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ON SEMIGROUPS OF ENDOMORPHISMS OF BIREGULAR ALGEBRAS

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Abstract

Let A be a finite dimensional algebra over a field F. Let R and S be biregular algebras over F such that $1_R \in R$ and $1_S \in S$. We show that if $R/P \simeq A \simeq S/M$ for each primitive ideal P in A and each primitive ideal M in S then $\mathrm{End}_F R \simeq \mathrm{End}_F S$ implies $R \simeq S$.

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1. Introduction

Magill (1964) showed that two Boolean rings R and S are isomorphic if and only if their respective semigroups of ring endomorphisms End R and End S are isomorphic. One kind of generalized Boolean rings is the kind of so-called p^k-rings (Foster). Let p be a prime integer and k a positive integer. A ring R is said to be a *p^k-ring* (*Foster*) if (i) $1_R \in R$, (ii) $x^{p^k} = x$ for all x in R, (iii) R has at least one subring F isomorphic to the Galois field GF(p^k) and (iv) $1_R \in F$. A subring F of R satisfying (iii) and (iv) is said to be a *normal subfield* of R. Note that if F is a normal subfield of R then R is an algebra over F. Luh and Smith (1974) showed that if R and S are p^k-rings (Foster) with normal subfields F and G respectively and if their respective semigroups of algebra endomorphisms $\mathrm{End}_F R$ and $\mathrm{End}_G S$ are isomorphic then R and S are isomporhic as rings. In this paper, we generalize their result to a class of biregular rings. A question raised by Luh and Smith is 'If R and S are p^k-rings (Foster) and their respective semigroups of ring endomorphisms End R and End S are isomorphic, does that imply R and S are isomorphic?'. We show, in this paper, the answer is affirmative in a more general setting.

2. Preliminaries

Let R be a ring and $\mathscr{P}$ the set of primitive ideals in R. Let A be any subset of $\mathscr{P}$. Set $\mathscr{D}_A = \cap\{P \mid P \in A\}$ and define $\mathscr{C}\ell A = \{P' \mid P' \in \mathscr{P}, P' \supseteq \mathscr{D}_A\}$. Then $\mathscr{P}$ is a topological space under the closure operator $\mathscr{C}\ell$. The topological space thus obtained will be called the *structure space* of R. If we use the symbol X to denote the structure space of R then $\mathscr{P} = \{P_x \mid P_x \text{ is primitive, } x \in X\}$.

A ring R is said to be *biregular* if every principal ideal in R is generated by a central idempotent. Let R and A be rings. If R is a left A-algebra and if R has an identity 1_R then the mapping $a \to a1_R$ is a homomorphism of A into R. We shall call this the *natural homomorphism* of A into R.

If R is a biregular ring which is a left A-algebra, every r in R has the form $r = er$ where e is a central idempotent. Then $ar = (ae)r$, $a \in A$. This shows that every ideal I of R is an A-ideal. Hence I and R/I are left A-algebras where R/I is the residue class ring $\{r+I \mid r \in R\}$.

We shall call a topological space X to be *totally disconnected* if any pair of points in X can be separated by two complementary closed sets in X.

LEMMA 2.1. *Let R be a ring, A a simple ring with an identity element. Then the following are necessary and sufficient conditions that R be isomorphic to the ring of continuous functions with compact supports on a locally compact totally disconnected space to A (which is considered to be a discrete space):*

(1) *R is biregular.*

(2) *R is a left A-algebra.*

(3) *For each primitive ideal P_x in R the natural homomorphism of A into the residue class ring (with identity) R/P_x is an isomorphism onto R/P_x.*

PROOF. See Jacobson (1968), p. 214.

Let R and A be as in 2.1, X the structure space of R, and $C(X, A)$ the ring of continuous functions of X into the discrete space A. For $f \in R$ and P_x a primitive ideal in R, we may write $f + P_x = a_x(1_x + P_x)$, where $a_x \in A$, and define f': $X \to A$ by $f'(x) = a_x$. Then $f' \in C(X, A)$. The correspondence $f \to f'$ is in fact the isomorphism of R onto $C(X, A)$ in 2.1. In the sequel we shall identify f with f' and write $f(x) = a_x$.

