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LOCAL MINIMALITY OF A LIPSCHITZ EXTREMAL

To the memory of Lamberto Cesari

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ABSTRACT. In this paper the question of weak and strong local optimality of a Lipschitz (as opposed to C^1) extremal is addressed. We show that the classical Jacobi sufficient conditions can be extended to the case of Lipschitz candidates. The key idea for this achievement lies in proving that the "generalized" strengthened Weierstrass condition is equivalent to the existence of a "feedback control" function at which the maximum in the "true" Hamiltonian is attained. Then the Hamilton-Jacobi approach is pursued in order to conclude the result.

1. **Introduction.** We are given an interval $[a, b]$, two points x_a, x_b in $\mathbb{R}^n$, and a function $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$. The problem of Bolza is:

$$(P) \qquad \text{minimize } J(x) = \int_a^b L\big(t, x(t), \dot{x}(t)\big)\, dt$$

over all *absolutely continuous* functions (*arcs*) x: $[a, b] \longrightarrow \mathbb{R}^n$ that satisfy the constraints $x(a) = x_a$, $x(b) = x_b$; we cali such functions *feasible* arcs.

A *tube* $T(x; \varepsilon)$ of radius ε about the arc x is the set of (t, y) in $[a, b] \times \mathbb{R}^n$ satisfying

$$|y - x(t)| < \varepsilon.$$

A *restricted* tube $RT(x; \varepsilon)$ of a *Lipschitz* arc x is the set of (t, y, v), where (y, v) in $\mathbb{R}^n \times \mathbb{R}^n$, and t in $[a, b]$ with $\dot{x}(t)$ exists (i.e., almost everywhere), such that

$$|y - x(t)| < \varepsilon \text{ and } |v - \dot{x}(t)| < \varepsilon.$$

An arc y is said to lie in $T(x; \varepsilon)$ if for all t in $[a, b]$, $\big(t, y(t)\big)$ is in $T(x; \varepsilon)$. An arc y is said to lie in $RT(x; \varepsilon)$ if for almost all t in $[a, b]$, $\big(t, y(t), \dot{y}(t)\big)$ is in $RT(x; \varepsilon)$. The feasible arc x is said to be a (strict) *weak local minimum* if for some $\varepsilon > 0$, one has $J(y) \geq J(x)$ for all feasible arcs $y(\neq x)$ in $RT(x; \varepsilon)$. A (strict) *strong local minimum* corresponds to replacing $RT(x; \varepsilon)$ by $T(x; \varepsilon)$ in this definition.

The fact that the minimum in (P) is searched for over the set of absolutely continuous functions x (as opposed to continuously differentiable), is not a feature of the classical

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setting. This class of functions, which was introduced by Tonelli, created a radical departure from earlier work. Concerning this problem one can find intensive studies starting with Tonelli's Existence Theorem. Recent works on the Euler-Legendre equation and the regularity of the solution of (P) were recently tackled by several people (e.g. Ball & Mizel [2], Cesari [5], Clarke & Loewen [10]–[11], Clarke & Vinter [8]–[9], Rockafellar [20]). A complete documentation of these results is given in [7].

In order to clearly situate the contribution of this article we give a brief survey of the necessary and sufficient conditions known for the problem (P).

1.1. *Necessary and sufficient conditions for a weak local minimum.* Let $\tilde{x}$ be a Lipschitz continuous feasible arc. The following assumptions are made:

(H_1) There exists $\varepsilon > 0$ such that, for $t \in [a,b]$ a.e., $L(t,\cdot,\cdot)$ is C^2 on the ε-neighborhood of $\big(\tilde{x}(t),\dot{\tilde{x}}(t)\big)$, L and its derivatives in (x,u) up to second order are measurable on $[a,b]$, $L_{ux}\big(\cdot,\tilde{x}(\cdot),\dot{\tilde{x}}(\cdot)\big)$ is $L^\infty[a,b]$, and there exists an integrable function $K\colon [a,b] \longrightarrow \mathbb{R}$ such that

$$|L(t,x,u)| + |\nabla_{(x,u)}L(t,x,u)| + |\nabla^2_{(x,u)}L(t,x,u)| \leq K(t).$$

It is worth mentioning that another type of nonsmoothness that is dealt with involves the Lagrangian itself as a function of t. In fact, Hypothesis (H_1) requires neither smoothness nor continuity in the t-variable. This sort of consideration is very modern in nature, (see eg. [7]).

The generalization of the classical Euler necessary condition can be found in [5] and [7]. It says that if $\tilde{x}$ is a weak local minimum then, for some constant C,

$$(E) \qquad \tilde{L}_u(t) = \int_a^t \tilde{L}_x(s)\,ds + C \text{ a.e. } t \text{ in } [a,b],$$

where (for example) $\tilde{L}_u(t)$ is an abbreviation of $L_u\big(t,\tilde{x}(t),\dot{\tilde{x}}(t)\big)$.

