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MONOTONE LIPSCHITZIAN SEMIGROUPS IN BANACH SPACES

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Abstract

We prove the existence of common fixed points for monotone contractive and monotone nonexpansive semigroups of nonlinear mappings acting in Banach spaces equipped with partial order. We also discuss some applications to differential equations and dynamical systems.

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1. Introduction

The question of existence of fixed points for contractions and nonexpansive mappings acting in Banach spaces, as well as of common fixed points for semigroups of such mappings, has been thoroughly investigated since the early 1960s. In recent years, we observe an emergence of a new research stream, which is focused on dealing with fixed point theorems in Banach, or more general, metric spaces equipped with partial order. In this case, the Lipschitzian assumption is only made for comparable (in the order sense) elements, unlike in the general theory, where such an assumption is made for all elements. Since the set of such comparable elements may be quite small, proofs often require the use of innovative techniques and special attention needs to be paid to avoid the forces of habit. For instance, monotone contractions or monotone nonexpansive mappings do not have to be continuous while the continuity of nonexpansive mappings is typically taken for granted. Moreover, some of the results are counter-intuitive as well. For example, the uniqueness of a fixed point takes a new and different form. However, an interesting and always evolving body of applications, as well as the linkage to graph theory (see [1, 8]), make this new field important and promising.

Ran and Reurings [17] initiated this research direction in relation to a class of matrix equations. The study of these matrix equations is motivated by applications, including stochastic filtering, control theory, and dynamic programming [7]. Nieto and Rodriguez-Lopez [16] improved the results of Ran and Reurings, using similar arguments to find periodic solutions for a class of differential equations. Khamsi and Khan [9] used this approach to prove the convergence of the Krasnoselskii–Ishikawa iteration process to fixed points of a monotone nonexpansive mapping acting in L_1. Bin Dehaish and Khamsi [6] proved analogs of Browder and Göhde fixed point theorems for monotone nonexpansive mappings acting in uniformly convex hyperbolic spaces and uniformly convex in every direction Banach spaces. For more information on the results in monotone fixed point theory, the reader is referred to the recent survey article by Bachar and Khamsi [3], see also [2].

So far, not much has been done for the development of the theory of common fixed points for semigroups of monotone contractions and monotone nonexpansive mappings. Bachar and Khamsi [5] showed nontrivial examples of such semigroups and proved some results on approximate fixed point sequences for such semigroups. However, the main issue slowing down the development of this theory is the fact that there have been no general existence theorems proven for such semigroups. The likely reason is that the iterative methods of construction used for a single monotone mapping [6] do not adapt well to the case of semigroups.

In the current paper, we were able to demonstrate, for the first time, the existence of common fixed points of monotone contractive semigroups and monotone nonexpansive semigroups of nonlinear mappings acting in Banach spaces. Therefore, the results presented in the paper provide a solid base for future investigations of monotone semigroups from the fixed point theory perspective.

The paper is organized as follows:

(a) Section 2 provides necessary preliminary material.
(b) Section 3 presents the main existence results for common fixed points of monotone contractive and monotone nonexpansive semigroups.
(c) Section 4 discusses applications to the theory of ordinary differential equations in ordered Banach spaces, shows when the solution set forms a monotone nonexpansive semigroup and interprets the common fixed points as the stationary points of an associated dynamical process.
(d) Section 5 touches upon the question of the algorithmic construction of such common semigroups referring the reader to the appropriate literature.
(e) Section 6 summarizes what has been achieved in the paper.

2. Preliminaries

Throughout this paper X will always denote a Banach space, and C a nonempty, bounded, closed and convex subset of X. Let us denote by $d(C)$ the diameter of C which, by boundedness of C is a finite number. Throughout this paper J will be a fixed parameter semigroup of nonnegative numbers, that is, a subsemigroup of $[0, \infty)$ with the normal addition. We assume that $0 \in J$ and that there exists $t > 0$ such that $t \in J$. The latter assumption implies immediately that $+\infty$ is a cluster point of J in the

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sense of the natural topology inherited by J from $[0, \infty)$. As a convention, by $t \to \infty$ we will always understand that t converges to ∞ over J. The typical examples of such parameter semigroups are: $J = [0, \infty)$ and $J = \{0, 1, 2, 3, \ldots\}$ but also ideals of the form $J = \{n\alpha; n = 0, 1, 2, 3, \ldots\}$ for a given $\alpha > 0$.