LEMMA 2.2. *Let R be a biregular algebra over a field F. If R has an identity and every primitive image R/P_x of R is isomorphic to a finite dimensional algebra A over F, then*

(1) *the structure space X of R is compact and totally disconnected,*

(2) *R contains a subalgebra* $\bar{A}$ *which is isomorphic to A over F such that* $\bar{A}+P_x = R$ *for every primitive ideal* P_x *of R, and*

(3) *R is isomorphic to* $C(X, A\bar{A})$ *the ring of continuous functions from X into the discrete space* $\bar{A}$.

PROOF. See Jacobson (1968), p. 215.

Note that the $\bar{A}$ in the above lemma is an algebra over the field F and $1_R \in \bar{A}$. The existence of such an $\bar{A}$ implies (2) and (3) of 2.1. We shall call it a *normal subalgebra* of R. If $\bar{a} \in \bar{A}$ then $\bar{a}+P_x = \bar{a}(1_R+P_x)$ for every primitive ideal P_x in R. Therefore $\bar{a}(x) = \bar{a}$ for every x in X. This also shows that $1_{\bar{A}} = 1_R$. For clarification, we shall use the symbol a to denote $\bar{a}(x)$ and σ_R the identity mapping $a \to \bar{a}$ from $\bar{A}$ into R. Also note that $\bar{A}$ is primitive since $\bar{A} \simeq R/P_x$. Consequently $\bar{A}$ is simple since R/P_x is biregular and $1_A \in \bar{A}$.

Let X be a topological space. Then the set of continuous selfmaps is a semigroup under operation composition. This semigroup will be denoted by $S(X)$.

LEMMA 2.3. *Let X and Y be compact and totally disconnected spaces. Then X is homeomorphic to Y if and only if* $S(X)$ *is isomorphic to* $S(Y)$.

PROOF. See Magill (1970), p. 987.

Note that if X is a compact and totally disconnected space then X has a base of open and closed sets by Simmons (1963), Theorem 33.C.

3. Endomorphisms and continuous selfmaps

Throughout this and the next sections, R will denote a biregular algebra over a field F such that $1_R \in R$ and every primitive image of R is isomorphic to a fixed finite dimensional algebra A over F, End R the semigroup of ring endomorphisms of R, $\mathrm{End}_{\bar{A}} R$ the semigroup of (left) algebra endomorphisms of R over $\bar{A}$, $\mathrm{End}_{\bar{A}}^1 R = \{\psi \in \mathrm{End}_{\bar{A}} R \mid \psi(\bar{1}_R) = \bar{1}_R\}$, ψ_0 the zero endomorphism, ψ_1 the identity endomorphism, X the structure space of R, b^{-1} the multiplicative inverse of b in an algebraic structure, $f^{\leftarrow}$ the inverse of a mapping f, and $\varphi g = \varphi(g)$ for g in the domain of a mapping φ.

LEMMA 3.1. *Let* $\tau \in S(X)$ *then there exists a unique* $\tau' \in \mathrm{End}_{\bar{A}}^1 R$ *such that* $\tau' f = f \circ \tau$ *for every f in R.*

PROOF. Let $\tau \in S(X)$ and define $\tau' \colon R \to R$ by $\tau' f = f \circ \tau$. Let $\bar{a} \in \bar{A}$ then

$$(\tau' \bar{a} f)(x) = (\bar{a} f \circ \tau)(x) = \bar{a} f(\tau x) = \bar{a}(\tau x) f(\tau x) = a f(\tau x) = \bar{a}(x)(f \circ \tau)(x)$$

$$= (\bar{a} \tau' f)(x) \quad \text{and} \quad \tau'(\bar{1}_R) = \bar{1}_R \circ \tau = \bar{1}_R.$$

If $\tau' = \sigma'$ then $f(\tau x) = f(\sigma x)$ for all f in R and all x in X. But X is totally disconnected, so $\tau x = \sigma x$ for every x in X. It is routine to show $\tau' \in \operatorname{End} R$. Hence $\tau' \in \operatorname{End}_{\bar{A}}^1 R$ and is unique.