The Legendre necessary condition states that

$$(L) \qquad \tilde{L}_{uu}(t) \geq 0 \text{ for } t \in [a,b] \text{ a.e. (positive semidefinite).}$$

The Jacobi necessary condition (J) for the problem (P) is recently proved in [26]. The strengthened Legendre condition $(L)'$ needs to be assumed:

$$(L)' \qquad \exists\delta > 0 : \tilde{L}_{uu}(t) \geq \delta I \text{ for } t \in [a,b] \text{ a.e.,}$$

where I is the $n \times n$-identity matrix. The Jacobi necessary condition states that there is no point c in (a,b) corresponding to which there is a nontrivial solution (η,ξ) on $[a,c]$ of the homogeneous first order system (called the Jacobi system)

$$(1.1) \qquad \begin{aligned} \dot{\eta}(t) &= A(t)\eta(t) + B(t)\xi(t) \\ \dot{\xi}(t) &= C(t)\eta(t) - A^T(t)\xi(t) \end{aligned}$$

with

$$(1.2) \qquad \eta(a) = \eta(c) = 0,$$

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where

$$A(\cdot) = -\tilde{L}_{uu}^{-1}(\cdot)\tilde{L}_{ux}(\cdot),\ B(\cdot) = \tilde{L}_{uu}^{-1}(\cdot),$$
$$C(\cdot) = \tilde{L}_{xx}(\cdot) - \tilde{L}_{xu}(\cdot)\tilde{L}_{uu}^{-1}(\cdot)\tilde{L}_{xu}(\cdot). \tag{1.3}$$

A point c for which a nontrivial solution of (1.1)–(1.2) does exist is called a *conjugate* point. Thus, the Jacobi condition is:

(J) there are no conjugate points in (a, b).

By modifying Legendre's and Jacobi's arguments for the classical setting, that is, expressing the second variation as the integral of a perfect square, it was proved in [27] that a slight strengthening of (E), (L), (J) is sufficient. The following notion was used.

DEFINITION 1. Let z be a function $z(\cdot)\colon [a, b] \longrightarrow \mathbb{R}^k$, and let $h(\cdot,\cdot)\colon T(z;\alpha) \longrightarrow M_{r\times s}$, where $M_{r\times s}$ is the space of $r \times s$-matrices. The function $h(t,\cdot)$ is said to be *continuous at* $z(\cdot)$ *uniformly* in t iff

$$\forall\varepsilon > 0,\ \exists\delta > 0 : \text{for } t \in [a, b] \text{ a.e., } \forall y : |y - z(t)| < \delta \text{ we have}$$
$$\big|h(t, y) - h\big(t, z(t)\big)\big| < \varepsilon.$$

The strengthened Jacobi condition is

$(J)'$ there exist no conjugate points in $(a, b]$.

THEOREM 1. *Let $\tilde{x}$ be a Lipschitz feasible arc satisfying (H_1). Assume that the Hessian $\nabla^2_{(x,u)}L(t,\cdot,\cdot)$ is continuous at $(\tilde{x},\dot{\tilde{x}})$ uniformly in t. Then, (E), $(L)'$ and $(J)'$ imply that $\tilde{x}$ is a weak local minimum for (P).*

1.2. *Necessary and sufficient conditions for a strong local minimum.* If $\tilde{x}$ is a strong local minimum for the problem (P) and satisfies (H_1), the conditions (E), (L) and (J) continue to be necessary. Moreover, as a special case of Pontryagin Maximum Principle [14], the generalized Weierstrass condition is also necessary, that is

$$E\big(t, \tilde{x}(t), \dot{\tilde{x}}(t), u\big) \geq 0 \quad \text{for } t \in [a, b] \text{ a.e., and for all } u \in \mathbb{R}^n, \tag{W}$$

where the "excess function" is given by

$$E(t, x, v, u) := L(t, x, u) - L(t, x, v) - L_u(t, x, v)\cdot(u - v).$$

For the case where L is C^1 this result is given in [5].

Concerning sufficient conditions, one can find the classical result which is only valid for $\tilde{x}$ *continuously differentiable* or, at least, piecewise smooth. It involves a strengthening of (W), where $\tilde{x}$ is C^1. For later purposes, let us state the *generalized* strengthened Weierstrass condition:

$$E(t, y, v, u) \geq 0 \text{ for } (t, y, v) \in RT(\tilde{x};\varepsilon), \text{ and for all } u \in \mathbb{R}^n, \tag{W'}$$

which reduces to the classical strengthened Weierstrass condition when $\tilde{x}$ is C^1.

THEOREM 2. *Suppose that $\tilde{x}$ is C^1 and that L is C^2. (E), $(L)'$, $(J)'$ and $(W)'$ imply that $\tilde{x}$ is a strong local minimum.*

This well-known result is classically proven by constructing a field of extremals (see eg., [1], [3], [4], [13], [15], [21]) or, as shown in [12], by constructing a solution a certain Hamilton-Jacobi inequality. This latter approach was used in [12] and [18] to prove both Theorems 1 and 2 when $\tilde{x}$ is C^1.

The aim of this paper is to complete the study of the problem (P). We will show using the generalized strengthened Weierstrass condition $(W)'$, that Theorem 2 can be extended to the case when $\tilde{x}$ is Lipschitz continuous (as opposed to C^1) and $L(t,x,u)$ is not continuous in t (as opposed to C^2). We require, instead, that the continuity in (x,u) is uniform it t. The method we use here allows us to prove Theorem 1 and the generalization of Theorem 2 at once. It is not a new approach per se, it was pioneered in [23], [24], [25], and used in [12] and [18]. The idea is to construct a function $V(t,x)$ which does not only satisfy a form of the Hamilton-Jacobi inequality,

$$\begin{aligned}&\min\{\, L^*(t,x,u) := L(t,x,u) - V_t(t,x) - V_x(t,x)u : (t,x,u) \in RT(\tilde{x};\alpha)\}\\&\qquad\text{is attained at } \big(\tilde{x}(t),\dot{\tilde{x}}(t)\big) \text{ for } t \in [a,b] \text{ a.e.,}\end{aligned}$$

but also gives rise to an "equivalent" problem (P^*) whose objective function $L^*(t,x,u)$ is jointly convex in (x,u). Then, by the convexity theory we conclude the weak local optimality of $\tilde{x}$ for (P^*), and hence for (P). Next, we prove that the generalized strengthened Weierstrass condition $(W)'$ is equivalent to the existence of a "feedback control" function $u(t,x,p)$ at which the maximum in the "true" Hamiltonian is attained. The function $u(t,\cdot,\cdot)$ turns out to be continuous uniformly in t, and thus a solution to the Hamilton-Jacobi inequality, where $(t,x) \in T(\tilde{x};\delta)$, is obtained. This leads to the strong local optimality of $\tilde{x}$ for (P^*), and hence for (P).

