conversely, if $y - \Omega(x) \in X_\theta$ then $y - \Omega(x) = g'(\theta)$ for some $g \in \ker\delta_\theta$ since $\delta'_\theta\colon \ker\delta_\theta \longrightarrow X_\theta$ is onto. Thus,

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$y = \Omega(x) + g'(\theta) = (F_x + g)'(\theta)$, and therefore $(y, x) = ((F_x + g)'(\theta), (F_x + g)(\theta)$. To prove the equivalence of norms, pick $(y, x) \in X_\theta \oplus_\Omega X_\theta$ so that x and $y - \Omega(x)$ belong to X_θ. Let F and G be the corresponding extremals:

$$x = F(\theta), \quad \|F\| \le (1+\varepsilon)\|x\|_{X_\theta}, \quad y - \Omega(x) = G(\theta), \quad \|G\| \le (1+\varepsilon)\|y - \Omega(x)\|_{X_\theta}.$$

If $\phi \colon \mathbb{S} \longrightarrow \mathbb{D}$ is a conformal map such that $\phi(\theta) = 0$ and we define $H(z) = \phi(\theta)^{-1}\phi(z)G(z) + F(z)$, then $H \in \mathcal{H}(X_0, X_1)$, with

$$\|H\| \le |\phi(\theta)|^{-1}\|G\| + \|F\| \le \max\left(|\phi(\theta)|^{-1}, 1\right)(1 + \varepsilon)\|(y, x)\|_\Omega,$$

and one has $H(\theta) = F(\theta) = x$ and $H'(\theta) = G(\theta) + F'(\theta) = y - \Omega(x) + \Omega(x) = y$. To prove the other inclusion, take $H \in \mathcal{H}(X_0, X_1)$ and set $(y, x) = (H'(\theta), H(\theta))$. Then $x \in X_\theta$ and $\|x\|_{X_\theta} \le \|H\|$. Besides $\Omega(x) = F_x'(\theta)$, with $\|F_x\| \le (1 + \varepsilon)\|x\|_{X_\theta}$. Hence $y - \Omega(x) = H'(\theta) - F_x'(\theta) = \delta_\theta'(H - F) \in X_\theta$ since $(H - F)(\theta) = 0$. Also,

$$\begin{aligned}\|y - \Omega(x)\|_{X_\theta} &\le \|\delta_\theta'\|\|H - F\| \le \|\delta_\theta'\|\Big(\|H\| + \|F\|\Big)\\ &\le \|\delta_\theta'\|\Big(\|H\| + (1 + \varepsilon)\|x\|_{X_\theta}\Big) \le (2 + \varepsilon)\|\delta_\theta'\|\|H\|,\end{aligned}$$

hence $\|(y, x)\|_\Omega = \|y - \Omega(x)\|_{X_\theta} + \|x\|_{X_\theta} \le (2 + \varepsilon)\|\delta_\theta'\|\|H\| + \|H\|$. □

And now, and this is the main event of the evening, the Kalton–Peck spaces appear as the derived spaces associated with the pair (ℓ_1, ℓ_∞). It is no exaggeration to say that complex interpolation theory is founded on the fact that if we interpolate the pair (ℓ_p, ℓ_q) by the complex method, with both spaces sitting in ℓ_∞ and $1 \le p, q \le \infty$, then $(\ell_p, \ell_q)_\theta = \ell_r$, where $r^{-1} = (1-\theta)p^{-1} + \theta q^{-1}$. Thus, for every $x \in \ell_r$ and every $\varepsilon > 0$, there is an $F_x \in \mathcal{H}(\ell_p, \ell_q)$ such that $F_x(\theta) = x$ with $\|F_x\| \le (1 + \varepsilon)\|x\|_r$. If $q < \infty$ then

$$F_x(z) = x\left(\frac{|x|}{\|x\|_r}\right)^{(r/q - r/p)(z-\theta)}$$

works even for $\varepsilon = 0$. If $q = \infty$, the same extremal can be used when x has finite support, but not in general, because F_x may be discontinuous on the right border of the strip. In any case, we get

$$\Omega(x) = \delta_\theta' F_x = \left(\frac{r}{q} - \frac{r}{p}\right) x \log \frac{|x|}{\|x\|_r} = \left(\frac{r}{p} - \frac{r}{q}\right)\mathsf{KP}(x).$$

Amazing, isn't it? This is the form in which Rochberg and Weiss [405] rediscovered the Kalton–Peck spaces. Many important features of the spaces Z_p can only be properly understood after realising that they arise as derived spaces in an interpolation schema.