LEMMA 3.2. *Let $\psi \in \operatorname{End}_{\bar{A}}^1 R$ and $\psi[R] = \bar{A}$. Then there exists a unique y in X such that $\sigma_R^{\leftarrow}(\psi f) = f(y)$ for every f in R.*

PROOF. Let $P_y = \ker \psi$. Since $\bar{A}$ is simple, P_y is a primitive ideal in R. Let $f \in R$ then $f + P_y = a_y(\bar{1}_R + P_y) = \bar{a}_y + P_y$ and $f(y) = a_y = \sigma_R^{\leftarrow}(\bar{a}_y) = \sigma_R^{\rightarrow}(\psi \bar{a}_y) = \sigma_R^{\leftarrow}(\psi f)$ for some $a_y \in \bar{A}$ since $\psi \in \operatorname{End}_{\bar{A}}^1 R$.

LEMMA 3.3. *Let Φ be a family of mappings that determines the topology of a space Y. A mapping σ from a space S into Y is continuous if and only if $\varphi \circ \sigma$ is continuous for every φ in Φ.*

PROOF. See Gillman and Jerison (1960), p. 42.

LEMMA 3.4. *Let $\psi \in \operatorname{End}_{\bar{A}}^1 R$. Then there exists a unique $\tau \in S(X)$ such that $\psi f = f \circ \tau$ for every f in R.*

PROOF. Let $x \in X$ and define $\alpha \colon R \to \bar{A}$ by $\alpha f = \sigma_R(\psi f(x))$. Clearly $\alpha \in \operatorname{End} R$. Since

$$\alpha \bar{1}_R = \sigma_R(\psi \bar{1}_R(x)) = \sigma_R(\bar{1}_R x) = \sigma_R(1_{\bar{A}}) = \bar{1}_R,$$

and since

$$\alpha(\bar{a} g) = \sigma_R(\psi \bar{a} g(x)) = \sigma_R((\bar{a} \psi g)(x)) = \sigma_R(\bar{a}(x) \psi g(x))$$

$$= \sigma_R(a \psi g(x)) = \bar{a} \sigma_R(\psi g(x)) = \bar{a} \alpha g$$

for every g in R, we have $\alpha \in \operatorname{End}_{\bar{A}}^1 R$. It follows that there exists a unique y in X such that $\sigma_R^{\leftarrow}(\alpha g) = g(y)$ for every g in R by 3.2. Define $\tau \colon X \to X$ by $\tau x = y$. Then $g(\tau x) = g(y) = \sigma_R^{\leftarrow}(\alpha g) = \sigma_R^{\leftarrow}(\sigma_R(\psi g(x))) = \psi g(x)$ for every g in R and every x in X. Since $\psi g \in R$ for all g in R and since X is compact and totally diconnected, we have $\tau \in S(X)$ by 3.3.

LEMMA 3.5. $\operatorname{End}_{\bar{A}}^1 R$ *is anti-isomorphic to* $S(X)$.

PROOF. Define π: $S(X) \to \mathrm{End}_{\bar{A}}^{1} R$ by $\pi(\tau) = \tau'$ where $\tau' f = f \circ \tau$ for every f in R. The π thus defined is a one-to-one mapping from $S(X)$ onto $\mathrm{End}_{\bar{A}}^{1} R$ by 3.1 and 3.4. Let τ, θ be in $S(X)$. Then

$$(\pi(\tau\theta))(f) = f \circ (\tau\theta) = (f \circ \tau)(\theta) = \theta'(f \circ \tau) = (\pi\theta)(\tau' f) = (\pi\theta)(\pi\tau)(f)$$

for every f in R. Therefore π is a semigroup anti-isomorphism from $S(X)$ onto $\mathrm{End}_{\bar{A}}^{1} R$.