2. Statement of the main result.

Let $\tilde{x}$ be a feasible Lipschitz continuous function. We provide a sufficiency criterion for strong local minimality of $\tilde{x}$ that extends the known one when $\tilde{x}$ is C^1. The following nonrestrictive assumptions will be made.

(H_2)
- (i) $\exists\varepsilon > 0$: for $t \in [a,b]$ a.e., $L(t,\cdot,\cdot)$ is C^2 on $\{(y,v) : (t,y,v) \in RT(\tilde{x};\varepsilon)\}$
- (ii) L and its derivatives in (x,u) up to second order are measurable in t and integrable along $(\tilde{x},\dot{\tilde{x}})$,
- (iii) For all functions $(x,u) \in T(\tilde{x},\dot{\tilde{x}};\varepsilon)$, $\nabla^2_{(x,u)}L(t,\cdot,\cdot)$ is continuous uniformly in t,
- (iv) $\nabla^2_{(x,u)}\tilde{L}(\cdot)$ is essentially bounded on $[a,b]$.

Note that in the classical setting, that is, when $\tilde{x}$ is C^1 and L is C^2, all the assumptions in Hypothesis (H_2) are automatically satisfied.

THEOREM 3. *Let $\tilde{x}$ be Lipschitz continuous and feasible for (P). Assume that for some $\varepsilon > 0$, Hypothesis (H_2) holds. Then, (E), $(L)'$, and $(J)'$ imply that $\tilde{x}$ is a strict weak local minimum. If in addition, $(W)'$ is satisfied then, $\tilde{x}$ is a strict strong local minimum.*

REMARKS. As we shall see in the proof of Theorem 3, we need only Hypothesis (H_1) along with the continuity of $\nabla^2_{(x,u)}L(t,\cdot,\cdot)$ uniformly in t at $(\tilde{x},\dot{\tilde{x}})$ in order to prove the first part of Theorem 3. Thus, Theorem 1 is a part of Theorem 3. It is clear that the second part of our result generalizes Theorem 2 to *Lipschitz continuous* candidates $\tilde{x}$ and to *nonsmooth* data $L(\cdot,x,u)$.

One can construct examples (see [26]) illustrating the indispensability of condition (iii) of (H_2).

Associated to the problem (P) there exists a matrix Riccati differential equation

$$(R) \qquad M(t,Q) := \dot{Q} - QBQ + QA + A^TQ + C = 0 \text{ a.e. } t.$$

where A, B and C are defined in (1.3). On page 319 of [19], it is shown that condition $(J)'$ is equivalent to the existence of a Lipschitz continuous solution Q_0 to (R). Thus, in Theorem 3 the strengthened Jacobi condition $(J)'$ can be replaced by the assumption that (R) has a solution Q_0 on $[a,b]$.

3. **Proof of the result.**

We introduce the notion of equivalent problems. Consider the problem

$$(P)^* \qquad \text{minimize } J^*(x) := \int_a^b L^*\big(t,x(t),\dot{x}(t)\big)\,dt$$

over the absolutely continuous functions $x\colon [a,b] \longrightarrow \mathbb{R}^n$ satisfying $x(a) = x_a$ and $x(b) = x_b$.

DEFINITION. We say that $(P)^*$ is equivalent to (P) if $\tilde{x}$ is a weak local, strong local, or global minimum for (P) if and only if $\tilde{x}$ is, respectively, weak local, strong local or global minimum for $(P)^*$.

An important class of equivalent problems to (P) is inspired by the Hamilton-Jacobi theory. Let $V(t,x)$ be a function defined on $[a,b]\times\mathbb{R}^n$ such that for almost all t, $V(\cdot,\cdot)$ is differentiable, and $V\big(\cdot,x(\cdot)\big)$ is absolutely continuous whenever $x(\cdot)$ is. Define for $t \in [a,b]$ a.e.,

$$(3.1) \qquad L^*(t,x,u) := L(t,x,u) - V_t(t,x) - V_x(t,x)u,$$

then, for any absolutely continuous function x with $x(a) = x_a$ and $x(b) = x_b$, it follows from (3.1) that

$$\int_a^b L^*\big(t,x(t),\dot{x}(t)\big)\,dt = \int_a^b L\big(t,x(t),\dot{x}(t)\big) - V(b,x_b) + V(a,x_a).$$

In other words $(P)^*$ defined through (3.1) is equivalent to (P). The proof of Theorem 3 proceeds to construct using (E), $(L)'$, $(J)'$ a function $V(t,x)$ for which L^*, defined by (3.1), satisfies the strict "Hamilton-Jacobi inequality"

$$(3.2) \qquad L^*(t,x,v) > L^*\big(t,\tilde{x}(t),\dot{\tilde{x}}(t)\big)$$

for $(t, x, v) \in RT(\tilde{x}; \alpha), (x, v) \neq \big(\tilde{x}(t), \dot{\tilde{x}}(t)\big)$, where $\alpha > 0$. Furthermore, if $(W)'$ holds, then (3.2) is true for $(t, x) \in T(\tilde{x}; \delta)$, $(x, v) \neq \big(\tilde{x}(t), \dot{\tilde{x}}(t)\big)$, where $\delta > 0$. This will then complete the proof of the result.