We assume in addition that X is endowed with a partial order '$\preceq$'. Let us recall that an order interval is any of the subsets $[a, \rightarrow) = \{y \in X : a \preceq y\}$ and $(\leftarrow, b] = \{y \in X : y \preceq b\}$. For $a \preceq b$ we will also use the notation $[a, b] = [a, \rightarrow) \cap (\leftarrow, b]$. Throughout this paper, we will always assume that the partial order '$\preceq$' and the linear structure of X are linked to each other by the following convexity property:

$$a \preceq b, \quad c \preceq d \Rightarrow \alpha a + (1 - \alpha)c \preceq \alpha b + (1 - \alpha)d \tag{2.1}$$

for any $\alpha \in [0, 1]$ and all $a, b, c, d \in X$. In view of the previous assumptions, it is clear that all order intervals are convex. We will also assume that all order intervals are closed. We will say that $x \in X$ and $y \in X$ are comparable if either $x \preceq y$ or $y \preceq x$.

Let us start with the more formal definitions of monotone Lipschitzian mappings and monotone Lipschitzian semigroups of mappings.

Definition 2.1. We say that $T : C \to C$ is a monotone (or order-preserving) mapping if

$$x \preceq y \Rightarrow T(x) \preceq T(y).$$

Definition 2.2. We say that $T : C \to C$ is a monotone Lipschitzian mapping if it is monotone and there is a constant $L > 0$ such that

$$\|T(x) - T(y)\| \leq L\|x - y\| \tag{2.2}$$

for any $x, y \in C$ such that x and y are comparable in the sense of the partial order '$\preceq$'. Such a mapping will also be called L-monotone Lipschitzian. If $L < 1$, then T will be called a monotone contraction. Similarly, if $L = 1$ then T is called a monotone nonexpansive mapping.

An element $w \in C$ is called a fixed point of T if $T(w) = w$. The set of all fixed points of T will be denoted by $\mathrm{Fix}(T)$.

Note that (2.2) needs to hold only for comparable x and y, which implies, among others, that monotone Lipschitzian mappings do not have to be continuous.

We illustrate this notion by the following example with an immediate application to the theory of integral equations.

Example 2.3 [4]. Let $X = L^2([0, 1], \mathbb{R})$ be equipped with the partial order defined by: $x \preceq y \Leftrightarrow x(t) \leq y(t)$ almost everywhere for $t \in [0, 1]$. Consider the integral equation

$$x(t) = g(t) + \int_0^1 F(t, s, x(s))\, ds, \quad t \in [0, 1], \tag{2.3}$$

where $g \in X$ and $F : [0, 1] \times [0, 1] \times X \to \mathbb{R}$ is measurable and satisfies

$$0 \leq F(t, s, x(s)) - F(t, s, y(s)) \leq x(t) - y(t),$$

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for $t, s \in [0, 1]$ and $x, y \in X$ such that $y \leq x$. Assume also that there exists a nonnegative function $h(\cdot, \cdot) \in L^2([0, 1] \times [0, 1])$ and $M < \frac{1}{2}$ such that for every $t, s \in [0, 1]$ and every $u \in L^2([0, 1]$,

$$|F(t, s, u)| \leq h(t, s) + M|u|,$$

almost everywhere. Let B be a closed ball in X centered at zero, with radius $\rho > 0$, where ρ is chosen so that

$$\frac{2}{(1 - 4M^2)} \int_0^1 |g(t)|^2 \, dt + \frac{4}{(1 - 4M^2)} \int_0^1 \int_0^1 h^2(t, s) \, ds \, dt \leq \rho^2.$$

Define the operator $J : X \to X$ by

$$J(x)(t) = g(t) + \int_0^1 F(t, s, x(s)) \, ds.$$

It is not difficult to show, using the Cauchy–Schwarz inequality, that J is a monotone nonexpansive mapping acting within B. Using a fixed point existence for monotone nonenxpansive mappings, [4], it can be proved that the integral equation (2.3) has a nonnegative solution provided that

$$g(t) + \int_0^1 F(t, s, 0) \, ds \geq 0$$

for almost everywhere $t \in [0, 1]$, which implies also that $J(0) \geq 0$.

For further examples, we refer the reader to [3]. Let us now introduce the notion of monotone Lipschitzian semigroups.