4. Semigroups of endomorphisms of a biregular algebra

Throughout this section, S will denote a biregular algebra over the field F such that $1_S \in S$ and every primitive image of S is isomorphic to A, $\bar{B}$ a normal subalgebra of S, $\mathrm{End}_{\bar{B}}^{1} S = \{\varphi \in \mathrm{End}_{\bar{B}} S \varphi(\bar{1}_S) = \bar{1}_S\}$, φ_0 the zero endomorphism of S, φ_1 the identity endomorphism of S, and Y the structure space of S.

THEOREM 4.1. *If* $\mathrm{End}_{\bar{A}} R \simeq \mathrm{End}_{\bar{B}} S$ *then* $R \simeq S$.

PROOF. Let π be an isomorphism from $\mathrm{End}_{\bar{A}} R$ onto $\mathrm{End}_{\bar{B}} S$. Let $\psi \in {}_{\bar{A}}^{1} R$ and $\varphi = \pi\psi$. If $\varphi(\bar{1}_S) \neq \bar{1}_S$ then $\varphi(\bar{1}_S)$ is an idempotent in S. Let $d = \varphi(\bar{1}_S)$ then d is nonzero since π is an isomorphism. Let (d) be the principal ideal generated by d then $(d) = (e)$ for a central idempotent e in S since S is biregular. Define φ_{1-e}: $S \to S$ by $\varphi_{1-e}(s) = (\bar{1}_S - e)s$. Then $\varphi_{1-e} \in \mathrm{End}_{\bar{B}} S$ since $\bar{1}_S - e$ is a central idempotent. Since $\varphi_{1-e}(e) = (\bar{1}_S - e)e = e - e^2 = \bar{0}_S$. So $(e) \subseteq \ker \varphi_{1-e}$. Therefore $\varphi_{1-e}\varphi = \varphi_0$. But $\psi_0 = \pi^{\leftarrow}(\varphi_0) = \pi^{\leftarrow}(\varphi_{1-e})\psi$ and $\psi\bar{1}_R = \bar{1}_R$, so $\pi^{\leftarrow}(\varphi_{1-e})(\bar{1}_R) = \psi_0 \bar{1}_R = 0_R$ and hence $\pi^{\leftarrow}(\varphi_{1-e}) = \psi_0$. This is a contradiction since π is an isomorphism. Therefore $\varphi \in \mathrm{End}_{\bar{B}} S$. Thus $\mathrm{End}_{\bar{A}}^{1} R$ is isomorphic to $\mathrm{End}_{\bar{B}}^{1} S$. Which implies that $S(X)$ is isomorphic to $S(Y)$ by 3.5. It follows that X is homeomorphic to Y by 2.3. Hence $R \simeq S$ by 2.2 and since $\bar{A}$ is homeomorphic as well as isomorphic to $\bar{B}$.

Let T be a p^k-ring (Foster). Then T may be viewed as an algebra over the Galois field GF(p). Since $t^{p^k} = t$ for every t in T, so T has no nonzero nilpotent elements. Therefore T is biregular by Jacobson (1968), Proposition 1, p. 210. Let K be a normal subfield of T. If M is a maximal ideal in T then T/M contains a copy of K since K is a field and $1_T \in K$. Since T/M is a field and $(t+M)^{p^k} = t+M$ and so T/M is isomorphic to K which is isomorphic to GF(p^k). Thus Theorem 4.1 is a generalization of the theorem of Luh and Smith.

For the rest of this section, π will denote an isomorphism of $\mathrm{End}_F R$ onto $\mathrm{End}_F S$. We shall show that if $\mathrm{End}_F R \simeq \mathrm{End}_F S$ then $R \simeq S$.

LEMMA 4.2. $\pi\psi_0 = \varphi_0$, $\pi\psi_1 = \varphi_1$.

LEMMA 4.3. *Let* $\psi \in \mathrm{End}_F R$. *Then* $\psi(\bar{1}_R) = \bar{1}_R$ *if and only if* $(\pi\psi)(\bar{1}_S) = \bar{1}_S$.

PROOF. See the proof of 4.1.