From the last remark following Theorem 3, $(J)'$ yields the existence of a Lipschitz continuous matrix function Q_0 satisfying the Riccati equation (R). Using the Embedding theorem of differential equations in the appendix of [16], or by extending the argument in [12], it follows that there exist $\lambda > 0$ and a Lipschitz continuous function Q satisfying for $t \in [a, b]$ a.e.,

$$M(t, Q) := \dot{Q} = QBQ + QA + A^T Q + C = \lambda I \tag{3.3}$$

Define

$$V(t, x) := \langle \tilde{L}_u(t), x - \tilde{x}(t) \rangle - \frac{1}{2}\big\langle x - \tilde{x}(t), Q(t)\big(x - \tilde{x}(t)\big)\big\rangle, \tag{3.4}$$

then, $V(t, x)$ is one of the functions used in (3.1). Using (3.1) and (E) we get:

$$\begin{aligned} L^*(t, x, u) = L(t, x, u) &- \big[\tilde{p}(t) - Q(t)\big(x - \tilde{x}(t)\big)\big] \cdot u \\ &- \tilde{L}_x(t)\big(x - \tilde{x}(t)\big) + \tilde{L}_u(t)\dot{\tilde{x}}(t) - \dot{\tilde{x}}^T(t)Q(t)\big(x - \tilde{x}(t)\big) \\ &+ \frac{1}{2}\big\langle x - \tilde{x}(t), \dot{Q}(t)\big(x - \tilde{x}(t)\big)\big\rangle, \end{aligned} \tag{3.5}$$

where

$$\tilde{p}(t) = \tilde{L}_u(t) \text{ a.e.} \tag{3.6}$$

Using (3.5) and (3.6) we obtain that, for $t \in [a, b]$ a.e.,

$$\nabla_{(x,u)}\tilde{L}^*(t) = (0, 0). \tag{3.7}$$

Moreover, a simple calculation shows that

$$\begin{aligned} \nabla^2_{(x,u)}\tilde{L}^*(t) &= \begin{bmatrix} \tilde{L}_{xx}(t) + \dot{Q}(t) & \tilde{L}_{xu}(t) + Q(t) \\ \tilde{L}_{ux}(t) + Q(t) & \tilde{L}_{uu}(t) \end{bmatrix}, \\ &= N^T(t)\begin{bmatrix} \lambda I & 0 \\ 0 & B^{-1}(t) \end{bmatrix} N(t) \end{aligned}$$

where

$$N(t) = \begin{bmatrix} I & 0 \\ B(t)Q(t) - A(t) & I \end{bmatrix}.$$

From $(L)'$ it results that, for $\beta = \min\{\delta, \lambda\}$ we have

$$\nabla^2_{(x,u)}\tilde{L}^*(t) \geq \beta N^T(t)N(t). \tag{3.8}$$

Note that N is invertible with inverse

$$N^{-1}(t) = \begin{bmatrix} I & 0 \\ -B(t)Q(t) + A(t) & I \end{bmatrix},$$

and that $B(\cdot)$ is in $L^\infty[a,b]$. Thus, there exists $\gamma > 0$ such that, for almost all t in $[a,b]$

$$\|N^{-1}(t)\| < \gamma,$$

where "$\|\cdot\|$" is any matrix norm. Hence, for $d \in \mathbb{R}^{2n}$,

$$|d|^2 = |N^{-1}(t)N(t)d|^2 \leq \|N^{-1}(t)\|^2|N(t)d|^2 \leq \gamma^2|N(t)d|^2,$$

where "$|\cdot|$" is the Euclidean norm.

It follows that

$$|N(t)d|^2 \geq \frac{1}{\gamma^2}|d|^2 \text{ a.e.t.}$$

and thus, using (3.8) we obtain

$$\nabla^2_{(x,u)}\tilde{L}^*(t) \geq \lambda_0 I \text{ a.e.t.,} \tag{3.9}$$

for $\lambda_0 = \frac{\beta}{\gamma^2}$.

Hypothesis (H_2) and (3.9) imply the existence of $\alpha > 0$ $(\alpha \leq \varepsilon)$ such that for $(t,x,u) \in RT(\tilde{x};\alpha)$

$$\nabla^2_{(x,u)}L^*(t,x,u) > 0 \text{ for almost all } t.$$

Therefore $L^*(t,\cdot,\cdot)$ is *strictly convex* on $RT(\tilde{x};\alpha)$. Using (3.7), it follows that for $(x,v) \neq \big(\tilde{x}(t),\dot{\tilde{x}}(t)\big)$ our L^* satisfies the strict Hamilton-Jacobi inequality (3.2). Thus $\tilde{x}$ is a strict weak local minimum for $(P)^*$, where L^* is defined by (3.5). Since $(P)^*$ is equivalent to (P) it results that $\tilde{x}$ is a weak local minimum for (P), proving the first part of Theorem 3.

