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ON THE ADJOINT GROUP OF A RADICAL RING

BERNHARD AMBERG AND OLIVER DICKENSCHIED

ABSTRACT. The relations between the adjoint group and the additive group of a radical ring and its nilpotency are investigated. It is shown that certain finiteness conditions carry over from the adjoint group to the additive group and that the converse holds for the class of minimax groups.

1. **Introduction.** A ring R is called *radical* if it coincides with its Jacobson radical, which means that R forms a group under the operation $a \circ b = a + b + ab$ for all a and b in R. This group is called the *adjoint group* R° of R. It is of some interest to study the relation between the adjoint group R° and the additive group R^+ of a radical ring. Watters has shown in [6] that R° satisfies the maximum condition on subgroups if and only if R^+ satisfies the maximum condition on subgroups and that R is a nilpotent ring in this case. In the following we prove a generalization of this result.

A group G is called a *minimax group* if it has a series of finite length whose factors satisfy the minimum or maximum condition on subgroups.

THEOREM A. *Let $\mathfrak{X}$ be a class of minimax groups which is closed under the forming of subgroups, epimorphic images and extensions. If R is a radical ring, then the following conditions are equivalent:*

(i) The additive group R^+ is an $\mathfrak{X}$-group,
(ii) The associated group $G(R)$ is an $\mathfrak{X}$-group,
(iii) The adjoint group R° is an $\mathfrak{X}$-group.
In this case R is a nilpotent ring.

In Theorem A, the class $\mathfrak{X}$ can in particular be the class of minimax groups itself or the class of π-minimax groups for a set of primes π in the sense of [3], Volume 2, p. 167. The *associated group* $G(R)$ of a radical ring R which appears in Theorem A is defined in the following way (see [4] and [1]). The adjoint group $A = R^\circ$ of a radical ring R operates on the additive group $M = R^+$ of R via the rule

$$m^a = m + ma$$

for all $a \in A$, $m \in M$. Let $G(R) = A \ltimes M$ be the semidirect product of A with M and identify A and M with its subgroups $\{(a, 0) \mid a \in R\}$ and $\{(0, m) \mid m \in R\}$, respectively. If $B = \{(r, r) \mid r \in R\}$ is the diagonal subgroup of $G(R)$, then

$$G(R) = A \ltimes M = B \ltimes M = AB \quad \text{and} \quad A \cap B = 1,$$

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where the subgroups A and B are isomorphic with R° and the normal subgroup M of $G(R)$ is isomorphic with R^+. This construction allows us to use results on factorized groups in order to study radical rings.

To explain our second result recall that a group G has *finite torsion-free rank* if it has a finite series whose factors are either periodic or infinite cyclic. The number of infinite cyclic factors in any such series is an invariant of G denoted by $r_0(G)$. The group G has *finite abelian subgroup rank* if each abelian subgroup of G has finite torsion-free rank and each abelian p-subgroup of G has finite Prüfer rank for every prime p. Here a group G is said to have *finite Prüfer rank* $r = r(G)$ if every finitely generated subgroup of G can be generated by r elements, and r is the least positive integer with this property. For the relation between these finiteness conditions see Chapter 6.3 of [3]. Note in particular that any group with finite Prüfer rank has finite abelian subgroup rank.

The additive group of the (commutative) radical subring

$$R = \left\{ \frac{u}{v} \;\middle|\; u \text{ is an even and } v \text{ is an odd integer} \right\}$$

of the ring of rationals has Prüfer rank 1, but R° has infinite torsion-free rank and R is not even a nil ring. Hence there is no analogue of Theorem A for the above finiteness conditions. However, the following theorem shows that these finiteness conditions are inherited from the adjoint group R° of a radical ring R to its additive group R^+ and that they imply some nilpotency conditions of the ring R.