10.9 The Properties of Z_2 Explained by Itself

The Kalton–Peck space Z_2 is *the* archetypal twisted Hilbert space, the archetypal twisted sum in fact. And, if twisted Hilbert spaces are the King's Landing of the theory of twisted sums, the space Z_2 sits on the Iron Throne. To study it, let us first formulate properties of general twisted Hilbert spaces before we pass to the specifics of Z_2. Twisted Hilbert spaces of course enjoy all the 3-space properties that Hilbert spaces enjoy. There is no need to make a complete list of such properties; we will just mention a few especially important ones:

⊛ *Twisted Hilbert spaces are ℓ_2-saturated.* That is, every closed infinite-dimensional subspace contains a copy of ℓ_2.

⊛ *Twisted Hilbert spaces are superreflexive.*

⊛ *Twisted Hilbert spaces are near-Hilbert*, i.e. they have type $2-\varepsilon$ and cotype $2+\varepsilon$ for all $\varepsilon > 0$ by Corollary 3.11.4. Near-Hilbert spaces were isolated by Szankowski [447] while studying Banach spaces all of whose subspaces have the approximation property (they must be near-Hilbert). On the other hand, it follows from the Maurey–Pisier Theorem 1.4.10 that near-Hilbert spaces have ℓ_2^n finitely represented in them. Near-Hilbert spaces are meaningful in the twisted context for at least two reasons: (a) the dichotomy theorem 7.2.10 for automorphic spaces – UFO spaces are either $\mathscr{L}_\infty$ or near-Hilbert – and (b) twisted Hilbert spaces are near-Hilbert.

⊛ *Non-trivial twisted Hilbert spaces do not have type* 2 *or cotype* 2 since type 2 spaces cannot contain uncomplemented Hilbert subspaces. Cotype 2 superreflexive spaces have type 2 duals, so the dual space (and thus the starting space) must be Hilbert. In the case of Z_2, this was explicitly shown during the proof of Proposition 3.2.7.

⊛ *Non-trivial twisted Hilbert spaces do not have unconditional bases*. The proof for Z_2 is in [280]. The state-of-the-art general proof is a complicated and not well-understood result of Kalton [268], see also [392], asserting that a twisted Hilbert space with unconditional basis must be Hilbert. It is also remarkable in this context that Z_2 admits an (obvious) unconditional 2-dimensional decomposition. It is not known whether the result remains valid for other p; that is, does there exist a non-trivial twisted sum of ℓ_p with unconditional basis? It can be proven that non-trivial twisted sums of ℓ_p spaces obtained from centralizers cannot have an unconditional basis [86, Theorem 3.9], but it is unknown whether non-trivial exact sequences $0 \longrightarrow \ell_p \longrightarrow \ell_p \longrightarrow \ell_p \longrightarrow 0$ exist for $p \neq 1, 2, \infty$.

⊛ *Twisted Hilbert spaces with additional properties*. The original Enflo–Lindenstrauss–Pisier example [167] has the form $\ell_2(\mathbb{N}, F_n)$, with F_n finite-dimensional. Twisted Hilbert spaces of the form $\ell_2(\mathbb{N}, F_n)$ enjoy property $\mathcal{W}_2$

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[122]: they are reflexive and weakly null sequences admit weakly 2-summable subsequences. Quotient operators $X \longrightarrow \ell_2$ defined on a $\mathcal{W}_2$ space X are 'strictly non-singular', with the meaning that every infinite-dimensional subspace of ℓ_2 contains a further infinite-dimensional subspace on which the quotient map is invertible (after all, weakly 2-summable sequences are exactly the linear continuous images of the canonical basis of ℓ_2). Any twisted Hilbert space with property $\mathcal{W}_2$ thus contains complemented copies of ℓ_2.