Let us prove the second part of Theorem 3. The idea is to show that adding condition $(W)'$ implies that, for some $\delta > 0$, (3.2) holds for $(t,x) \in T(\tilde{x};\delta)$, $(x,v) \neq \big(\tilde{x}(t),\dot{\tilde{x}}(t)\big)$. Let $Q(t), \tilde{p}(t)$ and α be as in the proof of the weak local minimality of $\tilde{x}$. Since $(W)'$ holds, then by Lemma 1 in the appendix, there exist $\varepsilon_0 > 0$ $(\varepsilon_0 \leq \varepsilon)$ and a unique function $u(t,x,p)$ with $u(t,\cdot,\cdot)$ continuous at $(\tilde{x},\tilde{p})$ uniformly in t, $u\big(t,\tilde{x}(t),\tilde{p}(t)\big) = \dot{\tilde{x}}(t)$, and

$$\min_{u\in\mathbb{R}^n}\{L(t,x,u) - p.u\} \text{ is attained at } u(t,x,p). \tag{$*$}$$

Define

$$p(t,x) := \tilde{p}(t) - Q(t)\big(x - \tilde{x}(t)\big), \tag{3.10}$$

then, since $p(t,\cdot)$ is continuous uniformly in t, there exists $\xi > 0$ $(\xi \leq \min\{\alpha,\varepsilon_0\})$ such that, for $(t,x) \in T(\tilde{x};\xi)$

$$|p(t,x) - \tilde{p}(t)| < \varepsilon_0.$$

Then, by $(*)$ and the continuity of $u(t,\cdot,\cdot)$ uniformly in t, it follows that, for some $\delta > 0$ $(\delta \leq \xi)$ and for $(t,x) \in T(\tilde{x};\delta)$

$$\min_{u\in\mathbb{R}^n}\{L(t,x,u) - p(t,x).u\}$$

is uniquely attained at $u\big(t,x,p(t,x)\big)$ with

$$|u\big(t,x,p(t,x)\big)-\tilde{u}(t)|<\alpha.$$

Thus, from (3.5) and (3.10) we have, for $(t,x)\in T(\tilde{x};\delta)$ and v in $\mathbb{R}^n$, $v\neq u\big(t,x,p(t,x)\big)$ that

$$L^*(t,x,v)>L^*\Big(t,x,u\big(t,x,p(t,x)\big)\Big).$$

But since $\Big(t,x,u\big(t,x,p(t,x)\big)\Big)\in RT(\tilde{x};\alpha)$ it follows from the proof of the first part of Theorem 3 that

$$L^*\Big(t,x,u\big(t,x,p(t,x)\big)\Big)>L^*\big(t,\tilde{x}(t),\dot{\tilde{x}}(t)\big)$$

for $\Big(x,u\big(t,x,p(t,x)\big)\Big)\neq\big(\tilde{x}(t),\dot{\tilde{x}}(t)\big)$. Hence, for $(t,x)\in T(\tilde{x};\delta)$, $v\in\mathbb{R}^n$ with $(x,v)\neq\big(\tilde{x}(t),\dot{\tilde{x}}(t)\big)$ we have (3.2) holds. This implies that $\tilde{x}$ is a strict strong local minimum for $(P)^*$. By the equivalence between (P) and $(P)^*$ we conclude that $\tilde{x}$ is also a strict strong local minimum for (P).

4. **Appendix.** The goal of this appendix is to establish the equivalence between the generalized strengthened Weierstrass condition $(W)'$ and the existence of a feedback control function $u(t,x,p)$ that is regular and at which the maximum in the true Hamiltonian is attained.

LEMMA 1. *Let $\tilde{x}$ be a Lipschitz continuous function at which (H_2) and $(L)'$ hold, and let $\tilde{p}$ be the function defined by (3.6). Then, $(W)'$ is equivalent to*

($*$) *there exist $\varepsilon_0>0$ and a unique function $u(t,x,p)$ defined on $T(\tilde{x},\tilde{p};\varepsilon_0)$, such that $u\big(t,\tilde{x}(t),\tilde{p}(t)\big)=\dot{\tilde{x}}(t)$, $u(t,\cdot,\cdot)$ is continuous at $(\tilde{x},\tilde{p})$ uniformly in t, and*

$$u(t,x,p)=\arg\min\{L(t,x,u)-p.u:u\in\mathbb{R}^n\}$$

for $(t,x,p)\in T(\tilde{x},\tilde{p};\varepsilon_0)$.

REMARKS. $(W)'$ can be written as

$$v\in\arg\min\{L(t,x,u)-L_u(t,x,v).u:u\in\mathbb{R}^n\}\text{ for }(t,x,v)\in RT(\tilde{x};\varepsilon).$$

The "true" Hamiltonian corresponding to L is defined to be

$$H(t,x,p):=\max\{p.u-L(t,x,u):u\in\mathbb{R}^n\}.$$

Thus, condition ($*$) can then be rephrased as:

$$H(t,x,p)=p.u(t,x,p)-L\big(t,x,u(t,x,p)\big),$$

where $u(t,x,p)$ is unique, $u(t,\cdot,\cdot)$ is continuous uniformly in t on $T(\tilde{x},\tilde{p};\varepsilon_0)$ and $u\big(t,\tilde{x}(t),\tilde{p}(t)\big)=\dot{\tilde{x}}(t)$.

PROOF OF LEMMA 1. Let $N_\varepsilon\big(z(\cdot)\big)=\{y(\cdot)\in L^\infty:\|y-z\|_\infty<\varepsilon\}$.