THEOREM B. *Let R be a radical ring. Then the following holds.*

(a) *If R° has finite torsion-free rank n, then also $r_0(R^+) = n$, and R is a nil ring.*
(b) *If R° has finite abelian subgroup rank, then so does R^+, and R is a two-sided T-nilpotent ring of class* $\mathrm{cl}(R) \leq \omega + r_0(R^+)$.
(c) *If R° has finite Prüfer rank, then so does R^+, and $r(R^+)$ is bounded by a function only depending on $r(R^\circ)$.*

The definition of the ring-theoretical terms used in Theorem B is as follows. The *transfinite two-sided annihilator series* $\big(\mathbf{B}_\alpha(R)\big)_\alpha$ of a ring R is defined by

$$\mathbf{B}_0(R) = 0,$$
$$\mathbf{B}_{\alpha+1}(R) = \{a \in R \mid aR + Ra \subseteq \mathbf{B}_\alpha(R)\}$$

for each ordinal number α and

$$\mathbf{B}_\lambda(R) = \bigcup_{\alpha<\lambda} \mathbf{B}_\alpha(R)$$

for each limit ordinal λ. The ring R is called *two-sided T-nilpotent* of *class* $\mathrm{cl}(R) = \alpha$ if $\mathbf{B}_\alpha(R) = R$ and if α is the least ordinal with this property. It is easy to see that the ring R satisfies $\mathbf{B}_n(R) = R$ for some finite ordinal n if and only if $R^{n+1} = 0$.

The other notation is standard and can for instance be found in [2] and [3].

2. **Proof of Theorem A.** The following lemma will be used frequently.

LEMMA 2.1. *Let R be a nilpotent ring and $\mathfrak{X}$ be a class of groups which is closed under the forming of subgroups, epimorphic images and extensions. Then the adjoint group R° of R is an $\mathfrak{X}$-group if and only if the additive group R^+ of R is an $\mathfrak{X}$-group.*

PROOF. Let $R^{n+1} = 0$. Then each factor ring R^i/R^{i+1} of the chain

$$0 = R^{n+1} \subseteq R^n \subseteq \cdots \subseteq R^2 \subseteq R^1 = R$$

has trivial multiplication and hence $(R^i/R^{i+1})^\circ = (R^i/R^{i+1})^+$. Thus the result follows by the hypotheses on the class $\mathfrak{X}$. ∎

The *radical join* of a subset X of a radical ring R is the intersection of all radical subrings of R which contain X. Clearly the radical join of X is likewise a radical ring. The following extension of Lemma 3 of [6] will be essential.

LEMMA 2.2. *Let the ring R be the radical join of an element a in R. Then the following holds:*

(a) R is commutative.

(b) If the group R° has finite torsion-free rank, then R is a nilpotent ring.

PROOF. (a) The centralizer $C = \{r \in R \mid ra = ar\}$ of a in R is a radical subring of R containing a, so that $C = R$. Thus a is contained in the center $Z(R) = \{r \in R \mid rs = sr$ for all $s \in R\}$ of R. As $Z(R)$ is a radical subring of R, it follows that $R = Z(R)$ is commutative.

(b) This runs along the same lines as the proof of Lemma 3 of [6], replacing 'finitely generated' by 'finite torsion-free rank'. ∎

COROLLARY 2.3. *Let $\mathfrak{X}$ be one of the finiteness conditions 'minimax', 'finite Prüfer rank', 'finite abelian subgroup rank' or 'finite torsion-free rank'. Then any radical ring R with $R^\circ \in \mathfrak{X}$ is a nil ring.*

PROOF. This follows from Lemma 2.2 by considering the radical join of every element a of R. ∎

The next lemma establishes a relation between the additive and the adjoint group of a nil ring.

LEMMA 2.4. *If R is a nil ring and p a prime, then the following holds:*

(a) R^+ is a p-group if and only if R° is a p-group.

(b) R^+ is torsion-free if and only if R° is torsion-free.

PROOF. (a) For an arbitrary element a of R, the radical join S of a is commutative by Lemma 2.2. Since R is a nil ring, we have $a^n = 0$ for some positive integer n. Hence $\mathbb{Z} \cdot a + Sa$ is a nilpotent subring of S containing a, so that

$$S = \mathbb{Z} \cdot a + Sa$$

is nilpotent. Thus by Lemma 2.1, S^+ is a p-group if and only if S° is a p-group. Hence (a) follows.

