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ON THE EXISTENCE OF SEQUENCES OF CO-PRIME PAIRS OF INTEGERS

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Abstract

We say that a positive integer d has property (A) if for all positive integers m there is an integer x, depending on m, such that, setting $n = m + d$, x lies between m and n and x is co-prime to mn. We show that infinitely many even d and infinitely many odd d have property (A) and that infinitely many even d do not have property (A). We conjecture and provide supporting evidence that all odd d have property (A).

Following A. R. Woods [3] we then describe conditions (A_u) (for each u) asserting, for a given d, the existence of a chain of at most $u + 2$ integers, each co-prime to its neighbours, which start with m and increase, finishing at $n = m + d$. Property (A) is equivalent to condition (A_1), and it is easily shown that property (A_i) implies property (A_{i+1}). Woods showed that for some u all d have property (A_u), and we conjecture and provide supporting evidence that the least such u is 2.

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In [3] Woods proved that there is a constant L such that if m, n are positive integers with $d = n - m > L$, then there is a sequence of numbers $m < x_1 < x_2 < \cdots < x_l < n$ with $1 \leq l \leq L$ having greatest common divisors satisfying $(m, x_1) = 1$, $(x_i, x_{i+1}) = 1$ for $1 \leq i < l$, $(x_l, n) = 1$. This led Woods to conjecture that $L = 1$, that is, to conjecture that all numbers $d > 1$ have

PROPERTY (A). *For all natural numbers m, n with $n - m = d$ there is some x with $m < x < n$ and $(x, mn) = 1$.*

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However, as Woods (private communication) has observed, this conjecture is false, the smallest counterexample being $d = 16$, $m = 2184 = 2^3.3.7.13$, $n = 2200 = 2^3.5^2.11$. This immediately gives infinitely many counterexamples, as we now show. Since $m < x < n$ implies $(x,m) < d$ and $(x,n) < d$, it follows that if $(x,mn) > 1$ then $p|(x,mn)$ for some prime $p < d$. Thus if $m = m_0$, $n = n_0$ is a counterexample to d having property (A) and P is the product of all prime numbers less than d, then $m = m_0 + tP$, $n = n_0 + tP$ gives another such counterexample for each natural number t.

It is thus natural to ask which values of d have property (A).

We answer this question for numbers d of certain forms, from which we show that property (A) holds for infinitely many even d (and for infinitely many odd d) and fails for infinitely many even d. We also modify the (incorrect) original conjecture to

CONJECTURE 1. *All odd $d > 1$ have property* (A); *that is, if $n - m > 1$ is odd, then there is some x with $m < x < n$ and $(x,mn) = 1$.*

NOTE. The author has proved this conjecture for all odd $d \leq 89$ and believes it to be true for all odd $d \leq 219$. A referee has checked the validity of the conjecture for $1 \leq m \leq 1000$, $d = 3,5,\ldots,501$.

THEOREM 1. *Let $t > 1$. Let $q_1 > 2$, $q_2 > q_3 > \cdots > q_t > 2$ be primes, $1 \leq i \leq t$. If $d < q_1^t$, $d < q_t \min(q_1,q_t)$, $q_2 = d - q_1$, $q_3 = d - q_1^2, \ldots, q_t = d - q_1^{t-1}$ and $d \equiv 1 \bmod q_i$, then d does not have property* (A). *Furthermore, a specific m and n illustrating the counterexample can be obtained by requiring that $q_1q_2\cdots q_t|n$ and that all other primes less than d divide m.*

PROOF. Initially requiring that all primes less than d divide m takes care of all numbers between m and n except $x = m+1$. Now, if we no longer require that $q_1|m$, nor that $q_2|m,\ldots,$ nor that $q_t|m$, then the only numbers between m and $n = m+d$ still requiring attention will be $m+1$, $m+q_1,\ldots,m+q_1^{t-1}$, $m+q_2,\ldots,m+q_{t-1}$ and $m+q_t$; that is, $n-(d-1)$, $n-q_2,\ldots,n-q_t$, $n-q_1,\ldots,n-q_1^{t-2}$ and $n-q_1^{t-1}$. The requirement that $q_1q_2\cdots q_t|n$ takes care of all of these since $d-1 \equiv 0 \bmod q_i$.

Theorem 1 gives us a method for producing d not satisfying property (A).

EXAMPLE 1: with $t = 2$, $i = 1$ and so $q_1 < q_2$.

