SOME RADICAL CONSTRUCTIONS FOR ASSOCIATIVE RINGS

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Introduction

Dickson's construction [1] of radical and semi-simple classes for certain abelian categories is a rather straightforward procedure in comparison with the methods traditionally used in more general situations. In §2 of the present paper we use a well-known characterization of the lower radical class to obtain, *via* consideration of maps with accessible images, a similar "homomorphic orthogonality" characterization of radical and semi-simple classes of associative rings. By substituting certain other subring properties for accessibility, we are then able to obtain simple constructions of various types of radical classes, including those which are *strict* in the sense first used by Kurosh [3] for groups.

1. Preliminaries

All rings considered are associative unless the contrary is clearly stated. In connection with a non-empty class $\mathscr{X}$ of rings we use the following notation: $L(\mathscr{X})$, $U(\mathscr{X})$ are respectively the lower and upper radical classes defined by $\mathscr{X}$; $S(\mathscr{X})$ is the smallest semi-simple class containing $\mathscr{X}$. For radical-theoretic terms we refer the reader to Divinsky's book [2]. If I is an ideal of a ring R we denote this by writing $I \lhd R$. A subring A of a ring R is *accessible* if there is a finite chain

$$A = A_0 \lhd A_1 \lhd \cdots \lhd A_n \lhd R.$$

Let Σ be a property of subrings, possession of which we indicate by the symbol $<$, such that

M1. if $A < B$ and $B < C$ then $A < C$;
M2. if $A \lhd B$ then $A < B$;
M3. if $A < B$ then $f(A) < f(B)$ for any homomorphism f from B;
M4. if $A \subseteq B \lhd C$ and $A < C$, then $A < B$.

For example we have properties satisfying M1–M4 when $A < B$ means

(i) A is a subring of B,
(ii) A is transfinitely accessible in B,
(iii) there is a chain

$$A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = B$$

where each A_i is a one-sided ideal of A_{i+1}.

Let $[A, B]$ be the set of homomorphisms $f : A \to B$ with $\mathrm{Im}(f) < B$ and for a non-void class $\mathscr{X}$ of rings let

$$\mathscr{X}^0 = \{A \mid [X, A] = \{0\} \forall X \in \mathscr{X}\}$$
$$\mathscr{X}^+ = \{B \mid [B, X] = \{0\} \forall X \in \mathscr{X}\}.$$

LEMMA 1.1. *For non-void classes $\mathscr{X}, \mathscr{Y}$ of rings, we have*

$$\mathscr{X} \subseteq \mathscr{Y} \Rightarrow \mathscr{X}^0 \supseteq \mathscr{Y}^0 \text{ and } \mathscr{X}^+ \supseteq \mathscr{Y}^+;$$
$$\mathscr{X} \subseteq (\mathscr{X}^0)^+; \ \mathscr{X} \subseteq (\mathscr{X}^+)^0.$$

COROLLARY 1.2. *For any non-void class $\mathscr{X}$ of rings we have $((\mathscr{X}^0)^+)^0 = \mathscr{X}^0$ and $((\mathscr{X}^+)^0)^+ = \mathscr{X}^+$.*

2. The standard constructions

In the special case of the situation described in §1 where Σ means accessibility, we write $\mathscr{X}^*, \mathscr{X}^\#$ for $\mathscr{X}^0, \mathscr{X}^+$ respectively.

THEOREM 2.1. *For any non-void class $\mathscr{X}$ of rings, $L(\mathscr{X}) = (\mathscr{X}^*)^\#$.*

PROOF. Let $\bar{\mathscr{X}}$ denote the homomorphic closure of $\mathscr{X}$. Then $L(\mathscr{X}) = L(\bar{\mathscr{X}}) = \{R \mid$ every non-zero homomorphic image of R has a non-zero accessible subring $\in \bar{\mathscr{X}}\}$ [5]. Now $\mathscr{X}^*$ is the class of rings with no non-zero accessible subrings in $\bar{\mathscr{X}}$, so

$$L(\mathscr{X}) = \{R \mid R \text{ has no non-zero homomorphic image in } \mathscr{X}^*\}.$$

Hence, clearly, $(\mathscr{X}^*)^\# \subseteq L(\mathscr{X})$. Since $\mathscr{X}^*$ is hereditary, the reverse inclusion holds also.

PROPOSITION 2.2. *Let $\mathscr{R}$ be a radical class, $\mathscr{S}$ the corresponding semi-simple class. Then* (i) $\mathscr{S} = \mathscr{R}^*$; (ii) $\mathscr{R} = \mathscr{S}^\#$.