(b) Let R^+ be torsion-free and assume that R° contains a non-trivial element of finite order. Then R° contains an element a of prime order p. Using a formal identity element 1 this implies that

$$1 = (1+a)^p = 1 + pa\sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}a^{i-1} + a^p,$$

where the coefficients $\frac{1}{p}\binom{p}{i}$ for $1 \le i < p$ are integers. It follows that

$$a^p = -pa(1+r)$$

for the element $r = \sum_{i=2}^{p-1}\frac{1}{p}\binom{p}{i}a^{i-1}$. Since R is a nil ring, there exists a non-negative integer n such that $a^{p^n} = 0$. Let n be minimal with this property. Then $n \ge 1$ and as a and r commute, it follows that

$$0 = a^{p^n} = \left(-pa(1+r)\right)^{p^{n-1}} = (-p)^{p^{n-1}}a^{p^{n-1}}(1+r)^{p^{n-1}}.$$

Since $(1+r)^{p^{n-1}}$ is invertible, we obtain $(-p)^{p^{n-1}}a^{p^{n-1}} = 0$. Hence

$$a^{p^{n-1}} = 0,$$

since R^+ is torsion-free. But this contradicts the minimal choice of n. Thus if R^+ is torsion-free, then so is R°. The converse follows from (a) by considering the ideal T formed by the torsion subgroup of R^+. The lemma is proved. ∎

REMARK 2.5. Let R be a radical ring. Then the torsion subgroup T of R^+ and its primary components T_p form ideals of R for each prime p. The natural decomposition

$$T^+ = \bigoplus_p {T_p}^+$$

of the additive group T^+ is also a decomposition

$$T = \bigoplus_p T_p$$

of the ring T as a direct sum of the ideals T_p and hence a decomposition

$$T^\circ = \bigotimes_p {T_p}^\circ$$

of the adjoint group T° as a direct product of the groups ${T_p}^\circ$.

If in this situation R is a nil ring (in particular if R° belongs to one of the classes listed in Corollary 2.3), then T° is a torsion group, each ${T_p}^\circ$ is a p-group and $R^\circ/T^\circ \cong (R/T)^\circ$ is a torsion-free group by Lemma 2.4.

Using this decomposition, we can reduce the study of radical rings R with finiteness conditions on R° to the investigation of such rings with R^+ and R° being either both torsion-free or both p-groups for some prime p. The first case is treated in the following lemma.

LEMMA 2.6. *Let R be a locally nilpotent ring with a torsion-free additive group. If the adjoint group R° of R has finite torsion-free rank, then R is a nilpotent ring.*

PROOF. Since R is a locally nilpotent ring, R° is a locally nilpotent group. By Lemma 2.4, R° is torsion-free of finite torsion-free rank and hence is nilpotent; see [3], Volume 2, Corollary 2 on p. 38. We use induction on $r = r_0(R^\circ)$, the result being clear if this number is zero. Suppose now that $r > 0$ and that the lemma holds for any locally nilpotent ring whose adjoint group has finite torsion-free rank less than r. Since R° is nilpotent, it contains a non-trivial central element a which hence belongs to the center of the ring R. As the element a of the locally nilpotent ring R is nilpotent, the ideal I of R generated by a is nilpotent, and so it remains to prove that R/I is likewise nilpotent. Since I° is a non-trivial normal subgroup of the torsion-free group R°, it has positive torsion-free rank. Thus $(R/I)^\circ \cong R^\circ/I^\circ$ has finite torsion-free rank less than r and the result follows from our inductive hypothesis. ∎

PROOF OF THEOREM A. Suppose first that the additive group R^+ of the radical ring R is an $\mathfrak{X}$-group. We have to prove that the group $G(R)$ is an $\mathfrak{X}$-group. Since $G(R) = A \ltimes M$ with $A \cong R^\circ$ and $M \cong R^+$, we only need to show that R° is an $\mathfrak{X}$-group. By Lemma 2.1 it suffices to show that the ring R is nilpotent. The torsion subgroup of R^+ obviously is an ideal T of R whose additive group T^+ is a periodic abelian minimax group and hence satisfies the minimum condition on subgroups. Thus T is a right artinian radical ring and so is nilpotent; see [2], Proposition 3.5.1. Therefore it suffices to show that R/T is nilpotent. Hence we may assume that R^+ is a torsion-free abelian minimax group.