The standard quasinorm of Z_2 is $\|(y, x)\| = \|y - \mathsf{KP}x\| + \|x\|$, which is equivalent to a norm because B-convex Banach spaces are $\mathscr{K}$-spaces (Corollary 3.11.3). We have:

★ *Z_2 is isomorphic to its dual.* To be honest, we don't know if this is a property of all twisted Hilbert spaces since all Kalton–Peck spaces $\ell_2(\varphi)$ and actually all twisted Hilbert spaces generated by centralizers have it [63]. This tree will grow in the Fiddler's Green of another book.

★ *The space Z_2 is 'self-similar'.* The feature of the Kalton–Peck maps KP_X formally described before Lemma 9.3.10, that they 'look the same everywhere', seems to be a property peculiar to Z_p. In fact, two-thirds of the authors of this book conjecture that it characterises Z_p. In the particular case $X = \ell_2$, Diagram (9.1) becomes

$$\begin{tikzcd}
0 \arrow[r] & \ell_2 \arrow[r] & Z_2 \arrow[r] & \ell_2 \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[u, equal] & U \oplus_{\mathsf{KP}_U} U \arrow[r] \arrow[u, "T_U"'] & U \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{10.15}$$

The middle operator is $T_U(u_n, 0) = (u_n, 0)$ and $T_U(0, u_n) = (\mathsf{KP}u_n, u_n)$ and is an isometry, as the proof of Lemma 9.3.10 clearly shows. Thus, its range is an isometric copy of Z_2. We show that it is complemented in Z_2. Let $D\colon Z_2 \longrightarrow Z_2^*$ be the isomorphism $D(y, x)(y', x') = \langle y, x'\rangle - \langle x, y'\rangle$ provided at Corollary 3.8.6. Let $D_U\colon U \oplus_{\mathsf{KP}_U} U \longrightarrow (U \oplus_{\mathsf{KP}_U} U)^*$ be the corresponding isomorphism whose action is determined by $\langle D_U(u_i, u_j), (u_k, u_l)\rangle = \delta_{il} - \delta_{jk}$. The diagram

$$\begin{tikzcd}
U \oplus_{\mathsf{KP}_U} U \arrow[r, "T_U"] \arrow[d, "D_U"'] & Z_2 \arrow[d, "D"] \\
(U \oplus_{\mathsf{KP}_U} U)^* & Z_2^* \arrow[l, "T_U^*"']
\end{tikzcd}$$

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is commutative since, for arbitrary $i, j, k, l \in \mathbb{N}$, we have

$$\begin{aligned}(T_U^* D T_U(u_i, u_j))(u_k, u_l) &= \langle D T_U(u_i, u_j), T_U(u_k, u_l)\rangle \\ &= \langle D(u_i + \mathsf{KP}u_j, u_j), (u_k + \mathsf{KP}u_l, u_l)\rangle \\ &= \langle u_i + \mathsf{KP}u_j, u_l\rangle - \langle u_j, u_k + \mathsf{KP}u_l\rangle \\ &= \delta_{il} - \delta_{jk}.\end{aligned}$$

It follows that $D_U^{-1} T_U^* D$ a projection onto the range of T_U.

★ *Operators on* Z_2. The space Z_2 enjoys a surprising '(V)-like' property:

Proposition 10.9.1 *Every operator defined on Z_2 is either strictly singular or an isomorphism on a complemented copy of Z_2.*

Proof Since KP is singular, Lemma 9.1.5 yields that an operator $\tau\colon Z_2 \longrightarrow X$ is strictly singular if and only if its restriction to the canonical copy of ℓ_2 also is. Thus, if τ is not strictly singular then there is a block subspace U of ℓ_2 where the restriction of τ is an isomorphism. Replacing Z_2 by the range of the operator T_U, we can continue with the proof by assuming that U is the whole of ℓ_2, so that the restriction $\tau|_{\ell_2}$ is an embedding, say $\|\tau(y, 0)\| \geq \|y\|$ for all $y \in \ell_2$. Let us stare for few seconds at the pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \ell_2 \arrow[r, equal] \arrow[d, "\imath"] & \ell_2 \arrow[d, "(\tau,\imath)"] & & \\
0 \arrow[r] & Z_2 \arrow[r, "(\tau,\mathbf{1}_{Z_2})"] \arrow[d, "\pi"] & X \oplus Z_2 \arrow[r] \arrow[d, "Q"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ell_2 \arrow[r] \arrow[d] & \mathrm{PO} \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