Define

$$\mathcal{F} : N_\varepsilon\big(\bar{x}(\cdot)\big) \times N_\varepsilon\big(\dot{\bar{x}}(\cdot)\big) \longrightarrow L^\infty[a,b]$$
$$\big(x(\cdot), u(\cdot)\big) \longrightarrow p(\cdot) = \mathcal{F}\big(x(\cdot), u(\cdot)\big)$$

where

$$\mathcal{F}\big(x(\cdot), u(\cdot)\big)(t) = L_u\big(t, x(t), u(t)\big).$$

Consider the equation

$$p(\cdot) = \mathcal{F}\big(x(\cdot), u(\cdot)\big).$$

(i) $\mathcal{F}\big(x(\cdot), \cdot\big)$ is differentiable on $\{u(\cdot) : \|u - \dot{\bar{x}}\|_\infty < \varepsilon\}$ for all $x(\cdot)$ in $\{x(\cdot) : \|x - \bar{x}\| < \varepsilon\}$:

Let $x(\cdot)$ and $u(\cdot)$ be in $N_\varepsilon(\bar{x})$ and $N_\varepsilon(\dot{\bar{x}})$, respectively, and let $\bar{\varepsilon} > 0$ be given.

By the continuity of $L_{uu}\big(t, x(t), \cdot\big)$ uniformly in t at $u(\cdot)$, there exists $\delta > 0$ such that, for $t \in [a,b]$ a.e.,

$$|L_{uu}\big(t, x(t), z(t)\big) - L_{uu}\big(t, x(t), u(t)\big)| < \bar{\varepsilon}/2 \tag{4.1}$$

whenever $\|z - u\|_\infty < \delta$.

Let $\bar{u}(\cdot) \in N_\delta\big(u(\cdot)\big)$. We need to show that

$$\|\mathcal{F}\big(x(\cdot), \bar{u}(\cdot)\big) - \mathcal{F}\big(x(\cdot), u(\cdot)\big) - \mathcal{F}_u\big(x(\cdot), u(\cdot)\big)\big(\bar{u}(\cdot) - u(\cdot)\big)\|_\infty < \bar{\varepsilon}\|u - \bar{u}\|_\infty$$

for some linear operator $\mathcal{F}_u$. In fact, for $t \in [a,b]$ a.e., the mean value Theorem gives

$$\begin{aligned}|L_u\big(t, x(t), \bar{u}(t)\big) - L_u\big(t, x(t), u(t)\big) - L_{uu}\big(t, x(t), u(t)\big)\big(\bar{u}(t) - u(t)\big)|\\ = |L_{uu}\big(t, x(t), \bar{\bar{u}}(t)\big) - L_{uu}\big(t, x(t), u(t)\big)\big(\bar{u}(t) - u(t)\big)|,\end{aligned}$$

where $\|\bar{\bar{u}} - u\|_\infty < \delta$. Using (4.1) twice we get the required inequality, where

$$\mathcal{F}_u\big(x(\cdot), z(\cdot)\big)(t) := L_{uu}\big(t, x(t), z(t)\big).$$

(ii) Let $\mathcal{B}(L^\infty, L^\infty)$ be the normed space of all bounded linear operators from L^∞ into L^∞. $\mathcal{F}_u : N_\varepsilon\big(\bar{x}(\cdot)\big) \times N_\varepsilon\big(\dot{\bar{x}}(\cdot)\big) \longrightarrow \mathcal{B}(L^\infty, L^\infty)$ is continuous, since $L_{uu}(t, \cdot, \cdot)$ is continuous uniformly in t at any $\big(x(\cdot), u(\cdot)\big) \in T(\bar{x}, \dot{\bar{x}}; \varepsilon)$.

(iii) $\mathcal{F}_u\big(\bar{x}(\cdot), \dot{\bar{x}}(\cdot)\big)$ is a homeomorphism of L^∞ onto L^∞: The strengthened Legendre condition yields that $\mathcal{F}_u\big(\bar{x}(\cdot), \dot{\bar{x}}(\cdot)\big)$ is a bijection and that $\tilde{L}_{uu}^{-1}(t)$ is essentially bounded. Since also $\tilde{L}_{uu}(t)$ is in $L^\infty[a,b]$, we get that $\mathcal{F}_u\big(\bar{x}(\cdot), \dot{\bar{x}}(\cdot)\big)$ is a homeomorphism.

(i)–(iii) allow us to apply the Implicit Function Theorem [22, II.3.8] to deduce the existence of ε_0, ε_1, and a *unique continuous* function $\mathcal{U}$ such that

$$\begin{aligned}&\mathcal{U} : N_{\varepsilon_0}\big(\bar{x}(\cdot)\big) \times N_{\varepsilon_0}\big(\bar{p}(\cdot)\big) \longrightarrow N_{\varepsilon_1}\big(\dot{\bar{x}}(\cdot)\big),\\ &p(\cdot) = \mathcal{F}\Big(x(\cdot), \mathcal{U}\big(x(\cdot), p(\cdot)\big)\Big)\end{aligned} \tag{4.2}$$

for all $\big(x(\cdot),p(\cdot)\big) \in N_{\varepsilon_0}\big(\tilde{x}(\cdot)\big) \times N_{\varepsilon_0}\big(\tilde{p}(\cdot)\big)$, and

$$\mathcal{U}\big(\tilde{x}(\cdot),\tilde{p}(\cdot)\big) = \dot{\tilde{x}}(\cdot).$$