For every prime p, the subring pR is an ideal of R. Since the Prüfer rank n of R^+ is finite, it follows that

$$|R/pR| \leq p^n.$$

In particular, R/pR is a nilpotent ring. Since the integer p^n has only $n+1$ positive divisors, we have $(R/pR)^{n+1} = 0$, so that $R^{n+1} \subseteq pR$ for any prime p. Hence

$$R^{n+1} \subseteq \bigcap_p pR = 0,$$

since R^+ is a torsion-free abelian minimax group (see [1], Lemma 6.6.3). This shows that R is a nilpotent ring and condition (ii) follows. Obviously (ii) implies (iii).

Suppose now that the adjoint group R° of the radical ring R is an $\mathfrak{X}$-group. It has to be shown that (i) holds. By Lemma 2.1, it suffices to show that R is a nilpotent ring. First, let the ring R be locally nilpotent. Then the additive group T^+ of the ideal T defined in Remark 2.5 is periodic, so that also T° is periodic. Thus every abelian subgroup of T° is a periodic minimax group and hence satisfies the minimum condition on subgroups. As R is a locally nilpotent ring, R° is a locally nilpotent group. Thus T° satisfies the minimum condition on subgroups; see [3], Volume 1, Theorem 3.32. Hence T is a right artinian radical ring and so is nilpotent by [2], Proposition 3.5.1. Now R/T is a locally nilpotent ring whose additive group is torsion-free and whose adjoint group is a locally nilpotent minimax group and hence is nilpotent of finite Prüfer rank by [3], Volume 2, Corollary 2 on p. 38. In particular $(R/T)^{\circ}$ has finite torsion-free rank, so that the ring R/T is nilpotent by Lemma 2.6. As T is likewise nilpotent, it follows that R is nilpotent, which completes the proof of the special case.

Consider now the general case. By Zorn's Lemma there exists a maximal locally nilpotent subring S of R, which is nilpotent by the special case. Assume now that $S \neq R$. Then it follows from a result of Szász ([5], Theorem 6), that the nilpotent subring S is properly contained in its idealizer

$$I = \{r \in R \mid rS + Sr \subseteq S\}.$$

Hence there exists an element a in the subring I of R which is not in S. The subring $\hat{S}$ generated by $S \cup \{a\}$ is contained in the idealizer I of S, and therefore S is an ideal of $\hat{S}$. The quotient ring $\hat{S}/S$ is generated by $a + S$. Since R is nil by Corollary 2.3, it follows that $\hat{S}/S$ is nilpotent. Hence $\hat{S}$ is a nilpotent subring of R containing S properly. This contradiction shows that $R = S$ is nilpotent. The proof of Theorem A is completed. ∎

3. **Proof of Theorem B.** Together with Lemma 2.1, the following result establishes a bound for the nilpotency class of the rings considered in Lemma 2.6.

LEMMA 3.1. *If R is a locally nilpotent ring whose additive group R^+ is torsion-free with finite torsion-free rank n, then $R^{n+1} = 0$.*

PROOF. Choose $n + 1$ arbitrary elements $x_1, \ldots, x_{n+1}$ of R and let S be the subring of R generated by them. Then S is nilpotent by hypothesis and it follows that the set $\mathcal{M}$ of all products of finitely many elements of $\{x_1, \ldots, x_{n+1}\}$ is finite. Since S^+ is the additive join of $\mathcal{M}$, it is a free abelian group of finite rank $r \leq n$.

By considering the factor rings S/pS for all primes p, it follows as in the proof of Theorem A, that

$$S^{n+1} \subseteq S^{r+1} \subseteq \bigcap_p pS = 0.$$

Hence $x_1 \cdot \cdots \cdot x_{n+1} = 0$, which proves that $R^{n+1} = 0$. ∎

The next result is mentioned in [4].