- The composition $Q\,(\tau, \mathbf{1}_{Z_2})$ is strictly singular since it factors through π.
- $Q\,(\tau, \mathbf{1}_{Z_2}) = Q(\tau, 0) + Q\,(0, \mathbf{1}_{Z_2})$.
- $Q\,(0, \mathbf{1}_{Z_2})$ is an embedding since

$$\begin{aligned}\|Q(0, z)\| &= \inf_{y \in \ell_2} \|(0, z) - (\tau, \imath)(y)\| \\ &= \inf_{y \in \ell_2} \|(-\tau y, z - y)\| \\ &= \inf_{y \in \ell_2} \{\|\tau(y, 0)\| + \|z - y\|\} \geq \|y\| + \|z\| - \|y\| = \|z\|.\end{aligned}$$

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Hence, $Q(\tau, 0)$, being the difference (or sum) between a strictly singular operator and an embedding, has to have closed range and finite-dimensional kernel [334, Proposition 2.c.10]. Therefore it must be an isomorphism on some finite codimensional subspace of Z_2, and the same is true for τ. All subspaces of Z_2 with even codimension are isomorphic to Z_2, and thus we are done. □

Thus, it is not only the quotient map $Z_2 \longrightarrow \ell_2$ that is strictly singular: every operator $Z_2 \longrightarrow \ell_2$ is strictly singular. In particular:

Corollary 10.9.2 *Z_2 does not contain complemented copies of ℓ_2.*

Old Ideas and Popular Problems

- In [266], Kalton used a version of Z_2 to obtain quite natural examples of non-isomorphic complex spaces whose underlying real spaces are isomorphic, thus providing alternative solutions to a problem first solved by Bourgain [50]. Recall that if X is a complex Banach space then the conjugate $\overline{X}$ is the same space X with the 'new' multiplication $cx = \overline{c}x$ for $c \in \mathbb{C}, x \in X$. The spaces X and $\overline{X}$ are always isomorphic as *real* spaces by means of the identity. Given $\alpha \in \mathbb{R}$, Kalton considers the Lipschitz map $\varphi_\alpha(t) = t^{1+i\alpha}$ and then the twisted Hilbert space $Z_2(\alpha)$ generated by the Kalton–Peck map induced by φ_α. It is easy to see that the conjugate of $Z_2(\alpha)$ is $Z_2(-\alpha)$; in particular, the underlying *real* spaces are isomorphic. In a real tour de force, Kalton shows that *if $\alpha \neq \beta$ then the complex spaces $Z_2(\alpha)$ and $Z_2(\beta)$ are not isomorphic*. In particular the complex space $Z_2(\alpha)$ is not isomorphic to $Z_2(-\alpha)$ if $\alpha \neq 0$. The proof is technically demanding. Fortunately, Benyamini and Lindenstrauss [42, Theorem 16.17] come to our aid: yes, their proof is still complicated, but it is far simpler than Kalton's original.
- There are non-separable versions of Z_2 used in [171, p. 576] to show that admitting an injection into Hilbert spaces is not a 3-space property.
- **Is Z_2 prime?** This is unknown [228; 254]. What is known [254] is that complemented subspaces of Z_2 that are isomorphic to their square are isomorphic to Z_2. Perhaps Z_2 fails to be prime for the simplest of reasons.
- **Is Z_2 isomorphic to its hyperplanes?** This is a still open problem that Kalton was very fond of [228]. The common belief is that it is not [228; 254], which would make of Z_2 the first natural Banach space that is not isomorphic to its hyperplanes. The first example was due to Gowers [195]. Codimension 2 subspaces of Z_2 are obviously isomorphic to Z_2.
- **Do hyperplanes of Z_2 admit a complex structure?** Connected to the previous question is Ferenczi's observation behind [96] that, since Z_2 admits

a complex structure, if we could prove that hyperplanes of Z_2 do not admit complex structures then it would follow that Z_2 is not isomorphic to its hyperplanes.

- **New problems.** If, ultimately, the origin of the problems in this book is the question 'is being Hilbert a 3-space property?' then a new problem could be suggested now: 'is being twisted Hilbert a 3-space property?' Unlike the Hilbert question, this one has its origins well documented: it is due to D. Yost, appears posed in [102], is considered in [58] and is negatively solved in [72]. One could write a book about all that.