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THE RANK THEOREM FOR LOCALLY LIPSCHITZ CONTINUOUS FUNCTIONS

BY

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ABSTRACT. The Rank Theorem is proved for locally Lipschitz continuous functions $f:R^n \to R^p$ with generalized derivative of constant rank.

1. **Introduction.** Let $f:R^n \to R^p$ be a locally Lipschitz continuous function. The properties of such functions are of interest in a wide variety of applications, many of which are described in Frank Clarke's monograph [6].

According to Rademacher's Theorem [7; p. 216, 3.1.6] a locally Lipschitz continuous function is differentiable almost everywhere. In [6; p. 70, 2.6.1] a generalized derivative is defined for such functions in the following manner: the *generalized Jacobian* or *derivative* of f at x, denoted $\partial f(x)$, is the convex hull of all $p \times n$ matrices obtained as the limit of a sequence of the form $Jf(x_i)$, where $x_i \to x$, x_i is a point at which f is differentiable, and $Jf(x_i)$ is the usual $p \times n$ matrix of partial derivatives of $f(x)$ at x_i.

Many properties of derivatives for differentiable functions can be extended to generalized derivatives of locally Lipschitz maps. For example, there are generalized versions of chain rules, the Mean Value Theorem, the Inverse Function Theorem, and the Implicit Function Theorem.

A version of the Rank Theorem is given by A. Auslender in [1].

In this work we present (3.1) a (somewhat) more general Rank Theorem. A key proposition, (2.1), allows us to drop the hypothesis in [1] that the same submatrix of each element A of $\partial f(x)$ has the rank of A.

We use the Rank Theorem to describe the local structure of certain Lipschitz continuous functions with singularities. The results can be regarded as generalizations of some of the results in [11] for the case $n = p$.

Our notation will follow that in [6].

2. **Convex sets of matrices of constant rank.** Let Ω be a compact, convex set of $p \times n$ matrices, all of rank k, and let Ω_1 be the compact, convex subset of R^n

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formed by the first rows of the elements of Ω. Let $\mathscr{R}(A)$ be the row vector space of $A \in \Omega$. The following proposition will be used to prove the Rank Theorem in Section 3.

2.1. PROPOSITION. *There is an* $(n - k)$*-dimensional subspace* H^{n-k} *of* R^n *transverse at* 0 *to the row vector space of any element of* Ω, *and a* $(p - k)$*-dimensional subspace* L^{p-k} *of* R^p *which is transverse at* 0 *to the column vector space of any element of* Ω.

The proof (2.4) will be preceded by two lemmas. The existence of L^{p-k} will follow from the existence of H^{p-k} for the set Ω^T consisting of the transposes of the matrices in Ω.

2.2. LEMMA. *No open neighborhood of* $0 \in R^n$ *is contained in* Ω_1.

PROOF. Suppose an open neighborhood of 0 were contained in Ω_1. Choose matrices $A_0, A_1, \ldots, A_n \in \Omega$ so that for $A = 1/(n + 1) \sum_{j=0}^n A_j$, $\alpha_1(A)$ = first row of $A = 0$, and $\{\alpha_1(A_j)\}_{j=0}^n$ are the vertices of an n-dimensional simplex in R^n.

Let $\alpha_{i_j}(A)$, $j = 1, 2, \ldots, k$ be rows of A which span its row vector space $\mathscr{R}(A)$. Without loss of generality we may assume that

$$\alpha_{i_j}(A) = e_j = (0, \ldots, \underset{\substack{\uparrow \\ j\text{th position}}}{1}, 0, \ldots, 0)$$

and relabel so that $i_j = j$, $j = 1, 2, \ldots, k$ and old row 1 becomes row $k + 1$.

Since $\alpha_{k+1}(A) = 0 \in \operatorname{int} \operatorname{co}\{\alpha_{k+1}(A_j)\}_{j=0}^n$ we can find an arbitrarily small perturbation A_ϵ of A such that $A_\epsilon \in \Omega$, distance $(A, A_\epsilon) < \lambda(\epsilon)$, and $\alpha_{k+1}(A_\epsilon) = \epsilon \cdot e_{k+1}$, where $\lambda(\epsilon) \to 0$ as $\epsilon \to 0$.