DEFINITION 2.4. A one-parameter family $\mathcal{S} = \{T_t : t \in J\}$ of mappings from C into itself is said to be a monotone Lipschitzian semigroup on C if $\mathcal{S}$ satisfies the following conditions:

(i) there exists $L > 0$ such that all T_t are monotone L-Lipschitzian mappings;
(ii) $T_0(x) = x$ for $x \in C$;
(iii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \in J$;
(iv) for each $x \in C$, the mapping $t \mapsto T_t(x)$ is norm continuous.

For each $t \in J$ let $\mathrm{Fix}(T_t)$ denote the set of its fixed points. Define then the set of all common set points for mappings from $\mathcal{S}$ as

$$\mathrm{Fix}(\mathcal{S}) = \bigcap_{t \in J} \mathrm{Fix}(T_t).$$

If $L < 1$ then $\mathcal{S}$ will be called a monotone contractive semigroup. Similarly, if $L = 1$ then $\mathcal{S}$ will be called a monotone nonexpansive semigroup.

For the discussion on examples and applications of such semigroups please refer to Section 4.

3. Existence theorems

In order to prove our existence results, we will need to assume some form of a linkage between the order structure and the semigroup $\mathcal{S}$. This is not really surprisingly, as otherwise we could have a situation where no element from set C was comparable with its images via mappings from $\mathcal{S}$ and hence it would be very difficult to expect an existence of a common fixed point. We will assume consequently that there exists an $x \in C$ such that $x \preceq T_t(x)$ for all $t \in J$ and we will use this element as the starting point of our construction of a common fixed point. The reader will note that we could replace this assumption by $T_t(x) \preceq x$ and the proofs would follow an analogous path.

Let us start with the question of an existence of a common fixed point for the monotone contractive semigroups.

Theorem 3.1. *Assume C is weakly compact. Let $\mathcal{S}$ be a monotone contractive semigroup on C. Assume in addition that there exists $x \in C$ such that $x \preceq T_t(x)$ for all $t \in J$. Then there exists a common fixed point $z \in \mathrm{Fix}(\mathcal{S})$ such that $x \preceq z$ and $\|T_t(x) - z\| \to 0$ as $t \to \infty$. Moreover, if w is another common fixed point, which is comparable with z, then $w = z$.*

Proof. Define

$$C_x = C \cap \Big(\bigcap_{s \in J} [T_s(x), \to) \Big).$$

In order to prove that C_x is not empty, note that for any finite family $t_1, \ldots, t_n \in J$,

$$T_{t_i}(x) \preceq T_{t_1 + \cdots t_n}(x),$$

because for any $s \in J$, $x \preceq T_s(x)$ and T_s is monotone. Hence

$$T_{t_1 + \cdots t_n}(x) \in C \cap \Big(\bigcap_{s \in \{t_1, \ldots, t_n\}} [T_s(x), \to) \Big).$$

The weak compactness of C and the fact that each $[T_s(x), \to)$ is closed convex force C_x to be nonempty. It is also clear that C_x is a closed and convex. Hence, C_x is weakly compact. Observe that $T_s(C_x) \subset C_x$ for every $s \in J$. Indeed, let $y \in C_x$. Then $T_t(x) \preceq y$ for every $t \in J$. Since T_t is monotone for every $t \in J$ and $x \preceq T_s(x)$

$$T_t(x) \preceq T_{t+s}(x) \preceq T_s(y),$$

hence, $T_s(y) \in C_x$. Define the type function $\rho : C_x \to [0, \infty)$ by

$$\rho(y) = \limsup_{t \to \infty} \|T_t(x) - y\|,$$

for $y \in C_x$. It is easy to show that ρ is a convex lower semicontinuous function on C_x. Hence, by Mazur's lemma, ρ is also weakly lower semicontinuous on C_x. Since C_x

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is weakly compact it follows that ρ attains its infimum at a point $z \in C_x$. Since $z \in C_x$ then $T_s(x) \preceq z$ for any $s \in J$. Let us fix temporarily $s \in J$.

$$\rho(T_s(z)) = \limsup_{t\to\infty} \|T_t(x) - T_s(z)\| = \limsup_{t\to\infty} \|T_{s+t}(x) - T_s(z)\|$$
$$\le L \limsup_{t\to\infty} \|T_t(x) - z\| = L\rho(z). \tag{3.1}$$

Using the minimality of z

$$\rho(z) \le \rho(T_s(z)) \le L\rho(z),$$

which implies that $\rho(z) = 0$ because $L < 1$. Then, by (3.1), for every $s \in J$

$$\lim_{t\to\infty} \|T_t(x) - T_s(z)\| = 0. \tag{3.2}$$

Taking $s = 0$

$$\lim_{t\to\infty} \|T_t(x) - z\| = 0. \tag{3.3}$$

From (3.2) and (3.3) it follows immediately that $T_s(z) = z$, that is, $z \in \mathrm{Fix}(\mathcal{S})$ as claimed.

