MAXIMAL DETERMINANTS IN COMBINATORIAL INVESTIGATIONS

H. J. RYSER

1. Introduction. Let Q be a matrix of order v, all of whose entries are 0's and 1's. Let the total number of 1's in Q be t, and let the absolute value of the determinant of Q be denoted by $|\det Q|$. In this paper we study the problem of determining the maximum of $|\det Q|$ for fixed t and v. It turns out that this problem is closely related to the v, k, λ problem, which has been extensively studied of late.

A v, k, λ configuration is defined as an arrangement of v elements $x_1, x_2, \ldots, x_v$ into v sets $S_1, S_2, \ldots, S_v$ such that each set contains exactly k distinct elements and such that each pair of sets has exactly λ elements in common $(0 < \lambda < k < v)$. If element x_j belongs to set S_i, let $a_{ij} = 1$; and if x_j does not belong to S_i, let $a_{ij} = 0$. The v by v matrix $A = [a_{ij}]$ is called the *incidence matrix* of the v, k, λ configuration. These matrices have been very useful in establishing the nonexistence of certain configurations (**1**; **2**). A general survey of the literature pertaining to v, k, λ configurations may be found in (**4**). In particular one proves that in a v, k, λ configuration,

$$k - \lambda = k^2 - \lambda v$$

and

$$AA^{\mathrm{T}} = A^{\mathrm{T}}A = B.$$

Here A^{T} denotes the transpose of the incidence matrix A, and the matrix B has k in the main diagonal and λ in all other positions. It is easy to see that $\det B = k^2(k - \lambda)^{v-1}$, whence it follows that

$$|\det A| = k(k - \lambda)^{\frac{1}{2}(v-1)}.$$

2. Theorems on maximal determinants.

THEOREM 1. *Let Q be a* 0, 1 *matrix of order v, containing exactly t 1's. Let k denote a positive real, and set $\lambda = k(k-1)/(v-1)$. If $t \leqslant kv$ and $0 < \lambda \leqslant k - \lambda$, or if $t \geqslant kv$ and $0 < k - \lambda \leqslant \lambda$, then*

$$|\det Q| \leqslant k(k - \lambda)^{\frac{1}{2}(v-1)}.$$

Let E be a 0, 1 matrix. Let $E(x, y)$ denote the matrix formed from E by replacing each 1 of E by x and each 0 of E by y, where x and y are indeterminates. Using this notation, we may write

$$Q_1 = Q(-(k - \lambda)/\lambda, 1).$$

Received May 31, 1955.

Now set $p = (k - \lambda)/\lambda$, and define the matrix $\bar{Q}$ of order $v + 1$ by

$$\bar{Q} = \begin{bmatrix} p & z \\ z^T & Q_1 \end{bmatrix}, \tag{1}$$

where $z = (\sqrt{p}, \ldots, \sqrt{p})$. By the Hadamard determinant theorem,

$$|\det \bar{Q}| \leqslant \sqrt{p^2 + vp} \prod_{i=1}^{v} \sqrt{p + s_i}\,, \tag{2}$$

where s_i denotes the sum of the squares of the ith row of Q_1. Now

$$p^2 + vp = p\left(\frac{k - \lambda + \lambda v}{\lambda}\right) = \frac{k^2}{\lambda^2}(k - \lambda).$$

Moreover,

$$s_1 + \ldots + s_v = tp^2 + (v^2 - t) = t(p^2 - 1) + v^2.$$

By hypothesis, $t \leqslant kv$ and $p^2 \geqslant 1$, or $t \geqslant kv$ and $p^2 \leqslant 1$. Hence we may conclude

$$s_1 + \ldots + s_v \leqslant kv(p^2 - 1) + v^2.$$

Now introduce quantities $\bar{s}_i$ such that

$$\bar{s}_i \geqslant s_i$$

and

$$\bar{s}_1 + \ldots + \bar{s}_v = v(kp^2 + v - k). \tag{3}$$

By (3),

$$\sum_{i=1}^{v} (p + \bar{s}_i) = v(kp^2 + v - k + p) = v[kp^2 + (\lambda v - \lambda k + k - \lambda)/\lambda]$$
$$= vkp(p + 1) = v(k - \lambda)k^2/\lambda^2.$$