If $\epsilon \neq 0$ is sufficiently small, then $\alpha_{k+1}(A_\epsilon)$ is linearly independent of $\alpha_1(A_\epsilon), \ldots, \alpha_k(A_\epsilon)$. For otherwise there exist $x_1, x_2, \ldots, x_k$ with

$$\alpha_{k+1}(A_\epsilon) = \sum_{i=1}^k x_i \alpha_i(A_\epsilon) = \epsilon \cdot e_{k+1}.$$

Now $\alpha_i(A_\epsilon) = e_i + \delta_i(\epsilon)$, where $\delta_i(\epsilon) = (\delta_{i1}, \ldots, \delta_{in})$ and $\delta_{ij}(\epsilon) \to 0$ as $\epsilon \to 0$. Thus we have the system of equations

$$\begin{pmatrix} 1 + \delta_{11} & \delta_{21} & \ldots & \delta_{k1} \\ \delta_{12} & 1 + \delta_{22} & \ldots & \delta_{k2} \\ \vdots & \vdots & \ldots & \vdots \\ \delta_{1k} & \vdots & \ldots & 1 + \delta_{kk} \\ \delta_{1k+1} & \vdots & \ldots & \delta_{kk+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \epsilon \end{pmatrix}$$

If ϵ is chosen small enough, the first k rows of the coefficient matrix still has rank k, so $x_1 = x_2 = \ldots = x_k = 0$. But that contradicts $\epsilon \neq 0$; thus $\alpha_{k+1}(A_\epsilon)$ is independent of $\{\alpha_i(A_\epsilon); i = 1, 2, \ldots, k\}$ and A_ϵ has rank $k + 1$, which also is a contradiction and we have shown no open neighborhood of $0 \in R^n$ is contained in Ω_1.

2.3. LEMMA. *If $\Omega_1 \neq 0$, then $\mathscr{R}(A) \cap \Omega_1$ contains nonzero vectors for all $A \in \Omega$.*

PROOF. It is trivial if $\alpha_1(A) \neq 0$, so suppose there exists $A_0 \in \Omega$ with $\alpha_1(A_0) = 0$ and $\mathscr{R}(A_0) \cap \Omega_1 = 0$. Then $\mathscr{R}(A_0)$ is a k-dimensional subspace of R^n which is transverse to span$\{\alpha_1(A)\}$ for all $A \in \Omega$ for which $\alpha_1(A) \neq 0$.

Without loss of generality we may assume that $\mathscr{R}(A_0)$ is generated by $\alpha_i(A_0) = e_i$, $i = 2, 3, \ldots, k + 1$, and that there exists $B \in \Omega$ with $\alpha_1(B) = e_1$. Then by an argument similar to that used in (2.2) we can show that if $\epsilon > 0$ is sufficiently small, then

$$\{\alpha_i(\epsilon B + (1 - \epsilon)A_0)\}_{i=1}^{k+1}$$

is an independent set, which contradicts the fact that dim $\mathscr{R}(A) = k$ for all $A \in \Omega$.

2.4. PROOF OF (2.1). It follows from (2.2) that there exists an $(n - 1)$-dimensional hyperplane H for which $H \cap \Omega_1 = 0$. Let $\widetilde{\mathscr{R}}(A) = \mathscr{R}(A) \cap H$.

If $A \in \Omega$ with $\alpha_1(A) \neq 0$, then since $\alpha_1(A)$ is transverse to H, $\widetilde{\mathscr{R}}(A)$ must have dimension $(k - 1)$.

If $A \in \Omega$ with $\alpha_1(A) = 0$, then by (2.3) there is a vector $v \in \mathscr{R}(A)$, $v \neq 0$ and v transverse to H. Since $\mathscr{R}(A)$ is spanned by $\widetilde{\mathscr{R}}(A) \cup \{v\}$, $\widetilde{\mathscr{R}}(A)$ must again have dimension $k - 1$.