Let us construct a function $u(\tau,x,p)$ on $T(\tilde{x},\tilde{p};\varepsilon_0)$. For a given $(\tau,x,p) \in T(\tilde{x},\tilde{p};\varepsilon_0)$, set

$$\big(x(s),p(s)\big) = \big(\tilde{x}(s)+x-\tilde{x}(\tau),\tilde{p}(s)+p-\tilde{p}(\tau)\big),$$

and define

$$u(\tau,x,p) = \mathcal{U}\big(x(\cdot),p(\cdot)\big)(\tau). \tag{4.3}$$

Then, $u(\cdot,\cdot,\cdot)$ is well-defined on $T(\tilde{x},\tilde{p};\varepsilon_0)$ with value in $T(\dot{\tilde{x}};\varepsilon_1)$. Also, (4.2) and (4.3) yield

$$\begin{aligned} p = L_u\big(\tau,x,u(\tau,x,p)\big) &\text{ for } (\tau,x,p) \in T(\tilde{x},\tilde{p};\varepsilon_0),\\ \text{and } u\big(t,\tilde{x}(t),\tilde{p}(t)\big) &= \dot{\tilde{x}}(t). \end{aligned} \tag{4.4}$$

Hypothesis (H_2) and condition $(L)'$ imply that $u(\cdot,\cdot,\cdot)$ is unique.

Now we will show that $u(t,\cdot,\cdot)$ is continuous uniformly in t at $(\tilde{x},\tilde{p})$. By the continuity of $\mathcal{U}$ at $(\tilde{x}(\cdot),\tilde{p}(\cdot))$ we have:

$$\forall\varepsilon > 0, \exists\delta > 0 : \text{ for } \|x-\tilde{x}\|_\infty < \delta \text{ and } \|p-\tilde{p}\|_\infty < \delta$$

we have

$$\|\,\mathcal{U}\big(x(\cdot),p(\cdot)\big) - \mathcal{U}\big(\tilde{x}(\cdot),\tilde{p}(\cdot)\big)\|_\infty < \varepsilon.$$

Thus, $\forall\varepsilon > 0, \exists\delta > 0$: for $t \in [a,b]$ a.e. and for all (y,p) with $|y-\tilde{x}(t)| < \delta$, $|p-\tilde{p}(t)| < \delta$ we have

$$\big|u(t,y,p) - u\big(t,\tilde{x}(t),\tilde{p}(t)\big)\big| < \varepsilon.$$

(4.4) shows that $(W)'$ is equivalent to $(*)$.

5. **Example.** Consider the problem

$$\text{minimize } \bar{J}(x) := \int_0^1 \Big\{\Big(\dot{x} - 2t|\sin\frac{\pi}{t}| + \pi\cos\frac{\pi}{t}\,\operatorname{sgn}(\sin\frac{\pi}{t})\Big)^2 - \frac{\dot{x}^2}{200}(x - t^2|\sin\frac{\pi}{t}|)^2\Big\}\,dt \tag{$\bar{P}$}$$

where $x\colon [a,b] \longrightarrow \mathbb{R}$ is absolutely continuous and satisfies

$$x(0) = x(1) = 0.$$

Set

$$L(t,x,u) = \Big(u - 2t|\sin\frac{\pi}{t}| + \pi\cos\frac{\pi}{t}\,\operatorname{sgn}\big(\sin\frac{\pi}{t}\big)\Big)^2 - \frac{u^2}{200}(x - t^2|\sin\frac{\pi}{t}|)^2.$$

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Define

$$\tilde{x}(t) = \begin{cases} t^2\left|\sin\dfrac{\pi}{t}\right| & \text{for } t \neq 0 \\ 0 & \text{for } t = 0 \end{cases}$$

it follows that $\tilde{x}(\cdot)$ is continuous on $[0, 1]$ with

$$\dot{\tilde{x}}(t) = 2t\left|\sin\frac{\pi}{t}\right| - \pi\cos\frac{\pi}{t}\,\mathrm{sgn}\left(\sin\frac{\pi}{t}\right)$$

for $t \neq 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and $|\dot{\tilde{x}}(t)| \leq 2 + \pi$ a.e. $t \in [0, 1]$. Thus, $\tilde{x}$ is Lipschitz.

We have for $t \in [0, 1]$ a.e.,

$$\begin{aligned}
L_x(t, x, u) &= -\frac{u^2}{100}\big(x - \tilde{x}(t)\big) \\
L_u(t, x, u) &= 2\big(u - \dot{\tilde{x}}(t)\big) - \frac{u}{100}\big(x - \tilde{x}(t)\big)^2 \\
L_{xx}(t, x, u) &= -\frac{u^2}{100} \\
L_{xu}(t, x, u) &= -\frac{u}{50}\big(x - \tilde{x}(t)\big) \\
L_{uu}(t, x, u) &= 2 - \frac{(x - \tilde{x}(t))^2}{100}.
\end{aligned}$$

It is easy to see that all the conditions of Hypothesis (H_2) are satisfied and that (E) and (L)$'$ hold. Moreover, for $t \in [0, 1]$ a.e.,

$$\begin{aligned}
E(t, x, v, u) &= \big(u - \dot{\tilde{x}}(t)\big)^2 - \frac{u^2}{200}\big(x - \tilde{x}(t)\big) - \big(v - \dot{\tilde{x}}(t)\big)^2 + \frac{v^2}{200}\big(x - \tilde{x}(t)\big)^2 \\
&\quad - 2\big(v - \dot{\tilde{x}}(t)\big)(u - v) + \frac{v}{100}\big(x - \tilde{x}(t)\big)^2(u - v) \\
&= (u - v)^2\left[1 - \frac{(x - \tilde{x}(t))^2}{200}\right].
\end{aligned}$$