To prove the uniqueness part, let us take another $w \in \mathrm{Fix}(\mathcal{S})$ which is comparable with z. Hence for each $t \in J$,

$$\|z - w\| = \|T_t(z) - T_t(w)\| \le L\|z - w\|,$$

which gives us $w = z$ because $L < 1$. This completes the proof of the theorem. □

Let us prove now the existence of common fixed points for monotone nonexpansive semigroups.

Theorem 3.2. *Assume X is uniformly convex. Let $\mathcal{S}$ be a monotone nonexpansive semigroup on C. Assume in addition that there exists $x \in C$ such that $x \preceq T_t(x)$ for all $t \in J$. Then $\mathcal{S}$ has a common fixed point $z \in \mathrm{Fix}(\mathcal{S})$ such that $x \preceq z$. Moreover, if $f_1, f_2 \in \mathrm{Fix}(\mathcal{S})$ are comparable then $f = cf_1 + (1 - c)f_2 \in \mathrm{Fix}(\mathcal{S})$ for every $c \in [0, 1]$.*

Proof. Let us define again

$$C_x = C \cap \Big(\bigcap_{s\in J} [T_s(x), \to)\Big).$$

Hence, as shown in the proof of Theorem 3.1, C_x is a nonempty, convex and weakly compact subset of C. Let us define on C_x the type function τ by the formula

$$\tau(y) = \limsup_{t\to\infty} \|T_t(x) - y\|^2.$$

Similarly as in the proof of Theorem 3.1, we can deduce that τ attains its infimum in C_x, that is, there exists a $z \in C_x$ such that $\tau(z) = \inf\{\tau(y) : y \in C_x\}$. Let us fix arbitrarily $t, s, u \in J$. Because X is uniformly convex it follows from [10, Proposition

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3.4] (see also [11, Theorem 3.4] and [18, Theorem 2]) that for each $d > 0$ there exists a continuous function $\lambda : [0, \infty) \to [0, \infty)$ such that $\lambda(t) = 0 \Leftrightarrow t = 0$, and

$$\|cw + (1-c)v\|^2 \leq c\|w\|^2 + (1-c)\|v\|^2 - c(1-c)\lambda(\|w - v\|), \tag{3.4}$$

for any $c \in [0, 1]$ and all $w, v \in X$ such that $\|w\| \leq d$ and $\|v\| \leq d$. Applying (3.4) to $w = T_{s+u+t}(x) - T_s(z)$, $v = T_{s+u+t}(x) - T_u(z)$, $d = d(C)$ and $c = \frac{1}{2}$ we obtain the following inequality

$$\begin{aligned} &\|T_{s+u+t}(x) - \tfrac{1}{2}(T_s(z) + T_u(z))\|^2 \\ &\quad \leq \tfrac{1}{2}\|T_{s+u+t}(x) - T_s(z)\|^2 + \tfrac{1}{2}\|T_{s+u+t}(x) - T_u(z)\|^2 - \tfrac{1}{4}\lambda(\|T_s(z) - T_u(z)\|). \end{aligned}$$

Letting $t \to \infty$ and using the fact that z and $T_p(x)$ are comparable for any $p \in J$,

$$\tau\Big(\frac{T_s(z) + T_u(z)}{2}\Big) \leq \tau(z) - \frac{1}{4}\lambda(\|T_s(z) - T_u(z)\|).$$

Using the convexity of C and the minimality property of τ at z,

$$\tau(z) \leq \tau\Big(\frac{T_s(z) + T_u(z)}{2}\Big)$$

for all $s, u \in J$. Hence,

$$\tau(z) \leq \tau(z) - \tfrac{1}{4}\lambda(\|T_s(z) - T_u(z)\|),$$

which immediately implies that

$$\lambda(\|T_s(z) - T_u(z)\|) = 0. \tag{3.5}$$

Using (3.5) with $u = 0$ we immediately see that $z \in \mathrm{Fix}(\mathcal{S})$.

