Correspondence

DEAR EDITOR,

In Note 81.26 (July 1997) there appears a version of an often-repeated incorrect statement about Pythagorean triples, namely that for integers $m > n > 0$ the formulae

$$x = m^2 - n^2, \qquad y = 2mn, \qquad z = m^2 + n^2 \tag{1}$$

give all the positive-integer solutions of $x^2 + y^2 = z^2$.

The correct result is, of course, that these formulae with *coprime* integers $m > n > 0$ *of opposite parity* give all the **primitive** Pythagorean triples (x, y, z), i.e. those positive-integer solutions x, y, z having no common divisor (apart from 1); and then all Pythagorean triples are given by

$$x = (m^2 - n^2)k, \qquad y = 2mnk, \qquad z = (m^2 + n^2)k \tag{2}$$

for any positive integer k.

If m and n have greatest common divisor d, say $m = m'd$ and $n = n'd$, then (1) becomes

$$x = (m'^2 - n'^2)d^2, \qquad y = 2m'n'd, \qquad z = (m'^2 + n'^2)d^2, \tag{3}$$

where $m' > n' > 0$ and m', n' are coprime; but (3) fails to yield *all* the Pythagorean triples because, even when m', n' have opposite parity, k in (2) need not be a perfect square.

If we employ (2) instead of (1), we find that the argument in Note 81.26 gives $b = \frac{1}{2}(m^2 + n^2)h$ and $ac = \frac{1}{4}mn(m^2 - n^2)h^2$, where $m > n > 0$ are coprime integers with opposite parity and h is now any *even* positive integer, say $h = 2k$. So all the desired monic quadratics (i.e. with $a = 1$) are

$$x^2 + (m^2 + n^2)kx + mn(m^2 - n^2)k^2 = 0.$$

They can be listed systematically in families (for given m, n and variable k), starting with $m = 2, n = 1$. For example, the (4, 1) family is

$$\{x^2 + 17kx + 60k^2 = 0 : k \in \mathbb{N}\}.$$

Yours sincerely,

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DEAR EDITOR,

I would like to comment on two recent notes.

1. Note 81.1 *A Pascal-like triangle for* $\alpha^n + \beta^n$.

Since α and β are roots of $ax^2 + bx + c = 0$ then

$$\alpha^{n+2} - l\alpha^{n+1} + m\alpha^n = 0,$$

where $\alpha + \beta = l$ and $\alpha\beta = m$ as defined in Note 81.1. An equivalent result holds for β.

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By adding these two equations, it follows

$$\left(\alpha^{n+2} + \beta^{n+2}\right) - l\left(\alpha^{n+1} + \beta^{n+1}\right) + m\left(\alpha^{n} + \beta^{n}\right) = 0. \qquad (1)$$

Now $C_{n,r}$ is the absolute value of the coefficient of $l^{n-2r}m^{r}$ in $\alpha^n + \beta^n$. By considering coefficients in (1), we obtain at once the result

$$C_{n,r} + C_{n+1,r+1} = C_{n+2,r+1}$$

which was proved in Note 81.1.

2. Note 81.38 *Going dotty with vectors.*

There are two definitions of scalar (or dot) product in common use:

(1) $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + \dots + a_nb_n$, where $\mathbf{a} = (a_1, a_2, \dots, a_n)$ etc

(2) $\mathbf{a}.\mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

These are equivalent, that is, (1) $\Leftrightarrow$ (2). Most good elementary texts take one of these as a 'definition' and deduce the other as a 'property'.

In Note 81.38 Taverner gives a proof that (2) follows from (1) for the case of two-dimensional vectors. It is not clear to me in what sense this is a 'derivation'. The simplest proof of equivalence (for any dimension) makes use of the cosine rule in triangle *OAB*.

For the vector (or cross) product, there are also two equivalent definitions in common use. Treatments of the proof of their equivalence may be found in the references. [1] and [2] show that the geometric definition follows from the algebraic; [3] and [4] adopt the converse approach.

References

1. Howard Anton, *Elementary linear algebra (Fifth edition)*, John Wiley, New York (1987).
2. D. Griffiths, *Pure mathematics*, Harrap (1984).
3. R. I. Porter, *A school course in vectors*, Bell and Hyman (1970).
4. Murray R. Spiegel, *Advanced calculus*, McGraw-Hill, NewYork (1963).

Yours sincerely,

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DEAR EDITOR,

With reference to Note 81.49 on the Steiner-Lehmus Theorem, it would appear that the second sentence of the first paragraph is missing. The missing sentence should read.

'If two internal bisectors are equal in length, the triangle is also isosceles, but the demonstration in this case is more challenging.'

Without this second sentence, it appears that the first sentence of Note