Choose $\varepsilon < \sqrt{200}$, it results that $(W)'$ is satisfied. From (1.3) we have, for $t \in [0, 1]$ a.e.,

$$A(t) \equiv 0,\ B(t) \equiv \frac{1}{2},\ C(t) \equiv -\frac{\dot{\tilde{x}}^2(t)}{100},$$

and hence (1.1) is:

$$\begin{aligned}
\dot{\eta}(t) &= \frac{\xi(t)}{2} \\
\dot{\xi}(t) &= -\frac{\dot{\tilde{x}}^2(t)}{100}\eta(t),
\end{aligned}$$

or equivalently,

$$\ddot{\eta} + \frac{\dot{\tilde{x}}^2(t)}{200}(t)\eta = 0. \tag{5.1}$$

Since $\dot{\tilde{x}}^2(t) < 36$, consider the equation

$$\ddot{\eta} + \frac{9}{50}\eta = 0.$$

Its solutions η such that $\eta(0) = 0$ are of the form

$$\eta(t) = A \sin \frac{3}{5\sqrt{2}} t$$

and thus, do not vanish anywhere in $(0, 1]$. Using the comparison Theorem and in particular the result of Problem 10 on page 238 of [19], it follows that (5.1) has no points in $(0, 1]$ conjugate to 0, proving that $(J)'$ holds. Therefore, by Theorem 3, $\tilde{x}$ is a strong local minimum for $(\bar{P})$.

Note that $\tilde{x}$ is only Lipschitz and not C^1, moreover, L_u is not continuous in t. Thus, the known related result (see eg. [12], [18]) cannot be used, but Theorem 3 of this paper implies the strong local minimality of $\tilde{x}$ for $(\bar{P})$.

REFERENCES

1. N. I. Akhiezer, *The calculus of variations*. Blaisdell Publishing, Boston, 1962.

2. J. Ball and V. Mizel, *One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation*, Arch. Rat. Mech. Anal. **90**(1985), 325–388.

3. O. Bolza, *Lectures on the calculus of variations*. University of Chicago Press, 1904; reprinted by Chelsea Press, 1961.

4. C. Caratheodory, *Calculus of variations and partial differential equations of the first order*. (Holden-Day, 1965, 1967); reprinted by Chelsea Press, 1982.

5. L. Cesari, *Optimization—theory and applications*. Springer-Verlag, New York, 1983.

6. F. H. Clarke, *Optimization and nonsmooth analysis*. Wiley and Sons, New York 1983.

7. ———, *Methods of dynamic and Nonsmooth optimization*. CBMS-NSF5, Society for Industrial and Applied Mathematics, 1989.

8. F. H. Clarke and R. B. Vinter, *On the conditions under which the Euler equation or the maximum principle hold*, Appl. Math. Optim. **12**(1984), 73–79.

9. ———, *Regularity properties of solutions to the basic problem in the calculus of variations*, Trans. Amer. Math. Soc. **289**(1985), 73–98.

10. F. H. Clarke and P. D. Loewen, *Intermediate existence and regularity in the calculus of variations*, Appl. Math. Letters **1**(1988), 193–195.

11. ———, *An intermediate existence theory in the calculus of variations*, Annali Scuola Norm. Sup., to appear.

12. F. H. Clarke and V. Zeidan, *Sufficiency and the Jacobi condition in the calculus of variations*, Can. J. Math. **38**(1986) 1199–1209.

13. G. M. Ewing, *Calculus of variations with applications*. W.W. Norton and Co. New York, 1969.

14. W. H. Fleming and R. W. Rishel, *Deterministic and stochastic optimal control*. Springer-Verlag, New York, 1975.

15. I. M. Gelfand and S. V. Fomin, *Calculus of variations*. Prentice-Hall, Inc., New Jersey, 1963.

16. M. R. Hestenes, *Calculus of variations and optimal control theory*. Wiley and Sons, New York, 1966.

17. G. Leitmann, *The calculus of variations and optimal control*. Plenum Press, New York, 1981.

18. P. D. Loewen, *Second-order sufficiency criteria and local convexity for equivalent problems in the calculus of variations*, J. Math. Anal. Appl. **146**(1990), 512–522.

19. W. T. Reid, *Ordinary differential equations*. John Wiley & Sons, New York, 1971.

20. R. T. Rockafellar, *Existence theorems for general control problems of Bolza and Lagrange*, Adv. in Math. **15**(1975), 312–333.

21. H. Sagan, *Introduction to the calculus of variations*. McGraw-Hill, New York, 1969.

22. J. Warga, *Optimal control of differential and functional equations*. Academic Press, New York, 1972.

23. V. Zeidan, *Sufficient conditions for the generalized problem of Bolza*, Trans. Amer. Math. Soc. **275**(1983), 561–586.

24. ———, *A modified Hamilton-Jacobi approach in the generalized problem of Bolza*, Appl. Math. Optim. **11**(1984), 97–109.

25. ———, *First and second order sufficient conditions for optimal control and the calculus of variations*, Appl. Math. Optim. **11**(1984), 209–226.

26. ———, *Sufficiency conditions with minimum regularity assumptions*, Appl. Math. Optim. **20**(1989), 19–31.

27. V. Zeidan and P. Zezza, *The Jacobi necessary and sufficient condition for Lipschitz extrema*, Bolletino dell'Unione Matematica Italiana VII. Series B4, **2**(1990), 275–284.

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