LEMMA 3.2. *The radical ring R is nilpotent if and only if the group $G(R)$ is nilpotent.*

PROOF. First suppose that R is a nilpotent ring. Then some simple calculations in the group $G(R)$ yield the commutator rule

$$[(a,b),(x,y)] = (r,s) \quad \text{for all } (a,b),(x,y) \in G(R),$$

where

$$r = (1 + a')(1 + x')(ax - xa)$$

and

$$s = bx - ya - (y + b + ya)r;$$

here a' denotes the adjoint inverse of a in R°.

Suppose that (a,b) lies in $G(R^m)$ for some positive integer m. Then a and b are elements of R^m and hence $r \in R^{m+1}$. This implies that also $s \in R^{m+1}$ and hence $(r,s) \in G(R^{m+1})$.

Thus, using induction on m, we obtain

$$\gamma_m G(R) \subseteq G(R^m)$$

for all positive integers m, where $\gamma_m G(R)$ denotes the m-th term of the lower central series of $G(R)$. Since R is nilpotent, it follows that $\gamma_n G(R) = 1$ for some n and thus $G(R)$ is nilpotent.

Now suppose that $G(R)$ is a nilpotent group. Then there is a positive integer n such that $[g_1,\ldots,g_n] = 1$ for all $g_i \in G(R)$. Given arbitrary $r_1,\ldots,r_n \in R$, choose $g_1 = (0,r_1)$ and $g_i = (r_i,0)$ for all $i > 1$. Then it follows by induction on i that

$$[g_1,\ldots,g_i] = (0, r_1 \cdot \cdots \cdot r_i) \quad \text{for each } i \leq n.$$

In particular

$$(0,0) = 1 = [g_1,\ldots,g_n] = (0, r_1 \cdot \cdots \cdot r_n).$$

Thus R is a nilpotent ring. ∎

PROOF OF THEOREM B. To prove (a) suppose that the adjoint group R° of the radical ring R has finite torsion-free rank. Then R is a nil ring by Corollary 2.3. As in Remark 2.5, let T be the ideal formed by the torsion subgroup of R^+, so that $(R/T)^+$ is a torsion-free group. By Zorn's Lemma, there exists a maximal locally nilpotent subring S of R/T. Then the additive group S^+ is torsion-free, while S° has finite torsion-free rank. Hence S is a nilpotent ring by Lemma 2.6. As in the proof of Theorem A, this implies that $S = R/T$ is a nilpotent ring. Since the torsion-free rank is additive on extensions, it follows as in the proof of Lemma 2.1 that the torsion-free ranks of $(R/T)^+$ and $(R/T)^\circ$ coincide. Since T° is a torsion group by Lemma 2.4, we obtain

$$r_0(R^+) = r_0\big((R/T)^+\big) = r_0\big((R/T)^\circ\big) = r_0(R^\circ/T^\circ) = r_0(R^\circ).$$

This proves (a).

To prove (b) let the adjoint group R° of the radical ring R have finite abelian subgroup rank. We first treat the special case that R is a locally nilpotent ring. Consider the ideals T and T_p (for all primes p) which were defined in Remark 2.5. Then the groups $(R/T)^+$ and $(R/T)^\circ$ are torsion-free, each ${T_p}^+$ and each ${T_p}^\circ$ is a p-group and

$$T = \bigoplus_p T_p.$$

Since R is locally nilpotent, it follows that $(R/T)^\circ$ is a torsion-free locally nilpotent group of finite abelian subgroup rank and hence is nilpotent of finite Prüfer rank; see [3], Volume 2, Theorem 6.36. In particular, $(R/T)^\circ$ has finite torsion-free rank, and thus R/T

is nilpotent by Lemma 2.6. Hence $r_0\big((R/T)^+\big)$ is finite by Lemma 2.1. Now Lemma 3.1 yields