Let $f_1, f_2 \in \mathrm{Fix}(\mathcal{S})$ be comparable. Let $c \in [0, 1]$, $f = cf_1 + (1-c)f_2$. We need to prove that $f \in \mathrm{Fix}(\mathcal{S})$. Set $t \in J$. Without any loss of generality we can assume that $f_1 \preceq f_2$. Using the properties of $\preceq$, $f_1 \preceq f \preceq f_2$. Hence,

$$\|T_t(f) - f_1\| = \|T_t(f) - T_t(f_1)\| \leq \|f - f_1\| = (1-c)\|f_1 - f_2\|, \tag{3.6}$$

and similarly

$$\|T_t(f) - f_2\| \leq c\|f_1 - f_2\|. \tag{3.7}$$

Since X, as uniformly convex, is strictly convex it follows from (3.6) and (3.7) that $T_t(f) = cf_1 + (1-c)f_2 = f$. □

4. Applications to differential equations and dynamical systems

In this section we assume that $(X, \preceq)$ is an ordered Banach space in the sense that $x \preceq y \Rightarrow \alpha x \preceq \alpha y$ for $\alpha \geq 0$ and that $a \preceq b$, $c \preceq d \Rightarrow a + c \preceq b + d$. Note that this is a very typical situation and that the condition (2.1) follows from the above assumptions. Let us fix $x \in C$.

In this section we consider the following initial value problem (IVP) for an unknown function $u(x,\cdot) : [0,\infty) \to C$.

$$\begin{cases} u(x,0) = x \\ u'(x,t) + (I - H_t)(u(x,t)) = 0, \end{cases} \tag{4.1}$$

where $H_t : C \to C$ are monotone nonexpansive mappings with respect to the order $\preceq$. The notation $u'(x,t)$ denotes the derivative of the function $t \mapsto u(x,t)$. We assume that $x \preceq H_t(x)$ for every $t \in [0,\infty)$. Our aim is to construct a solution $u(x,\cdot)$ for the IVP, (4.1), such that $x \preceq u(x,t)$ for every $t \in [0,\infty)$. To achieve this we need to construct $u(x,\cdot)$ such that

$$u(x,t) = e^{-t}x + \int_0^t e^{s-t} H_s(u(s))\, ds.$$

It is easy to show, using the standard methods of the Bochner integration, that the above formula gives us the required solution. Define then $T_t : C \to C$ by $T_t(x) = u(x,t)$ for all $t \geq 0$. We will show that $\mathcal{S} = \{T_t : t \geq 0\}$ forms a monotone nonexpansive semigroup in the sense of Definition 2.4.

Let us introduce the following convenient notations which will be used throughout this section. For any $t > 0$ we define

$$K(t) = 1 - e^{-t} = \int_0^t e^{s-t}\, ds.$$

Let us fix $A > 0$. For a Bochner measurable function $v : [0,A] \to X$, $t \in [0,A]$, and any $\tau = \{t_0, \ldots, t_n\}$, a subdivision of the interval $[0,A]$, we define

$$S_\tau(v)(t) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} v(t_i).$$

Let us start with the following technical lemma.

Lemma 4.1 [12, Lemma 4.4], [5, Example 2.1]. *Let $f, g : [0,A] \to X$ be two Bochner-integrable $\|\cdot\|$-bounded functions. Then for every $t \in [0,A]$*

$$\|e^{-t}g(t) + \int_0^t e^{s-t} f(s)\, ds\| \leq e^{-t}\|g(t)\| + K(t) \sup_{s\in[0,t]} \|f(s)\|.$$

Define the sequence of functions $u_n : C \times [0,A] \to C$ by the following inductive formula:

$$\begin{cases} u_0(x,t) = x \\ u_{n+1}(x,t) = e^{-t}x + \displaystyle\int_0^t e^{s-t} H_s(u_n(x,s))\, ds. \end{cases}$$

Observe that all $u_n(x,t)$ belong to C. Indeed, for a given subdivision τ of $[0,A]$ define

$$u^\tau_{n+1}(x,t) = e^{-t}x + S_\tau(H_{t_i}(u_n(x,t_i)))(t).$$

Since C is convex it is easy to prove by induction that $u_n^\tau(x,t) \in C$. By the properties of the Bochner integral,

$$\|u_n^\tau(x,t) - u_n(x,t)\| \to 0,$$

as $|\tau| \to 0$. Since C is closed it follows that $u_n(x,t) \in C$.