PROOF. (i) $\mathscr{S} = \mathscr{R}^*$ since $\mathscr{S}$ is hereditary and $\mathscr{R}$ is homomorphically closed.
(ii) $\mathscr{R} = L(\mathscr{R}) = (\mathscr{R}^*)^\# = \mathscr{S}^\#$.

COROLLARY 2.3. $\mathscr{R} = (\mathscr{R}^*)^\#$ *and* $\mathscr{S} = (\mathscr{S}^\#)^*$.

PROPOSITION 2.4. *For any non-void class $\mathscr{X}$ of rings,* (i) *$\mathscr{X}^\#$ is a radical class and* (ii) *$\mathscr{X}^*$ is a semi-simple class.*

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PROOF. (i) $\mathscr{X}^\# = ((\mathscr{X}^\#)^*)^\# = L(\mathscr{X}^\#)$.
(ii) $\mathscr{X}^* = ((\mathscr{X}^*)^\#)^* = L(\mathscr{X})^*$.

THEOREM 2.5. *Let $\mathscr{X}$ be a non-void class of rings. Then*
(i) $S(\mathscr{X}) = (\mathscr{X}^\#)^*$
(ii) $U(\mathscr{X}) = \mathscr{X}^\#$.

PROOF. (i) Let $\mathscr{S}$ be a semi-simple class with $\mathscr{X} \subseteq \mathscr{S}$. Then $(\mathscr{X}^\#)^* \subseteq (\mathscr{S}^\#)^* = \mathscr{S}$.
(ii) $U(\mathscr{X})^* = S(\mathscr{X}) = (\mathscr{X}^\#)^*$, so $U(\mathscr{X}) = (U(\mathscr{X})^*)^\# = ((\mathscr{X}^\#)^*)^\# = \mathscr{X}^\#$.

3. The general constructions

We now revert to consideration of an arbitrary property Σ satisfying M1–M4. A radical class $\mathscr{R}$ is *Σ-strict* if for each ring, $R, \mathscr{R}(R)$ contains all subrings A such that $A < R$ and $A \in \mathscr{R}$. A class $\mathscr{C}$ is *Σ-hereditary* if $A \in \mathscr{C}$ whenever $A < R \in \mathscr{C}$.

PROPOSITION 3.1. *Let $\mathscr{R}$ be a radical class, $\mathscr{S}$ the corresponding semi-simple class. Then $\mathscr{R}$ is Σ-strict if and only if $\mathscr{S}$ is Σ-hereditary.*

We now obtain constructions for Σ-strict radical classes analogous to those of §2. We find it convenient to begin with semi-simple classes.

LEMMA 3.2. *Let R be a ring, $0 \neq J < R$, and let I be the ideal of R generated by J. Then $[J, I/K] \neq \{0\}$ for all $K \lhd I$, $K \neq I$.*

PROOF. If $J \nsubseteq K$, then $0 \neq (J + K)/K < I/K$, so the natural map $I \to I/K$ induces a non-trivial element of $[J, I/K]$. If $J \subseteq K$, then I is the ideal of R generated by K, so by Andrunakievich's Lemma ([2], page 107), $I^3 \subseteq K$. If $I^2 \subseteq K$ and $I = (J + RJ)R^1 \nsubseteq K$, where R^1 is the Dorroh extension of R, define $f: J \to I/K$ by

$$\begin{cases} f(x) = axb + K \text{ if } \exists\, a \in R, b \in R^1 \text{ with } aJb \nsubseteq K \\ f(x) = xb + K \text{ where } b \in R^1 \text{ is such that } Jb \nsubseteq K, \text{ if } RJR^1 \subseteq K. \end{cases}$$

Then for any $x, y \in J$, we have $f(xy) = 0 = f(x)f(y)$ and f is a non-zero ring homomorphism. If $I^2 = (J + RJ)I \nsubseteq K$ define $f: J \to I/K$ by

$$\begin{cases} f(x) = axb + K \text{ if } \exists\, a \in R, b \in I \text{ with } aJb \nsubseteq K \\ f(x) = xb + K \text{ where } b \in I \text{ is such that } Jb \nsubseteq K, \text{ if } RJI \subseteq K. \end{cases}$$

As before, f is a non-zero ring homomorphism. (This argument is due to Stewart [8]; see also [9]). But in both cases $\mathrm{Im}(f) \lhd I/K$: in the first case, I/K is a zeroring and in the second $\mathrm{Im}(f)$ is contained in the annihilator of I/K. Thus $f \in [J, I/K]$.