Recoordinatize and identify H with R^{n-1}. This induces a set $\widetilde{\Omega}$ of matrices which is again a compact convex set: a nonsingular affine transformation of Ω. The corresponding set of first rows $\widetilde{\Omega}_1$ has H as a support (or separating) plane and $\widetilde{\mathscr{R}}(\widetilde{A}) = \mathscr{R}(\widetilde{A}) \cap H$ is $(k - 1)$-dimensional for all $\widetilde{A} \in \widetilde{\Omega}$. Also, $\widetilde{\mathscr{R}}(\widetilde{A})$ is generated by the last $p - 1$ rows of $\widetilde{A}$, and the rank of the lower left $(p - 1) \times (n - 1)$ submatrix $\hat{A}$ of $\widetilde{A} = k - 1 =$ rank of the last $p - 1$ rows.

The rest of the proof is by induction. Suppose that we have proved the row vector space portion of (2.1) for $(p - 1) \times (n - 1)$ matrices of rank $k - 1$. By this inductive hypothesis there exists a subspace H^{n-k} of $H = R^{n-1}$ with dimension $n - k$ such that

$$\begin{aligned}\mathscr{R}(\hat{A}) \cap H^{n-k} &= \mathscr{R}(\widetilde{A}) \cap H^{n-k} = \mathscr{R}(\widetilde{A}) \cap H \cap H^{n-k} \\ &= \widetilde{\mathscr{R}}(\widetilde{A}) \cap H^{n-k} = 0 \text{ for all } \widetilde{A} \in \widetilde{\Omega}.\end{aligned}$$

We claim that $\mathscr{R}(A) \cap H^{n-k} = 0$ for all $\widetilde{A} \in \widetilde{\Omega}$.

For suppose that $W \in \mathscr{R}(A) \cap H^{n-k}$. Then $w = x \cdot \alpha_1(A) + y \cdot \widetilde{\alpha}$ for some vector $\widetilde{\alpha} \in \widetilde{\mathscr{R}}(\widetilde{A}) = \mathscr{R}(\hat{A})$ and scalars x, y. If $\alpha_1(A) \neq 0$, then since

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$w \in \mathscr{R}(A) \cap H^{n-k}$, $w \in H$ so $x = 0$ since the first row is transverse to H. Thus $w = y \cdot \tilde{\alpha} \in \mathscr{R}(\tilde{A}) \cap H^{n-k} = 0$, and $y = 0$ also. If $\alpha_1(\tilde{A}) = 0$, then $w = y \cdot \tilde{\alpha}$ too, which is as before.

To complete the induction, set $n - p = q \geqq 0$, $p - k = s \geqq 0$, and start the induction with $(1 + s) \times (1 + q + s)$ matrices of rank 1. Denote $1 + s$ by M, $1 + q + s$ by N. The result for this case will be a consequence of the proposition (2.5), below.

The result is immediate if 2.5 (a) applies (see below). If 2.5 (b) applies, then without loss of generality we may suppose that all columns of A are multiples of $e_1^T = (1, 0, \ldots, 0)^T$ for each $A \in \Omega$.

A separating hyperplane H^{N-1} in R^N for the set $\Omega_1 = \{\alpha_1(A), A \in \Omega\}$ can be found as a consequence of (2.2), and this hyperplane is the required $(N - 1)$-dimensional subspace which is transverse to $\mathscr{R}(A)$ for each $A \in \Omega$.

2.5. PROPOSITION. *Let Ω be a compact convex set of $M \times N$ matrices of rank* 1. *Then either* (*a*) *the rows of every matrix $A \in \Omega$ are multiples of some fixed vector, or* (*b*) *the columns of every matrix $A \in \Omega$ are multiples of some fixed vector.*

PROOF. By [8], if A, B and $(1/2)(A + B)$ are matrices of rank 1, then either (a) or (b) applies to A and B (and hence the segment joining them). On account of this result we see that the set of ordered pairs $\Omega \times \Omega$ is the union of two equivalence classes $\mathscr{R}$ and $\mathscr{C}$, where $(A, B) \in \mathscr{R}$ if the rows of A and B are all multiples of a fixed vector; likewise $\mathscr{C}$ is defined in terms of columns.