LEMMA 4.4. *Let* $\psi_p \in \mathrm{End}_{\bar{A}}{}^1 R$ *such that* $\psi_p[R] = \bar{A}$. *Then* $\psi\psi_p = \psi_p$ *for every* $\psi \in \mathrm{End}_{\bar{A}}{}^1 R$. *Furthermore, if* $\varphi_p = \pi\psi_p$ *then* $\varphi\varphi_p = \varphi_p$ *for every* $\varphi \in \pi[\mathrm{End}_{\bar{A}}{}^1 R]$ *and* $\varphi_p^2 = \varphi_p$.

PROOF. Since $\psi(\bar{1}_R) = \bar{1}_R = \psi_p(\bar{1}_R)$ and ψ, $\psi_p \in \mathrm{End}_{\bar{A}} R$, so $\psi|_{\bar{A}} = \psi_p|_{\bar{A}} = id$ (the identity function on $\bar{A}$). But $\psi_p[R] = \bar{A}$, so $\psi\psi_p = \psi_p$. Now if $\varphi = \pi\psi$ for some $\psi \in \mathrm{End}_{\bar{A}} R$ then $\varphi\varphi_p = (\pi\psi)(\pi\psi_p) = \pi(\psi\psi_p) = \pi(\psi_p) = \varphi_p$. Since $\psi_p = \psi_p \psi_p = \psi_p$, so $\varphi_p^2 = \varphi_p$.

LEMMA 4.5. *Let* $\psi_p \in \mathrm{End}_{\bar{A}}{}^1 R$ *such that* $\psi_p[R] = \bar{A}$. *If* $\psi \in \mathrm{End}_F{}^1 R$ *and* $\psi_p \psi = \psi$ *then* $\psi|_{\bar{A}}$ *is an automorphism and* $\psi[R] = \bar{A}$.

PROOF. $\psi[R] = \psi_p \psi[R] = \psi_p[\psi[R]] \subseteq \psi_p[R] = \bar{A}$. Let $\delta = \psi|_{\bar{A}}$. Then δ is a ring isomorphism since $\bar{A}$ is simple and $\delta(1_{\bar{A}}) = 1_{\bar{A}}$. Since $\psi \in \mathrm{End}_F{}^1 R$ so δ is also a vector space homomorphism over F. But $\bar{A}$ is finite dimensional over F and so δ is onto.

LEMMA 4.6. *Let* $\delta \in \mathrm{Aut}_F \bar{A}$ *the automorphism group of* $\bar{A}$ *over* F. *Then there exists* $\delta_R \in \mathrm{Aut}_F R$ *such that* $\delta_R|_{\bar{A}} = \delta$.

PROOF. Let $\theta = \sigma_R^{\leftarrow} \circ \delta \circ \sigma_R$ (recall that R is the ring of continuous functions of X into $\bar{A}$ and σ_R the identity mapping of $\bar{A}$ into R). Then $\theta \in \mathrm{Aut}_F \bar{A}$ and is a homeomorphism of the discrete space $\bar{A}$ onto $\bar{A}$. Define δ_R: $R \to R$ by $\delta_R(f) = \theta \circ f$ for $f \in R$. Clearly $\delta_R \in \mathrm{Aut}_F R$. Let $\bar{a} \in \bar{A}$; then $\delta(\bar{a}) \in \bar{A}$. Let $\bar{a}_1 = \delta(\bar{a})$. For $x \in X$, we have

$$\begin{aligned}(\delta_R \bar{a})(x) &= (\theta \circ \bar{a})(x) \\ &= \theta(\bar{a}x) = \theta(a) = (\sigma_R^{\leftarrow} \circ \delta \circ \sigma)_R(a) = (\sigma_R \circ \delta)(\sigma_R a) = (\sigma_R^{\leftarrow} \circ \delta)(\bar{a}) = \sigma_R^{\leftarrow}(\delta\bar{a}) \\ &= \sigma_R^{\leftarrow}(\bar{a}_1) = a_1 = \bar{a}_1(x) = (\delta\bar{a})(x).\end{aligned}$$

Therefore $\delta_R|_{\bar{A}} = \delta$.

