THE AUTOMATA THEORY OF SEMIGROUP EMBEDDINGS

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1. Abstract

The theorem of Trevor Evans [1] that every countable semigroup can be embedded in a two-generator semigroup becomes obvious in automata theory as the statement that every countable automaton can be embedded in one with binary inputs. Standard techniques of automata theory [1], [3] yield a proof of the Evans Theorem using wreath products, as in Neumann [4].

2.

An automaton M (without output) is a triple (Q, X, δ) where Q is the set of states, X is the input set, and $\delta : Q \times X \to Q$. The semigroup $S(M)$ is the subsemigroup of Q^Q generated by the maps $\delta(\cdot, x) : Q \to Q$. M is called countable [finite] if Q is countable [finite].

Given a countable semigroup S with generators $G(S)$, we may represent it as the semigroup of the machine $M_S = (S^1, G(S), \delta_S)$ where δ_S is multiplication in the semigroup S, and S^1 is S with a unit adjoined only if S is not a monoid.

We then replace M_S by a machine which has input set $\{0, 1\}$ and which reads in strings until a code (i.e., a monomorphism of $\mathscr{F}_{G(S)}$, the free semigroup generated by $G(S)$, into $\mathscr{F}_{\{0,1\}}$) for an element of $G(S)$ has been read, and then acts accordingly.

We now give two examples of codes and the corresponding constructions.

One which works whether or not $G(S) = \{s_1, s_2, \cdots\}$ is finite is to code s_j as $1^j 0$, i.e. a string of j ones followed by a zero. Then

$$M_1 = (N \times S^1, \{0, 1\}, \delta_2) \qquad (\text{taking } N = \{1, 2, 3, \cdots\})$$

with

$$\delta_1((n, s), 0) = (0, s \cdot s_n)$$
$$\delta_1((n, s), 1) = (n+1, s)$$

and the map $s_j \to 1^j 0$ yields an embedding of S in the two-generator semigroup $S(M_1)$. If $G(S) = \{s_1, \cdots, s_d\}$ is finite, we can replace N by $\{1, \cdots, d\}$, addition then being modulo d to handle sequences not coding words of S, so that $S(M_1)$ *is finite if* S *is finite.*

If $G(S) = \{s_1, \cdots, s_d\}$ is finite and $2^{m-1} < d \leqq 2^m$ we may give a 'faster' construction, encoding each s_j as a distinct string t_j of m 0's and 1's. Then if A is the set of strings of at most $m-1$ 0's and 1's (including the empty string Λ) we may set

$$M_2 = (A \times S^1, \{0, 1\}, \delta_2)$$

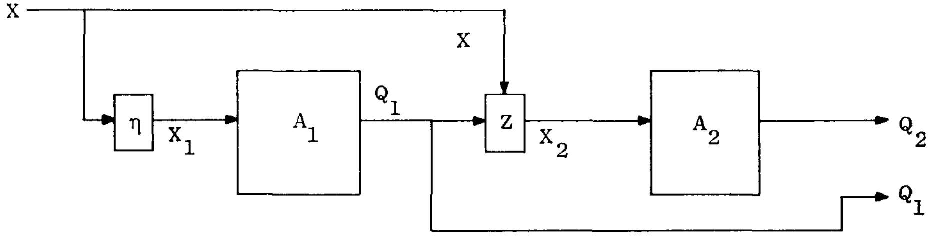
with

$$\delta_2([u_1 \cdots u_k, s], u) = \begin{cases} [u_1 \cdots u_k u, s] & \text{if } \ 0 \leqq k < n-1 \\ [\Lambda, s \cdot s_j] & \text{if } \ k = n-1 \text{ and } t_j = u_1 \cdots u_k u \\ [\Lambda, s] & \text{else} \end{cases}$$

and the map $s_j \to t_j$ yields an embedding of S in the two-generator semigroup $S(M_2)$.

It should be clear how this construction can be extended to arbitrary codes.

We now show how our result yields Neumann's wreath-product approach. The cascade $A_2 \times_Z^\eta A_1$ of any two automata $A_j = (Q_j, X_j, \lambda_j)$ with connecting map $Z : Q_1 \times X \to X_2$ and encoder $\eta : X \to X_1$ is the automaton with $\lambda_Z^\eta([q_2, q_1], x) = [\lambda_2(q_2, Z(q_1, x)), \lambda_1(q_1, \eta(x))]$ and (as was first pointed out by Krohn and Rhodes [3], see also [1]) the semigroup of $A_2 \times_Z^\eta A_1$ can be embedded in the wreath product of $S(A_2)$ and $S(A_1)$.



$(Q_2 \times Q_1, X, \lambda_Z)$

Now it is clear from the definitions of δ_1 and δ_2 that both M_1 and M_2 are cascades of some machine A_1 with M_{S^1}. In the case of the first construction, the semigroup T_0 of A_1 is $N^0[\{1, \cdots, d\}^0$ if $G(S)$ is finite] — that is, the integers under addition [modulo d] with a *multiplicative* zero adjoined. In either case, we see that S can be embedded in a two-generator semigroup which can in turn be embedded in the wreath product of $S(A_1)$ and S^1. The first construction yields what is essentially Neumann's proof. However, where Neumann [4] took the order m of his cyclic group T to be at least $3d$

in the finite case, we only need take $m = d$, though we must adjoin our 0 to T whereas he adjoins his zero to S. It then becomes a simple exercise to obtain the results of Neumann's Section 5 with improved bounds. We hope this note will encourage further applications of automata theory to the algebraic theory of semigroups.

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