Now suppose there exist A, B, C, $\in \Omega$ such that $(A, B) \in \mathscr{R} - \mathscr{C}$ and $(A, C) \in \mathscr{C} - \mathscr{R}$. If $(B, C) \in \mathscr{R}$, then $(A, C) \in \mathscr{R}$; if $(B, C) \in \mathscr{C}$, then $(A, B) \in \mathscr{C}$. Both of these are a contradiction. Thus for any $A \in \Omega$, and for any B, $C \in \Omega$, either (A, B), (A, C) belong to $\mathscr{R}$ or to $\mathscr{C}$, so $\Omega \times \Omega = \mathscr{R}$ or $\mathscr{C}$ and the assertion is proved. (We wish to thank E. Leonard for referring us to [8]).

3. **The Rank Theorem.**

3.1. THEOREM. *Suppose that $f: R^n \to R^p$ is Lipschitz in an open set $W \subset R^n$, and that each element of $\{\partial f(x); x \in W\}$ has rank k.*

Then there exist neighborhoods U of $x \in W$, V of $f(x)$, and Lipschitz homeomorphisms (*with Lipschitz inverses*) *$\alpha: U \to R^n$, $\beta: V \to R^p$ such that*

$$\beta f \alpha^{-1}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_k, 0, \ldots, 0).$$

PROOF. We may as well assume $x = 0$, $f(x) = 0$, since otherwise the map f may be replaced by $\bar{f}(z) = f(x + z) - f(x)$. Then according to the chain rule [6; p. 75, Corollary] $\partial \bar{f}(z) \cdot v \subset \mathrm{co}\{\partial f(x + z) \cdot I_n \cdot v\} = \partial f(x + z) \cdot v$ for any $v \in R^n$, I_n the $n \times n$-identity map. Thus every element of $\partial \bar{f}(z)$ in a neighborhood of $z = 0$ has rank k.

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By Lemma (2.1), there is an $(n - k)$-dimensional subspace H^{n-k} of R^n transverse at 0 to the row vector space of each element of $\partial f(0)$, and a $(p - k)$-dimensional subspace L^{p-k} of R^p which is transverse at 0 to the range of each element of $\partial f(0)$.

In fact, it follows from the upper-semicontinuity property of the derivative that there exists an open set $X \subset W$ about 0 so that H^{n-k} and L^{p-k} are transverse to the row and column vector spaces, respectively, of each element of $\partial f(x)$, $x \in X$.

Let H_k, L_k be the orthogonal complements of H^{n-k} and L^{p-k}. Choose orthonormal bases $\{v_1, v_2, \ldots, v_n\}$ for R^n and $\{w_1, w_2, \ldots, w_p\}$ for R^p so that $\{v_1, v_2, \ldots, v_k\}$ spans H_k and $\{w_1, w_2, \ldots, w_k\}$ spans L_k.

Define

$$Q:R^n = H_k \times H^{n-k} \rightarrow R^n = R^k \times R^{n-k} \text{ and}$$

$$P:R^p = L_k \times L^{p-k} \rightarrow R^p = R^k \times R^{p-k}$$

to be the orthogonal transformations generated by $Q(v_i) = e_i$ and $P(w_i) = e_i$, where e_i is the ith member of the canonical bases for R^n and R^p.

Define $g:R^k \times R^{n-k} \rightarrow R^k \times R^{p-k}$ by $g = PfQ^{-1}$. Then in some neighborhood of 0 each element of $\partial g(y) = P\partial f(Q^{-1}(y))Q^{-1}$ has rank k. In addition, there is a neighborhood Y of 0 so that $0 \times R^{n-k}$ is transverse to the row vector space and $0 \times R^{p-k}$ is transverse to the range of every element of $\partial g(y)$, $y \in Y$.