Since the geometric mean of v positive quantities is less than or equal to their arithmetic mean, we may write

$$\prod_{i=1}^{v} (p + \bar{s}_i) \leqslant \left(\frac{1}{v}\sum_{i=1}^{v} (p + \bar{s}_i)\right)^v, \tag{4}$$

whence

$$\prod_{i=1}^{v} (p + \bar{s}_i) \leqslant (k - \lambda)^v k^{2v}/\lambda^{2v}. \tag{5}$$

Hence by (2),

$$|\det \bar{Q}| \leqslant \frac{k}{\lambda}\sqrt{k - \lambda} \prod_{i=1}^{v} \sqrt{p + \bar{s}_i} \tag{6}$$
$$\leqslant \frac{k}{\lambda}\sqrt{k - \lambda}\left(\frac{k}{\lambda}\sqrt{k - \lambda}\right)^v = \left(\frac{k}{\lambda}\sqrt{k - \lambda}\right)^{v+1}.$$

To evaluate det $\bar{Q}$, multiply row one by $-1/\sqrt{p}$ and add the resulting row to each of the other rows. From (6) it follows that

$$|\det \bar{Q}| = p|\det Q(-k/\lambda, 0)| \leqslant (k\sqrt{k-\lambda}/\lambda)^{v+1}. \tag{7}$$

But

$$|\det Q(-k/\lambda, 0)| = (k/\lambda)^v|\det Q|, \text{ whence}$$

$$p|\det Q| \leqslant \frac{k}{\lambda}(\sqrt{k-\lambda})^{v+1},$$

and

$$|\det Q| \leqslant k(\sqrt{k-\lambda})^{v-1}.$$

Using the notation of Theorem 1, we have

THEOREM 2. *If* $|\det Q| = k(k-\lambda)^{\frac{1}{2}(v-1)}$, *then* Q *is the incidence matrix of a* v, k, λ *configuration.*

If equality holds in Theorem 1, then

$$p\left|\det Q\left(-\frac{k}{\lambda}, 0\right)\right| = \left(\frac{k\sqrt{k-\lambda}}{\lambda}\right)^{v+1},$$

and by (7),

$$|\det \bar{Q}| = (k\sqrt{k-\lambda}/\lambda)^{v+1}. \tag{8}$$

Equality in (6) implies equality in (5) and (4). But for equality to hold in (4), we must have

$$p + \bar{s}_i = (k-\lambda)k^2/\lambda^2.$$

But then the equality in (6) implies

$$\bar{Q}\bar{Q}^T = \frac{k^2(k-\lambda)}{\lambda^2} I, \tag{9}$$

where I is the identity matrix of order $v+1$. Thus

$$Q_1Q_1^T = \frac{k^2}{\lambda^2}(k-\lambda) I - pS, \tag{10}$$

where $Q_1 = Q(-p, 1)$, and S is the v by v matrix of all 1's. Let e denote the number of 1's in row r of Q. Then

$$p^2e + (v-e)\cdot 1 = \frac{k^2}{\lambda^2}(k-\lambda) - p,$$

and

$$(p^2 - 1)e = \frac{k^2}{\lambda^2}(k-\lambda) - p - v,$$

whence we conclude that $e = k$. Let f denote the inner product of rows r and s of Q, where $r \neq s$. Then

$$fp^2 - 2(k - f)p + v - 2k + f = -p,$$

whence

$$f(p^2 + 2p + 1) = 2kp - p + 2k - v,$$

and $fk^2/\lambda^2 = k^2/\lambda$. Thus $f = \lambda$, and Q is the incidence matrix of a v, k, λ configuration.

