MAXIMAL SUBSETS OF A GIVEN SET HAVING NO TRIPLE IN COMMON WITH A STEINER TRIPLE SYSTEM ON THE SET

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Introduction. Let E be a finite set containing n elements, $n \equiv 1, 3 \pmod 6$, $S = S(E)$ a Steiner triple system on E, i.e. each unordered pair of elements of E is a subset of exactly one triple in S. Let T be a subset of E such that none of the triples of elements of T is a member of S. Erdős has asked (in a recent letter to the authors) for the maximal size of such a set T. Denote $\max|T|$ for fixed n and S by $f(n,S)$. We prove in this note the following result:

$$\text{(i)} \qquad f(n,S) \leq \begin{cases} \frac{n+1}{2} & \text{if } n \equiv 3,7 \pmod{12} \qquad (1) \\ \frac{n-1}{2} & \text{if } n \equiv 1,9 \pmod{12} \qquad (2) \end{cases}$$

(ii) for every $n \equiv 1, 3 \pmod 6$ there exists a Steiner triple system S^0 such that equality holds in i.

The upper bound of $f(n,S)$. Let E, $|E| = n$, $S = S(E)$ and T be defined as in the introduction. In order to prove (1) consider some fixed element x of T. The number of triples in S which contain x and also at least one element of T different from x, is equal to $|T| - 1$. Indeed, the number of pairs of T which contain x is $|T| - 1$ and no two pairs of T can occur, by the definition of T, in the same triple of S. Moreover each pair has to occur in one of the triples of S. Since the total number of triples in S containing x is $\frac{n-1}{2}$, clearly $|T| - 1 \leq \frac{n-1}{2}$. Thus $|T| \leq \frac{n+1}{2}$.

This upper bound may be slightly improved in case (2). In this

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case the assumption that $|T| = \frac{1}{2}(n+1)$ leads to a contradiction. Equality (3) implies that each triple in S which contains the element x must contain another element of T. Hence each pair of E - T has to occur in a triple of S consisting of elements of E - T. This implies the existence of a Steiner triple system S(E - T), which is impossible because $|E - T| = n - \frac{1}{2}(n+1) = \frac{1}{2}(n-1) \equiv 0 \pmod 2$. Therefore

$$|T| < \frac{n+1}{2}, \text{ which proves } |T| \leq \frac{n-1}{2}.$$

<u>The existence of S^0</u>. In order to prove (ii), we observe that the classical construction of Steiner triple systems [1] is done, partitioning the set E into subsets T and E - T, with $|T| = \frac{1}{2}(n+1)$ in case (1) and $T = \frac{1}{2}(n-1)$ in case (2) and then constructing the triples which contain the pairs of T by assigning to each pair of T a certain element of E - T. Hence T has the required number of elements as well as the property prescribed in the introduction. Consequently we can choose S_0 to be one of these Steiner triple systems.

REFERENCE

1. Eugene Netto, Lehrbuch der Combinatorik. Chelsea Publishing Company, New York, N.Y.

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