Let $\pi:R^k \times R^{p-k} \rightarrow R^k = R^k \times \{0\}$ be projection. If $y \in Y$ and $M \in \partial g(y)$, then the rows of πM are the first k rows of M. The set $\{\pi Me_i, i \leqq k\}$ spans $R^k \times \{0\}$. This is so since if

$$0 = \sum_{i=1}^{k} a_i \pi M e_i = \pi M \sum_{i=1}^{k} a_i e_i, \text{ then } M \sum_{i=1}^{k} a_i e_i \subset \ker \pi = \{0\} \times R^{p-k}.$$

The range of M is transverse to $\{0\} \times R^{p-k}$, so $\sum_{i=1}^{k} a_i e_i \in \ker M$. Since the row vector space of M is transverse to $\{0\} \times R^{n-k}$, the kernel of M is transverse to $R^k \times \{0\}$ and $\sum_{i=1}^{k} a_i e_i = 0$. Thus $a_i = 0$ for all i and $\{\pi Me_i, i \leqq k\}$ is a linearly independent set.

Let $y = (y_1, y_2, \ldots, y_n)$ and define $G:R^k \times R^{n-k} \rightarrow R^k \times R^{n-k}$ by

$$G(y) = (\pi g(y), y_{k+1}, \ldots, y_n).$$

If $S_G \subset R^n$ is the set of points at which G is not differentiable and $JG(y)$ is the Jacobian matrix of G at $y \notin S_G$, then

$$JG(y) = \begin{pmatrix} A(y) & B(y) \\ 0 & I_{n-k} \end{pmatrix}$$

where $(A(y)B(y))$ is the $k \times n$-matrix $J(\pi g) = \pi Jg(y)$.

By definition, $\partial G(0) = \mathrm{co}\{\lim_{q_i \to 0} JG(q_i), q_i \notin S_G\}$, so that each element of $\partial G(0)$ has the form

$$\begin{pmatrix} A & B \\ 0 & I_{n-k} \end{pmatrix}$$

where $(A\ B) \in \partial(\pi g)(0) = \pi \partial g(0) =$ first k-rows of an element of $\partial g(0)$, and A is invertible. Thus each element of $\partial G(0)$ has rank n, and by the Inverse Function Theorem [6; p. 253, 7.1.1] there exists a neighborhood $Z \subset R^n$ of 0 and a Lipschitz function $F: Z \to R^k \times R^{n-k}$ such that $GF(x) = x$.

We may as well assume that Z is the open unit ball centered at 0, since for some $r > 0$ the ball of radius r about 0 is contained in Z and we could replace G by $\bar{G}(y) = (1/r)G(y)$.

If $x = (x_1, x_2, \dots, x_n)$, then $x = GF(x) = (\pi g(F(x)), x_{k+1}, \dots, x_n)$, so that $\pi gF(x) = (x_1, x_2, \dots, x_k)$. Thus $gF(x) = (x_1, x_2, \dots, x_k, h(x))$ where $h: R^n \to R^{p-k}$.

If $x \notin S_{gF} \cup S_F = S$, then G is differentiable at $F(x) = y$ and $gFG(y) = g(y)$ is differentiable. Define

$$\begin{aligned} \partial_S(gF)(x) &= \mathrm{co}\left\{\lim_{q_i \to x} J(gF)(q_i), q_i \notin S\right\} \\ &= \mathrm{co}\left\{\lim_{q_i \to x} Jg(p_i)JF(q_i), q_i \notin S\right\}, \text{ where } p_i = F(q_i). \end{aligned}$$

At each q_i, $JF(q_i)$ is the inverse of $JG(p_i)$, so $JF(q_i)$ has the form

$$\begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix}$$

where

$$Jg(p_i) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and $D = CA^{-1}B$. Thus

$$Jg(p_i)JF(q_i) = \begin{pmatrix} I_k & 0 \\ CA^{-1} & 0 \end{pmatrix},$$

which implies that the elements of $\partial_S(gF)(x)$ have their last $n - k$ columns zero.

Since

$$\partial_S(gF)(x) = \left(\frac{I_k\ 0}{\partial_S h(x)}\right),$$

it must be the case that the last $n - k$ columns of elements of $\partial_S h(x)$ are zero.