PROPOSITION 3.3. *For any non-empty class $\mathscr{X}$ of rings, $\mathscr{X}^0$ is a Σ-hereditary semi-simple class.*

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PROOF. It is clear that $\mathscr{X}^0$ is Σ-hereditary (and hence hereditary), so we need only check condition F of [2]. Let R be a ring every non-zero ideal of which has a non-zero homomorphic image in $\mathscr{X}^0$, X a member of $\mathscr{X}$ and $f \in [X, R]$. Let I be the ideal of R generated by $\mathrm{Im}(f)$. Then if $I \neq 0$, I/K belongs to $\mathscr{X}^0$ for some $K \neq I$. By Lemma 3.2, $[\mathrm{Im}(f), I/K]$ has a non-zero element g. But $\mathrm{Im}(g\bar{f}) = \mathrm{Im}(g) < I/K$, where $\bar{f}: X \to \mathrm{Im}(f)$ is the map induced by f, so $g\bar{f} \in [X, I/K] = \{0\}$. Hence $g = 0$ and we conclude that $I = 0$ and $R \notin \mathscr{X}^0$.

PROPOSITION 3.4. *Let $\mathscr{R}$ be a Σ-strict radical class, $\mathscr{S}$ the corresponding semi-simple class. Then* (i) $\mathscr{S} = \mathscr{R}^0$ *and* (ii) $\mathscr{R} = \mathscr{S}^+$.

PROOF. (i) If $R \in \mathscr{R}^0$, it has no non-zero Σ-subrings from $\mathscr{R}$ and in particular no non-zero ideals from $\mathscr{R}$, so $R \in \mathscr{S}$. If $R \notin \mathscr{R}^0$, we have $A < R$ for some non-zero $A \in \mathscr{R}$, since $\mathscr{R}$ is homomorphically closed. But $\mathscr{S}$ is Σ-hereditary, so $R \notin \mathscr{S}$.

(ii) $\mathscr{R} = \mathscr{S}^* = \mathscr{S}^+$.

COROLLARY 3.5. *If $\mathscr{R}$ is a Σ-strict radical class, then $\mathscr{R} = (\mathscr{R}^0)^+$, and if $\mathscr{S}$ is a Σ-hereditary semi-simple class, then $\mathscr{S} = (\mathscr{S}^+)^0$.*

PROPOSITION 3.6. *For any non-void class $\mathscr{X}$ of rings, $\mathscr{X}$ is a Σ-strict radical class.*

PROOF. $\mathscr{X}^+ = ((\mathscr{X}^+)^0)^+$ is the radical class whose semi-simple class is $(\mathscr{X}^+)^0$.

THEOREM 3.7. *Let $\mathscr{X}$ be a non-void class of rings. Then*

(i) $(\mathscr{X}^0)^+$ *is the smallest Σ-strict radical class containing $\mathscr{X}$.*

(ii) $(\mathscr{X}^+)^0$ *is the smallest Σ-hereditary semi-simple class containing $\mathscr{X}$.*

(iii) $\mathscr{X}^+$ *is the largest Σ-strict radical class with respect to which all members of $\mathscr{X}$ are semi-simple.*

The proof follows that of Theorem 2.5.

4. Generalizations and limitations

The results of §2 can be transcribed for any category where radical theory is viable and semi-simple classes are hereditary, e.g. the categories of alternative rings and groups. The relevant characterization of the lower radical for these two situations was given in [4] and [7] respectively. The results of §3 can be obtained for groups by the rearrangement of some results of Kurosh [3]. Kurosh's proofs not lend themselves to direct translation into ring-theoretic terms.

In any category suitable for radical theory, $\mathscr{X}^{\#}$ is a radical class for any non-void $\mathscr{X}$, but $\mathscr{X}^*$ need not be semi-simple. Some constructions due to Ryabukhin [6] provide examples of non-associative rings A_1, A_2, A_3 such that the only proper ideal of A_{i+1} is $A_i \oplus A_i$, $i = 1, 2$ and $A_1, A_2/(A_1 \oplus A_1)$, $A_3/(A_2 \oplus A_2)$ are pairwise non-isomorphic simple rings. Let $\mathscr{X} = \{A_1\}$. Then $(A_2 \oplus A_2)/(A_1 \oplus A_1 \oplus A_2)$,

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$A_3/(A_2 \oplus A_2) \in \mathscr{X}^*$, i.e. every non-zero ideal of A_3 has a non-zero homomorphic image in $\mathscr{X}^*$. But from the chain

$$A_1 \lhd A_1 \oplus A_1 \lhd A_2 \lhd A_2 \oplus A_2 \lhd A_3$$

we see that $A_3 \notin \mathscr{X}^*$.

In general $\mathscr{X}^\#$ is the upper radical class defined by the *hereditary closure* of $\mathscr{X}$.

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