$q_1 = 5$; $q_2 = 11$. This gives $2.3.7.13|m$, $2.5.11|n = m+d = m+5+11 = m+16$ and we have seen this one before.
$q_1 = 7$; $q_2 = 29$.
$q_1 = 11$; $q_2 = 23, 67, 89$.
Etc.

EXAMPLE 2: with $t = 3$ and $i = 1$.

$q_1 = 3$; $(q_2, q_3) = (13, 7)$ ($d = 16$; this gives the 'reverse' of the other $d = 16$ example),

$(q_2, q_3) = (19, 13)$.

$q_1 = 5$; $(q_2, q_3) = (31, 11)$ (this is different from our other counter-examples with $d = 36$),

$(q_2, q_3) = (61, 41)$.

Etc.

As we might suspect from the examples, property (A) fails for infinitely many even values of d.

Let $P(k, l)$ be the least prime in the arithmetic progression $n \equiv l \pmod{k}$, where $\gcd(k, l) = 1$.

LEMMA 2 [2]. *Given $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ and infinitely many primes q such that $P(q, 1) < c(\varepsilon)\ q^{\theta+\varepsilon}$, where $\theta = 2e^{1/4}(2e^{1/4} - 1)^{-1} = 1.63773\ldots$.*

COROLLARY 3. *There exist infinitely many pairs of primes p, q satisfying $p \equiv 1 \bmod q$ and $p < q^2 - q$.*

It follows from Theorem 1 (with $t = 2$ and $i = 1$) and Corollary 3 that property (A) fails for infinitely many even values of d.

It turns out that property (A) holds for infinitely many even values of d (and infinitely many odd values of d).

THEOREM 4. *If either*

(a) $d = q^\gamma + 1$, *q a prime,* $\gamma \geq 0$,

or

(b) $d = p_1^{\beta_1} + p_2^{\beta_2} = p_1^{\alpha_1} p_2^{\alpha_2} + 1$, *where p_1, p_2 are distinct primes,* $\beta_1, \beta_2, \alpha_1, \alpha_2 > 0$,

then d has property (A).

PROOF. (a) Let $d = q^\gamma + 1$. If $\gamma = 0$, we can take $x = m + 1$. If $\gamma > 0$, then if $q \nmid n$ we can take $x = m + 1$, while if $q \nmid m$ we can take $x = n - 1$.
(b) If $p_1 \nmid m$ and $p_2 \nmid n$, we can take $x = m + p_1^{\beta_1}$. Similarly, if $p_2 \nmid m$ and $p_1 \nmid n$, we can take $x = m + p_2^{\beta_2}$. Finally, if $p_1 p_2 | m$ we can take $x = m + 1$; while if $p_1 p_2 | n$, then $x = n - 1$ suffices.

It follows from Case (a) of Theorem 4 with q an odd prime that there are infinitely many even values of d with property (A); and with $q = 2$ it follows that there are infinitely many odd values of d with property (A).

Between them, Theorems 1 and 4 go some way toward classifying all values of d. The cases unclassified by Theorems 1 and 4 for $d \leq 38$ are $d = 11, 23, 27, 29, 31, 35, 37$. These can be all shown to have property (A).

We note that Theorems 1 and 4 classified all even values of $d \leq 38$.

QUESTION. Do Theorems 1 and 4 classify all even values of d?

As we mentioned at the start of the paper, Woods [3] proved that there is a constant L such that if m, n are positive integers with $d = n - m > L$, then there is a sequence of numbers $m < x_1 < \cdots < x_l < n$ with $1 \leq l \leq L$ having greatest common divisors satisfying $(m, x_1) = 1$, $(x_i, x_{i+1}) = 1$ for $1 \leq i < l$, $(x_l, n) = 1$. We have shown that the smallest such L is at least 2; we now try to find it.

First, we generalize the notion of property (A).

DEFINITIONS. *Say* $x \prec y$ *if and only if* $\gcd(x, y) = 1$ *and* $x < y$.
Say $x \preccurlyeq y$ *if and only if* $(\gcd(x, y) = 1$ *and* $x < y)$ *or* $x = y$.