Let $\bar{x} = (x_1, x_2, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_n)$. According to the Mean Value Theorem [6; p. 72, 2.6.5] and [6; p. 71, 2.6.4]

$$h(\bar{x}) - h(x) \in \text{co}\ \partial_S h[\bar{x}, x](\bar{x} - x)$$

where the right hand side is the convex hull of the set of all points of the form $\partial_S h(q) \cdot (\bar{x} - x)$, q a point on the line joining $\bar{x}$ and x. Since $\bar{x} - x \in \{0\} \times R^{n-k}$ and the last $n - k$ columns of $\partial_S h(q)$ are zero, $h(\bar{x}) = h(x)$. As a consequence we may consider h to be a function of the first k variables alone.

If $z = (z_1, z_2, \ldots, z_p)$, define $\gamma: R^k \times R^{p-k} \to R^k \times R^{p-k}$ by

$$\gamma(z) = (z_1, z_2, \ldots, z_k, h(z_1, z_2, \ldots, z_k)) - (0, \ldots, 0, z_{k+1}, \ldots, z_p).$$

Then each element of $\partial\gamma(z)$ has the form

$$\begin{pmatrix} I_k & 0 \\ * & -I_{p-k} \end{pmatrix}$$

so γ is a Lipschitz homeomorphism of a neighborhood of 0 in R^p to a neighborhood of 0 in R^p with a Lipschitz inverse.

For some ball Z_r of radius r about 0 in Z, $\gamma g F$ is defined and $\gamma g F(x_1, \ldots, x_n) = (x_1, x_2, \ldots, x_k, 0, \ldots, 0)$.

Let φ_k be a diffeomorphism of the open ball of radius r in R^k to all of $R^k \times \{0\}$, and let φ_n be a diffeomorphism extending φ_k from the open ball of radius r in R^n to all of R^n. Let φ_p be a diffeomorphism of the open ball of radius r in R^p that extends φ_k.

Now define

$$U = Q^{-1}F(Z_r),\ V = P^{-1}\gamma^{-1}\varphi_p^{-1}(R^p)$$

and $\alpha: U \to R^n$ by $\alpha = \varphi_n G Q$, $\beta: V \to R^p$ by $\beta = \varphi_p \gamma P$.
Then $\beta f \alpha^{-1}$ has the properties desired.

3.2. Example. Let $h: R^2 \to R$ be defined by $h(x, y) = 0$ if $x < 0$, and $h(x, y) = x$ if $x \geqq 0$. Define $F: R^2 \to R^2$ by $F(x, y) = (|x|, h(x, y))$.

If $x > 0$,

$$\partial F(x, y) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix};$$

if $x < 0$,

$$\partial F(x, y) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix};$$

at any point $(0, y)$,

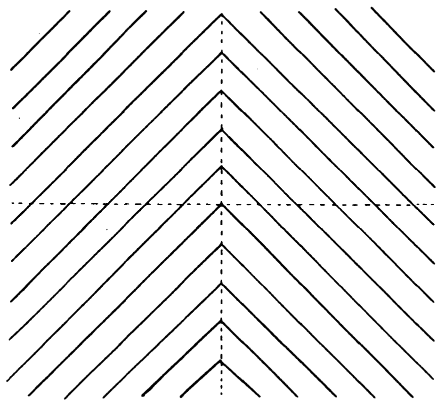
$$\partial F(0, y) = \left\{ t\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + (1 - t)\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \leqq t \leqq 1 \right\}.$$

In particular, rank of $\partial F(x, y)$ is 1 at all points (x, y), so (3.1) applies. However, F does not satisfy the hypothesis stated in Theorem 2 of [1] on the y axis.

4. **Some consequences of the Rank Theorem.** The map $f:R^n \to R^p$ at x is *locally topologically equivalent* (resp., *locally Lipschitz equivalent*) to $g:K \to L$ if there are open neighborhoods U of x, and v of $f(x)$, and homeomorphisms (resp., locally Lipschitz homeomorphisms with locally Lipschitz inverses) α and β such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f|U"] \arrow[d, "\alpha"'] & V \arrow[d, "\beta"] \\ K \arrow[r, "g"] & L \end{tikzcd}$$

4.1. Example. Consider $f:R^2 \to R^1$ defined by $f(x, y) = |x| + y$. Then if $x \neq 0$, $\partial f(x, y) = (1, 1)$ or $(-1, 1)$. If $x = 0$, then $\partial f(x, y) = ([-1, 1], 1)$. Thus in a neighborhood of any point, f satisfies the hypothesis of the Rank Theorem (3.1) and f is locally Lipschitz equivalent to the projection map $\rho:R^2 \to R^1$. The level curves for f are illustrated in the figure.