LEMMA 4.7. *If* α: $\bar{B} \to S$ *is an isomorphism leaving* F *fixed then* $M_y + \alpha[\bar{B}] = S$ *for every primitive ideal* M_y *in* S. (*We identify the natural image of* F *with* F.)

PROOF. Define α^*: $S/M_y \to S/M_y$ by $\alpha^*(\bar{b} + M_y) = \alpha\bar{b} + M_y$. The domain of α^* is S/M_y since $\bar{B} + M_y = S$. Since $S/M_y \simeq \bar{B}$ and $\bar{B}$ is simple so S/M_y is simple.

It follows that α^* is an isomorphism since $\alpha^*(\bar{1}_S+M_y) = \bar{1}_S+M_y$ the identity of S/M_y. Since $\bar{B}$ is finite dimensional over F so is S/M_y. Hence α^* is onto since α^* is also a vector space homomorphism. Therefore $\{\alpha\bar{b}\}_{\bar{b}\in\bar{B}}+M_y = S$ or $\alpha[\bar{B}]+M_y = S$.

LEMMA 4.8. *If* α: $\bar{B}\to S$ *is an isomorphism leaving F fixed then* $\mathrm{End}_{\bar{B}}^{-1} S \simeq \mathrm{End}_{\bar{D}}^{-1} S$, *where* $\bar{D} = \alpha[\bar{B}]$.

PROOF. $\bar{D}+M_y = S$ for every primitive ideal M_y in S by 4.7. It follows that S is isomorphic to the ring of continuous functions from X into $\bar{D}$ by 2.1 and the remarks following 2.2. Therefore $\mathrm{End}_{\bar{B}} S \simeq \mathrm{End}_{\bar{D}} S$ and hence $\mathrm{End}_{\bar{B}}^{-1} S \simeq \mathrm{End}_{\bar{D}}^{-1} S$.

LEMMA 4.9. $\pi[\mathrm{End}_{\bar{A}}^{-1} R] = \mathrm{End}_{\bar{D}}^{-1} S$ *for some normal subalgebra* $\bar{D}$ *if* S.

PROOF. Let $\psi_p \in \mathrm{End}_{\bar{A}}^{-1} R$ such that $\psi_p[R] = \bar{A}$. Let $\varphi_p = \pi\psi_p$ and $\bar{D} = \varphi_p[S]$. We want to show that $\bar{D}$ is isomorphic to $\bar{B}$. Since $\psi_p(\bar{1}_R) = \bar{1}_R$ so $\varphi_p(\bar{1}_S) = \bar{1}_S$ by 4.3. Therefore $\varphi_p[\bar{B}]$ is isomorphic to $\bar{B}$ since $\bar{B}$ is simple. Since $S/M_y \simeq \bar{B}$ for every primitive ideal M_y in S so there exists $\varphi_M \in \mathrm{End}_F^{-1} S$ such that $\varphi_M[S] = \varphi_p[\bar{B}]$. Clearly $\varphi_M|_{\bar{B}}$ is an isomorphism of $\bar{B}$ into $\varphi_p[\bar{B}]$ which is contained in $\bar{D}$. Since $\psi_p \in \mathrm{End}_{\bar{A}}^{-1} R$ and $\psi_p[R] = \bar{A}$ so $\varphi_p^2 = \varphi_p$ by 4.4. Therefore $\varphi_p|_{\bar{D}} = id$ and hence $\varphi_p\varphi_M = \varphi_M$. Let $\psi_M = \pi^{\leftarrow}(\varphi_M)$ then $\psi_p\psi_M = \psi_M$. Let $\delta = \psi_M|_{\bar{A}}$ then δ is an automorphism of $\bar{A}$ and $\psi_M[R] = \bar{A}$ by 4.5. Therefore δ has an extension $\delta_R \in \mathrm{Aut}_F R$ by 4.6. Clearly $\delta_R^{-1}\psi_M$ is identity on $\bar{A}$. So

$$\delta_R^{-1}\psi_M \in \mathrm{End}_{\bar{A}}^{-1} R \quad \text{and} \quad \pi(\delta_R^{-1}\psi_M) \in \pi[\mathrm{End}_{\bar{A}} R].$$