DEFINITION. *For each* $u \in \mathbf{N}$ *we say that* $d > 1$ *has property* (A_u) *if and only if*

$$\forall m \forall n (m < n = m + d \to \exists z_1, z_2, \ldots, z_u, m \preccurlyeq z_1 \preccurlyeq z_2 \preccurlyeq \cdots \preccurlyeq z_u \preccurlyeq n).$$

DEFINITION. *For each* $u \in \mathbf{N}$ *we say that* $d > u$ *has property* (B_u) *if and only if*

$$\forall m \forall n (m < n = m + d \to \exists z_1, z_2, \ldots, z_u, m \prec z_1 \prec z_2 \prec \cdots \prec z_u \prec n).$$

NOTE. For all d, d has property (A) if and only if d has property (A_1) and if and only if d has property (B_1). For all k and for all d, d has property (B_k) implies d has property (A_k) which implies d has property (A_{k+1}). For all k and for all d, d has property (B_k) implies $d + 1$ has property (B_{k+1}), which implies $d + 1$ has property (A_{k+1}).

It follows from the above note that if Conjecture 1 is true then all $d > 1$ have property (A_2). It will follow from Theorem 5 and Corollary 8 that if Conjecture 1 is true then all $d > 2$ have property (B_2).

We now gather further evidence to suggest that all $d > 2$ have property (B_2), in turn providing even stronger evidence that all $d > 1$ have property (A_2).

Our next result is based on Theorem 4.

THEOREM 5. *Let* d_1 *have property* (A). *If* p *is a prime such that* $p \nmid d_1$ *and* $k \geq 0$, *then* $d_2 = d_1 + p^k$ *has property* (B_2).

PROOF. Consider m with $m < z_1 < m + d_1$ illustrating property (A). If $p | m$ we have $m < z_1 < z_2 = m + d_1 < n = z_2 + p^k$. If $p \nmid m$ we have $m < m + p^k = z_1 < z_2 < n = z_1 + d_1$.

COROLLARY 6. *If q_1 and q_2 are primes (not necessarily distinct), then $d_2 = q_1 + q_2 + 1$ has property* (B_2).

PROOF. *Case* 1. $q_1 + q_2 = 5$ and so $d_2 = 6$. If $2|m$ and $2|n$ then $m < z_1 = m + 1 < z_2 = m + 5 < n$ does the job. If $2 \nmid mn$, then $m < z_1 = m + 2 < z_2 = m + 4 < n$ does the job.

Case 2. $q_1 + q_2 \neq 5$. Without loss of generality, suppose $q_1 \geq q_2$. Then $q_1 \nmid q_2 + 1$. By Theorem 4, $d_1 = q_2 + 1$ has property (A). So, by Theorem 5, $d_2 = q_1 + q_2 + 1$ has property (B_2).

COROLLARY 7. *If Goldbach's conjecture is true, then all odd $d_2 \geq 3$ have property* (B_2).

COROLLARY 8. *If d_1 is odd and has property* (A), *and $k \geq 1$, then $d_2 = d_1 + 2^k$ has property* (B_2).

These results tend to suggest that all odd $d \geq 3$ have property (B_2). (This would in turn imply that all $d > 1$ have property (A_3).) Evidence that all even $d \geq 4$ have property (B_2) follows again from Theorem 5 requiring d_1 and p to be odd (and possibly k to be zero).

Having gathered our evidence, we finish with two conjectures.

CONJECTURE 2. *All $d \geq 3$ have property* (B_2).

CONJECTURE 3. *All $d \geq 2$ have property* (A_2).

We recall that Conjecture 1 implies Conjecture 2, which implies Conjecture 3.

Note added in proof

The author has written a computer program whose output to date tells us that Conjecture 1 holds for $1 \leq m < n \leq 3{,}000{,}000$. Furthermore, the output tells us that the only value of d shown not to have property (A) from inspecting $1 \leq m < n \leq 3{,}000{,}000$ is $d = 16$.

Recalling the note after Conjecture 1, for a given d let $\pi(d)$ equal the product of all primes less than d. We note that if d does not have property (A) *and if* the relevant (counter-)example (m, n) has each prime less than d either dividing m or dividing n, then clearly $\pi(d)|mn = m(m + d)$ and so $m > \sqrt{\pi(d)} - d/2$. Now, since $\pi(53) > 5{,}000{,}000{,}053^2$ and since Conjecture

1 holds for all odd $d \leq 89$, the evidence that Conjecture 1 likewise holds for $1 \leq m < n \leq 5,000,000,000$ is overwhelming.

We conclude that the approach of sequentially checking m and n (as in the author's program) is sluggish in the extreme compared to the alternative approach of checking each value of d in turn; although the latter would undoubtedly constitute a more difficult programming exercise. A copy of the author's program (written in Pascal), which sequentially checks m and n, is available from the author upon request.

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An earlier version of this paper appears in the author's Ph.D. thesis [1].

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