A map f is *proper* if $f^{-1}(K)$ is compact for all compact sets K. This is equivalent to requiring that if $|x| \to \infty$ so does $|f(x)|$.

We will say a locally Lipschitz continuous function f is *nonsingular* if every element of $\partial f(x)$, $x \in R^n$, has maximal rank. It is singular if $\partial f(x)$ contains a singular matrix for at least one x; the set of points at which $\partial f(x)$ contains a singular matrix is the *critical set* of f.

4.2. Proposition. *A proper locally Lipschitz map $f:R^n \to R^p$, $n > p$, must be singular.*

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Example (4.1) shows that (4.2) is not true if the condition "proper" is removed.

PROOF. Suppose f is nonsingular. According to the Rank Theorem (3.1), at each $x \in R^n$, f is locally topologically equivalent to the projection map $\rho: R^n \to R^p$. Since f is proper, it is a fiber bundle map ([2; p. 151]: they assume f is monotone, but the hypothesis is not used in the proof) with a compact manifold for fiber and base space R^p. (The map f must be onto since it is both an open and closed map.) Thus R^n would be homeomorphic to the product of R^p and a compact manifold, which is a contradiction unless $n = p$.

The following two theorems are a consequence of (3.1) and the results in [3], [4], [5], and [12].

4.3. THEOREM. *Let $f: R^n \to R^p$ be a locally Lipschitz continuous function with $n - p = 0, 1$, or 2, $p \neq 1$, and $(n, p) \neq (4, 2)$. Suppose the set of critical points is discrete. Then at each $x \in R^n$, f is locally topologically equivalent to one of the following maps*:

(a) *the projection map $\rho: R^n \to R^p$*;

(b) *$\sigma: C \to C$ defined by $\sigma(z) = z^d$ $(d = 2, 3, \ldots)$ where C is the complex plane*;

(c) *$T: C \times C \to C \times R$ defined by $T(z, w) = (2z \cdot \bar{w}, |w|^2 - |z|^2)$, where $\bar{w}$ is the complex conjugate of w.*

In case $(n, p) = (4, 2)$, there are many (distinct) examples of maps f with a single point in the critical set at which f is not locally topologically equivalent to ρ. For example, complex polynomial maps η defined by $\eta(z, w) = z^j + w^k$, $j > k \geqq 2$.

In case (1, 1) and (2, 1), f at x is locally topologically equivalent to ρ, $f: R^1 \to R^1$ defined by $f(x) = x^2$, $f: C \to R$ by $f(z) = |z|^2$ or by $f(z) = \mathrm{Re}\, z^d$ ([10] and [13]).

4.4. THEOREM. *Let $f: R^n \to R^p$ be a locally Lipschitz continuous function with $n \geqq p$. Suppose that the critical set is discrete and that each critical point x is a component of $f^{-1}(f(x))$. Then at each $x \in R^n$, f is locally topologically equivalent to*

(a) *the projection map $\rho: R^n \to R^p$, or*

(b) *$(n, p) = (n, 1)$ and f at x is locally topologically equivalent to $\sigma(x) = |x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$, or*

(c) *$(n, p) = (2, 2)$ and f at x is locally topologically equivalent to $\sigma: C \to C$ defined by $\sigma(z) = z^d$, $d = 2, 3, \ldots$ where C is the complex plane, or*

(d) *$(n, p) = (4, 3)$, $(8, 5)$ or $(16, 9)$ and f at x is locally topologically equivalent to*

$T: A \times A \to A \times R$ defined by

$T(x, y) = (2\overline{xy}, |y|^2 - |x|^2)$, where A is the complex numbers, the quaternions or the Cayley numbers. (*See* [9; p. 102] *for a discussion of the map* T).

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