We will show now that for every $n = 0, 1, 2, \dots$ and every $t \in [0, A]$

$$u_n(x,t) \preceq u_{n+1}(x,t). \tag{4.2}$$

Let us prove (4.2) by induction. It follows from $x \preceq H_s(x)$ that

$$u_0(x,t) = x = e^{-t}x + (1 - e^{-t})x \preceq e^{-t}x + \int_0^t e^{s-t} H_s(x)\, ds = u_1(x,t).$$

Assume now that (4.2) holds for $n = k$. Because each H_s is monotone then

$$u_{k+1}(x,t) = e^{-t}x + \int_0^t e^{s-t} H_s(u_k(x,s))\, ds \preceq e^{-t}x + \int_0^t e^{s-t} H_s(u_{k+1}(x,s))\, ds = u_{k+2}(x,t).$$

Note that (4.2) can be rewritten as

$$x = u_0(x,t) \preceq u_1(x,t) \preceq \cdots u_n(x,t) \preceq u_{n+1}(x,t) \preceq \cdots,$$

which implies that x and all $u_n(x,t)$ are comparable to each other (in a monotone way) and that they belong to the order interval $[x, \to)$.

Now we will prove that $u_n(x,t)$ is a Cauchy sequence. To do this it is enough to validate the following inequality:

$$\|u_{n+p}(x,t) - u_n(x,t)\| \le K^n(A)\, d(C), \tag{4.3}$$

for any $p \in \mathbb{N}$ and all $t \in [0, A]$. We prove this by induction with respect to n with p fixed arbitrarily. For $n = 0$ (4.3) is trivial. Assume now (4.3) holds for $n = k$. Noting that

$$u_{k+1+p}(x,t) - u_{k+1}(x,t) = \int_0^t e^{s-t}(H_s(u_{k+p}(x,s)) - H_s(u_k(x,s)))\, ds$$

and applying (4.1) with $g(t) = 0$ and $f(t) = H_s(u_{k+p}(x,s)) - H_s(u_k(x,s))$, then

$$\begin{aligned}
\|u_{k+1+p}(x,t) - u_{k+1}(x,t)\| &\le K(A) \sup_{s\in[0,A]} \|H_s(u_{k+p}(x,s)) - H_s(u_k(x,s))\| \\
&\le K(A) \sup_{s\in[0,A]} \|u_{k+p}(x,s) - u_k(x,s)\| \le K(A)K(A)^n\, d(C) = K(A)^{n+1}\, d(C),
\end{aligned}$$

finishing the proof of (4.3). Hence, $u_n(x,t)$ is Cauchy and, since C is closed, there exists $u(x,t) \in C$ such that $\|u_n(x,t) - u(x,t)\| \to 0$ as $n \to \infty$. Since $u_n(x,t) \in [x, \to)$ and $[x, \to)$ is closed it follows that $x \preceq u(x,t)$. Following an inductive proof similar to the one above, it is easy to show that if $x \preceq y$ then $u(x,t) \preceq u(y,t)$. In addition,

$$\|H_s(u_n(x,s)) - H_s(u(x,s))\| \le \|u_n(x,s) - u(x,s)\| \le K(A)^n\, d(C),$$

which implies that both $\|u_n(x,s)-u(x,s)\|$ and $\|H_s(u_n(x,s))-H_s(u(x,s))\|$ tend to zero uniformly in $t \in [0,A]$. Therefore,

$$u(x,t) = e^{-t}x + \int_0^t e^{s-t} H_s(u(s))\,ds,$$

as claimed. It is clear that the solution of the IVP (4.1) can be extended to a solution $u(x,t)$ defined on $[0,+\infty)$ such that its restriction to the interval $[0,A]$ is the limit of the sequence $\{u_n(x,t)\}$.