Let $\delta_S = \pi\delta_R^{-1}$; then

$$\delta_S\varphi_M = (\pi\delta_R^{-1})(\pi\psi_M) = \pi(\delta_R^{-1}\psi_M) \in \pi[\mathrm{End}_A^{-1} R].$$

Let $\psi = \pi^{\leftarrow}(\delta_S\varphi_M)$; then $\psi \in \mathrm{End}_{\bar{A}}^{-1} R$ and hence $\psi\psi_p = \psi_p$ by 4.4. Therefore $(\delta_S\varphi_M)\varphi_p = \varphi_p$ and

$$\bar{D} = \varphi_p[S] = (\delta_S\varphi_M)\varphi_p[S] = \delta_S\varphi_M[\bar{D}] \subseteq \delta_S\varphi_M[S] = \delta_S\varphi_p[\bar{B}].$$

Since $\delta_S = \pi\delta_R^{-1}$ and π is an isomorphism so $\delta_S \in \mathrm{Aut}_F S$. Therefore $\bar{D}$ is contained in an isomorphic image of $\bar{B}$. But $\varphi_p[\bar{B}] \subseteq \bar{D}$ and both φ_p and $\delta_S\varphi_p$ are vector space homomorphisms and so $\bar{D}$ is isomorphic to $\bar{B}$ since $\bar{B}$ is finite dimensional. Thus $\varphi_p[\bar{B}] = \bar{D}$ is a normal subalgebra of S by 4.7. Let $\varphi \in \mathrm{End}_{\bar{D}}^{-1} S$ then $\varphi|_{\bar{D}} = id$, and $\varphi\varphi_p = \varphi_p$. Let $\psi = \pi^{\leftarrow}\varphi$. Then $\psi_p = \pi^{\leftarrow}(\varphi_p) = \pi^{\leftarrow}(\varphi\varphi_p) = (\pi^{\leftarrow}\varphi)(\pi^{\leftarrow}\varphi_p) = \psi\psi_p$. Since $\psi_p \in \mathrm{End}_{\bar{A}}^{-1} R$ so $\psi_p|_{\bar{A}} = id$. Let $\bar{a} \in \bar{A}$ then $\psi(\bar{a}) = \psi(\psi_p\bar{a}) = \psi\psi_p(\bar{a}) = \psi(\bar{a}) = \bar{a}$. Therefore $\psi|_{\bar{A}} = id$ and hence $\psi \in \mathrm{End}_{\bar{A}}^{-1} R$. This implies that $\varphi = \pi\psi \in \pi[\mathrm{End}_{\bar{A}}^{-1} R]$. Thus $\mathrm{End}_{\bar{D}}^{-1} S \subseteq \pi[\mathrm{End}_{\bar{A}}^{-1} R]$. Let $\varphi \in \pi[\mathrm{End}_{\bar{A}}^{-1} R]$ then $\varphi\varphi_p = \varphi_p$ by 4.4. Since $\varphi_p^2 = \varphi_p$ so $\varphi_p|_{\bar{D}} = id$. Therefore $\varphi|_{\bar{D}} = id$ and hence $\varphi \in \mathrm{End}_{\bar{D}}^{-1} S$. This shows that $\pi[\mathrm{End}_{\bar{A}}^{-1} R] \subseteq \mathrm{End}_{\bar{D}}^{-1} S$. Thus $\mathrm{End}_{\bar{D}}^{-1} S = \pi[\mathrm{End}_{\bar{A}}^{-1} R]$.

THEOREM 4.10. *If* $\mathrm{End}_F R \simeq \mathrm{End}_F S$ *then* $R \simeq S$.