$$l = \mathrm{cl}(R/T) \le r_0\big((R/T)^+\big) = r_0(R^+/T^+) = r_0(R^+)$$

for the nilpotency class l of R/T. For every prime p the primary p-component $T_p{}^\circ$ of R° is a Chernikov-group (see [3], Volume 2, Corollary 1 on p. 38), so that the ideal T_p of R is nilpotent by Theorem A. Therefore $\mathbf{B}_\omega(T) = T$. By induction on n, it follows that

$$\mathbf{B}_{2ln}(R) \supseteq \mathbf{B}_n(T)$$

for every integer $n \ge 1$. Forming unions, we obtain $\mathbf{B}_\omega(R) \supseteq \mathbf{B}_\omega(T) = T$. Thus $R = \mathbf{B}_{\omega+l}(R)$. Moreover, since each $T_p{}^+$ has finite abelian subgroup rank by Lemma 2.1, it follows that T^+ has likewise finite abelian subgroup rank. As R/T is nilpotent, the group $R^+/T^+ = (R/T)^+$ also has finite abelian subgroup rank by Lemma 2.1, and hence so does R^+. This proves (b) for the special case of a locally nilpotent ring.

In the general case let S be a maximal locally nilpotent subring of R. Then S is two-sided T-nilpotent by the special case, and it follows as in the proof of Theorem A that $S = R$. Thus (b) is proved.

To prove (c) let R be a radical ring whose adjoint group has finite Prüfer rank r. Then R is two-sided T-nilpotent by (b) and hence locally nilpotent. As in the proof of (b) the rings R/T and T_p (for each prime p) are nilpotent. As each $T_p{}^\circ$ has finite Prüfer rank $r(T_p{}^\circ) \le r$, also each $T_p{}^+$ has finite Prüfer rank by Lemma 2.1. It follows that the groups

$$G_p = G(T_p) = A_p \ltimes M_p = B_p \ltimes M_p = A_pB_p$$

have finite Prüfer ranks. Now each group G_p is nilpotent by Lemma 3.2. Hence by a theorem of Zaitsev and Robinson (see [1], Theorem 4.3.5), there is an integer k only depending on r such that $r(G_p) \le k$ for all primes p. Hence

$$r(T_p{}^+) = r(M_p) \le r(G_p) \le k$$

for all primes p. Since the group T^+ is the direct sum of the groups $T_p{}^+$, it has finite Prüfer rank. As the ring R/T is nilpotent, the group R^+/T^+ has likewise finite Prüfer rank by Lemma 2.1, so that also R^+ has finite Prüfer rank. Thus the associated group $G(R)$ of R has finite Prüfer rank. As the ring R is locally nilpotent, the group $G(R)$ is locally nilpotent by Lemma 3.2. Applying the theorem of Zaitsev and Robinson to $G(R)$, we obtain that $r(R^+)$ is bounded by a function of $r(R^\circ)$. The theorem is proved. ∎

Finally we mention a simple example of a radical ring R which is not nilpotent, but the two groups R^+ and R° are both abelian torsion groups with Prüfer rank 1. This shows that the bound given in Theorem B(b) can not be replaced by a finite one.

EXAMPLE. Let $(p_n)_n$ be a strictly increasing series of odd primes and let $R_n = p_n\mathbb{Z}/p_n{}^n\mathbb{Z}$ for every integer $n > 0$. Then the ring

$$R = \bigoplus_n R_n$$

is obviously two-sided T-nilpotent and its additive group R^+ is the direct sum of cyclic p_n-groups and hence has Prüfer rank 1. Since each group $R_n{}^\circ$ is isomorphic with a subgroup of the group of units of the ring $\mathbb{Z}/p_n{}^n\mathbb{Z}$, which is cyclic, it follows that R° has likewise Prüfer rank 1. But obviously, R is not a nilpotent ring, so that $\mathrm{cl}(R) = \omega$.

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Fachbereich Mathematik
der Universität Mainz
D–55099 Mainz
Germany
e-mail: amberg@mzdmza.zdv.uni-mainz.de
e-mail: dickenschied@mat.mathematik.uni-mainz.de