Let us denote $T_t(x) = u(x,t)$. We will show that $\mathcal{T} = \{T_t\}_{t\geq 0}$ is a monotone nonexpansive semigroup in the sense of Definition 2.4. Observe first that if $x \preceq y$ then $x \preceq u(x,t) = T_t(x) \preceq u(y,t) = T_t(y)$, which proves that each T_t is monotone. By an easy induction one can show that $\|u_n(x,t)-u_n(y,t)\| \leq \|x-y\|$ which implies that $\|u(x,t)-u(y,t)\| \leq \|x-y\|$ and consequently that $\|T_t(x)-T_t(y)\| \leq \|x-y\|$, that is, that each T_t is monotone nonexpansive. Clearly, $T_0(x) = u(x,0) = x$ proving part (ii). The function $t \mapsto T_t(x) = u(x,t)$ is norm continuous as the almost uniform limit of continuous functions $t \mapsto u_n(x,t)$, which justifies (iv). It remains to be proved that $T_{t+s}(x) = T_t(T_s(x))$. Let us fix temporarily $n \in \mathbb{N}$. We know that

$$\|u_{n+m}(x,t+s) - u(x,t+s)\| \to 0, \tag{4.4}$$

as $m \to \infty$. On the other hand, it can be proved by induction on n and by the use of Lemma 4.1 (see [13, Lemma 5.1]) that

$$\|u_n(u(x,s),t) - u(_{n+m}(x,t+s)\| \leq \sum_{i=n+1}^{n+m+1} K^i(s)\,d(C) + K^{n+1}(t)\,d(C). \tag{4.5}$$

Using (4.4) with (4.5),

$$\begin{aligned}\|u_n(u(x,s),t) - u(x,t+s)\| &= \lim_{m\to\infty} \|u_n(u(x,s),t) - u_{n+m}(x,t+s)\| \\ &\leq \lim_{m\to\infty} \sum_{i=n+1}^{n+m+1} K^i(s)\,d(C) + K^{n+1}(t)\,d(C) \to 0,\end{aligned}$$

as $n \to \infty$. On the other hand,

$$\|u_n(u(x,s),t) - T_t(T_s(x))\| = \|u_n(T_s(x),t) - T_t(T_s(x))\| \to 0.$$

From the uniqueness of the limit if follows that $T_t(T_s(x)) = u(x,t+s) = T_{t+s}(x)$, as claimed.

Let us summarize the results of this section. First we proved that the IVP, (4.1), has the solution $u(x,t)$ such that $x \preceq u(x,t)$ for a given $x \in C$. Let us emphasize again that we only assume nonexpansiveness of each of H_t on comparable elements of C, which can be a much smaller set than C. Recall that in Section 2 of this paper we presented nontrivial examples of such monotone nonexpansive mappings. Next we proved that the solution set for this IVP forms a monotone nonexpansive semigroup, giving us a nontrivial example of such a semigroup. Moreover, such a situation is quite typical

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in mathematics and applications. For instance, in the theory of dynamical systems, the Banach space X would define the state space and the mapping $(t, x) \to T_t(x)$ would represent the evolution function of a dynamical system. The question about the existence of common fixed points, and about the structure of the set of common fixed points (addressed in our Theorems 3.1 and 3.2), can be interpreted as a question of whether there exist stationary points for this process, that is, elements of C that are fixed during the state space transformation T_t at any given point of time t, and if yes, what the structure of a set of such points may look like and how such points can be constructed algorithmically. In the setting of this paper, the state space may be an infinite dimensional. Therefore, it is natural to apply these results not only to deterministic dynamical systems but also to stochastic dynamical systems.

5. Notes on the construction of common fixed points

In [6], the authors proved convergence of some iterative processes to a fixed point of a monotone nonexpansive mapping using an algorithm based on the Krasnoselskii–Ishikawa iterations. It is, however, much harder to obtain such methods for finding a common fixed point of a monotone nonexpansive semigroup. For some special cases when this can be done, we refer the reader to [14], where the author introduced new algorithms while applying some of the techniques from [15].

6. Conclusions

We have proved the existence of common fixed points for monotone contractive and monotone nonexpansive semigroups of nonlinear mappings acting within a convex and weakly compact subset of a Banach space equipped with a partial order. We also discussed some properties of the set of common fixed points: a form of uniqueness in the contraction case, and a form of convexity in the case of nonexpansiveness. In both cases, an existence of an $x \in C$ such that x is comparable with any $T_t(x)$ was the critical assumption. In the nonexpansive case, we also used several geometrical properties of uniformly convex Banach spaces. In each case we demonstrated an existence of a common fixed point by showing an existence of an element of C on which a suitably defined type function attains its infimum. We also provided a generic example of a monotone nonexpansive semigroup and touched upon its application to the theory of differential equations and dynamic systems. Knowing that the common fixed points actually do exist, while important on its own, is also fundamental for the construction of such algorithms, see [14].

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