PROOF. By 4.9, we have $\pi[\mathrm{End}_{\bar{A}}^1 R] = \mathrm{End}_{\bar{D}}^1 S$. Therefore $\mathrm{End}_{\bar{A}}^1 R \simeq \mathrm{End}_{\bar{D}}^1 S$. But $\mathrm{End}_B{}^1 S \simeq \mathrm{End}_D{}^1 S$ by 4.8. So $\mathrm{End}_A{}^1 R \simeq \mathrm{End}_B{}^1 S$. Thus $R \simeq S$ by 4.1.

If T is a p^k-ring (Foster) and K the Golois field $\mathrm{GF}(p)$ then T is a biregular algebra over F such that $T/M \simeq \mathrm{GF}(p^k)$ for every maximal ideal M in T. Therefore if U is another p^k-ring (Foster) and $\mathrm{End}_K T \simeq \mathrm{End}_K U$ then $T \simeq U$ by 4.10. But $\mathrm{End}_K T = \mathrm{End}\, T$ and $\mathrm{End}_K U = \mathrm{End}\, U$ so $\mathrm{End}\, T \simeq \mathrm{End}\, U$ implies $T \simeq U$.

5. Endomorphisms of rings of continuous real-valued functions

Throughout this section, all spaces are assumed to be completely regular and Hausdorff. The symbol $\mathbf{R}$ will denote the real field with natural topology, and $C(X)$ the ring of continuous functions from a space X into $\mathbf{R}$. If X is a space, then $\mathrm{End}\, C(X) = \mathrm{End}_{\mathbf{R}} C(X)$ by Gillman and Jerison (1960), 1I, p. 23. A space X is said to be *realcompact* if $C(X)/M \simeq \mathbf{R}$ for every maximal ideal M in $C(X)$.

LEMMA 5.1. *Let X and Y be realcompact spaces. If* $\mathrm{End}\, C(X) \simeq \mathrm{End}\, C(Y)$ *then* $S(X) \simeq S(Y)$.

PROOF. Essentially the same as that of 4.1.

A class of topological spaces is said to be *S-admissible* if for each pair of spaces X and Y from the class, any isomorphism from $S(X)$ onto $S(Y)$ is induced by a homeomorphism. There are extensive classes of spaces which are S-admissible and at the same time are such that the spaces belonging to them are all realcompact, for example, the class of compact totally disconnected spaces. For a survey of known results on S-admissible classes, one may consult Magill (1975/76).

THEOREM 5.2. *Let X and Y be realcompact spaces and suppose they both belong to the same S-admissible class. Then the following statements are all equivalent.*

(1) $\mathrm{End}\, C(X) \simeq \mathrm{End}\, C(Y)$.
(2) $C(X) \simeq C(Y)$.
(3) *X is homeomorphic to Y.*
(4) $S(X) \simeq S(Y)$.

PROOF. (1) implies (2): immediate following from 5.1.
(2) implies (3): See Gillman and Jerison (1960), Chapter 8, Theorem 8.2.

(3) implies (4): obvious.

(4) implies (1): Since both X and Y are in the same admissible class, so $X \simeq Y$ by definition. Hence End $C(X) \simeq$ End $C(Y)$.

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References

L. Gillman and M. Jerison (1960), *Rings of continuous functions* (Van Nostrand, New York).

N. Jacobson (1968), *Structure of rings* (Amer. Math. Soc., Providence, R.I.).

J. Luh and D. B. Smith, Jr. (1974), 'On semigroups of endomorphisms of generalized Boolean rings', *J. Austral. Math. Soc.* **18**, 411–418.

K. D. Magill, Jr. (1964), 'Semigroups of continuous functions', *Amer. Math. Monthly* **71**, 984–988.

K. D. Magill, Jr. (1970), 'The semigroup of endomorphisms of a Boolean ring,' *J. Austral. Math. Soc.* **11**, 411–416.

K. D. Magill, Jr. (1975/76), 'A survey of semigroups of continuous selfmaps', *Semigroup Forum* **11**, 189–282.

G. F. Simmons (1963), *Introduction to topology and modern analysis* (McGraw-Hill, New York, San Francisco, Toronto, London).

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