It is now clear that we have established the following:

THEOREM 3. *Let Q be a 0, 1 matrix of order v, containing exactly t 1's. Let $k = t/v$ and set $\lambda = k(k-1)/(v-1)$, with $0 < \lambda < k < v$. Then*

$$|\det Q| \leqslant k(k-\lambda)^{\frac{1}{2}(v-1)},$$

and equality holds if and only if Q is the incidence matrix of a v, k, λ configuration.

Consider once again Theorem 1. Note that $(k - \lambda)/\lambda = (v - k)/(k - 1)$. Thus the requirement $\lambda \leqslant k - \lambda$ means $k \leqslant \frac{1}{2}(v + 1)$, and $k - \lambda \leqslant \lambda$ means $k \geqslant \frac{1}{2}(v + 1)$. Suppose that $k = \frac{1}{2}(v + 1)$. Then if Q is a 0, 1 matrix with no restriction on the number of 1's, we must have

$$|\det Q| \leqslant \frac{(v+1)^{\frac{1}{2}(v+1)}}{2^v}. \tag{11}$$

The incidence matrix associated with the case of equality has parameters $v = 4\lambda - 1$, $k = 2\lambda$, $\lambda = \lambda$. These incidence matrices give rise to the Hadamard matrices of order 4λ **(3)**. The determination of the maximum of $|\det Q|$, where Q is of arbitrary order v, is an unsolved problem of considerable difficulty **(5)**.

If we place no restriction on the number of 1's in the 0, 1 matrix Q of order v and assume that $|\det Q| = k(k - \lambda)^{\frac{1}{2}(v-1)}$, then we may not conclude in general that Q is the incidence matrix of a v, k, λ configuration. For example, let A be an incidence matrix of a v, k, λ configuration with $v - 2k > 0$. Define its complement C by $A + C = S$, where S is the matrix of all 1's. The complement of A is again a v, k, λ configuration with parameters $\bar{v} = v$, $\bar{k} = v - k$, and $\bar{\lambda} = v - 2k + \lambda$. Note that

$$|\det C| = (v - k)(k - \lambda)^{\frac{1}{2}(v-1)}.$$

It is easy to check that

$$A^{-1} = \frac{1}{(k-\lambda)}\left(A^{T} - \frac{\lambda}{k} S\right),$$

where A^{-1} denotes the inverse of A. Thus in $A = [a_{rs}]$, if $a_{rs} = 1$, then the cofactor of a_{rs},

$$A_{rs} = \frac{1}{k} \det A.$$

Similarly for the complement $C = [c_{rs}]$, if $c_{rs} = 1$, then the cofactor of c_{rs},

https://doi.org/10.4153/CJM-1956-028-4 Published online by Cambridge University Press

$$C_{rs} = \frac{1}{v - k} \det C.$$

We are assuming that $v - 2k > 0$. Thus we may replace $v - 2k$ of the 1's in the first row of C by 0's. The resulting matrix Q is a 0, 1 matrix satisfying

$$|\det Q| = k(k - \lambda)^{\frac{1}{2}(v-1)},$$

but Q is not an incidence matrix of a v, k, λ configuration.

References

1. R. H. Bruck and H. J. Ryser, *The nonexistence of certain finite projective planes*, Can. J. Math., *1* (1949), 88–93.
2. S. Chowla and H. J. Ryser, *Combinatorial problems*, Can. J. Math., *2* (1950), 93–99.
3. R. E. A. C. Paley, *On orthogonal matrices*, J. Math. Phys., *12* (1933), 311–320.
4. H. J. Ryser, *Geometries and incidence matrices*, Slaught Memorial Papers (Suppl. Amer. Math. Monthly), *62* (1955), 25–31.
5. John Williamson, *Determinants whose elements are* 0 *and* 1, Amer. Math. Monthly, *53* (1946), 427–434.

Ohio State University